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NOTE

Inequalities for the Crank

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Garvan first defined certain "vector partitions" and assigned to each such partition a "rank." Denoting by $N_V(r, m, n)$ the (weighted) count of the vector partitions of n with rank r modulo m, he gave a number of relations between the numbers $N_V(r, m, mn+k)$ when $m=5$, 7 and 11, $0 \leqslant r$, $k<m$. The true crank whose existence was conjectured by Dyson was discovered by Andrews and Garvan who also showed that $N_V(r, m, n) = M(r, m, n)$ unless $n=1$, where $M(r, m, n)$ denotes the number of partitions of n whose cranks are congruent to r module m. In the case of module 11, a simpler form of Garvan's results have been found by Hirschhorn. In fact, the Hirschhorn result was derived using Winquist's identity, but the details were omitted. In this work, from the simpler form we deduce some new inequalities between the $M(r, 11, 11n+k)$'s and give the details of Hirschhorn's result. We also prove some conjectures of Garvan in the case of module 7.

1. INTRODUCTION

First, we introduce some standard notations

$$(z; q)_\infty = \prod_{n=1}^{\infty} (1 - zq^{n-1}),$$
$$[z; q]_\infty = (z; q)_\infty (z^{-1}q; q)_\infty, \qquad |q| < 1. \tag{1}$$

Note that

$$-z[zq; q]_\infty = -z[z^{-1}; q]_\infty = [z; q]_\infty = [z^{-1}q; q]_\infty. \tag{2}$$

For simplicity, we also write, as in [G1],

$$P(a) := [y^a; y^{11}]_\infty (y^{11}; y^{11})_\infty, \qquad 1 \leqslant a < 11 \tag{3}$$

$$P(0) := (y^{11}; y^{11})_\infty \tag{4}$$

$$Q(b) := [y^b; y^{33}]_\infty (y^{33}; y^{33})_\infty, \qquad 1 \leqslant b < 33. \tag{5}$$

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The variables y and q are always related by $y=q^{11}$. From (2), we have

$$\begin{aligned} P(11-a)&=P(a), \qquad & P(11+a)&=-y^a P(a) \\ Q(33-a)&=Q(a), \qquad & Q(33+a)&=-y^a P(a) \end{aligned} \tag{6}$$

which we shall use without explicit mention below. We have [G1]

$$\sum_{k=0}^{t-1} \zeta^k \sum_{n=0}^{\infty} M(k, t, n)\, q^n = \frac{(q;q)_\infty}{(\zeta q;q)_\infty\, (\zeta^{-1}q;q)_\infty}, \tag{7}$$

where t is a prime and $\zeta=\exp(2\pi i/t)$. The main problem is to express the right-hand side of (7) as a polynomial in q of degree $t-1$ whose coefficients are power series in $y=q^t$. When $t=5$, 7 and 11, the expressions were given in [G1].

Considering the obvious relation

$$(y;y)_\infty = P(0)\, P(1)\, P(2)\, P(3)\, P(4)\, P(5), \tag{8}$$

Hirschhorn's identity [H] can be stated as follows

$$\begin{aligned} &\frac{(q;q)_\infty}{(\zeta q;q)_\infty\, (\zeta^{-1}q;q)_\infty} \\ &\quad = \frac{P(0)}{P(1)} + (\zeta+\zeta^{-1}-1)\, \frac{P(5)\, P(0)}{P(2)\, P(3)}\, q + (\zeta^2+\zeta^{-2})\, \frac{P(3)\, P(0)}{P(1)\, P(4)}\, q^2 \\ &\qquad + (\zeta^3+\zeta^{-3}-1)\, \frac{P(2)\, P(0)}{P(1)\, P(3)}\, q^3 + (\zeta^4+\zeta^{-4}+\zeta^2+\zeta^{-2}+1)\, \frac{P(0)}{P(2)}\, q^4 \\ &\qquad - (\zeta^4+\zeta^{-4}+\zeta^2+\zeta^{-2})\, \frac{P(4)\, P(0)}{P(2)\, P(5)}\, q^5 + (\zeta^4+\zeta^{-4}+\zeta+\zeta^{-1})\, \frac{P(0)}{P(3)}\, q^7 \\ &\qquad + (\zeta^4+\zeta^{-4}+\zeta^3+\zeta^{-3}+\zeta+\zeta^{-1})\, \frac{P(1)\, P(0)}{P(4)\, P(5)}\, y q^8 \\ &\qquad - (\zeta^4+\zeta^{-4}+1)\, \frac{P(0)}{P(4)}\, q^9 - (\zeta^3+\zeta^{-3})\, \frac{P(0)}{P(5)}\, q^{10}. \end{aligned} \tag{9}$$

In the next section, we show how (9) can be derived from Winquist's identity. We write, as in [G1],

$$R_{ij}(k) := \sum_{n \geqslant 0} (M(i, 11, 11n+k) - M(j, 11, 11n+k))\, q^n. \tag{10}$$

By using arguments analogous to that of Section 4 in [G1], (9) allows calculation of the $R_{ij}(k)$

THEOREM 1.

$$R_{01}(0) = \frac{(q^{11}; q^{11})_\infty}{[q; q^{11}]_\infty} \tag{11}$$

$$-\frac{1}{2} R_{01}(1) = R_{12}(1) = \frac{[q^5; q^{11}]_\infty (q^{11}; q^{11})_\infty}{[q^2; q^{11}]_\infty [q^3; q^{11}]_\infty} \tag{12}$$

$$R_{12}(2) = -R_{23}(2) = -\frac{[q^3; q^{11}]_\infty (q^{11}; q^{11})_\infty}{[q; q^{11}]_\infty [q^4; q^{11}]_\infty} \tag{13}$$

$$R_{01}(3) = -R_{23}(3) = R_{34}(3) = \frac{[q^2; q^{11}]_\infty (q^{11}; q^{11})_\infty}{[q; q^{11}]_\infty [q^3; q^{11}]_\infty} \tag{14}$$

$$\begin{aligned} R_{01}(4) &= -R_{12}(4) = R_{23}(4) = -R_{34}(4) = R_{45}(4) \\ &= \frac{(q^{11}; q^{11})_\infty}{[q^2; q^{11}]_\infty} \end{aligned} \tag{15}$$

$$\begin{aligned} R_{12}(5) &= -R_{23}(5) = R_{34}(5) = -R_{45}(5) \\ &= \frac{[q^4; q^{11}]_\infty (q^{11}; q^{11})_\infty}{[q^2; q^{11}]_\infty [q^5; q^{11}]_\infty} \end{aligned} \tag{16}$$

$$R_{01}(7) = -R_{12}(7) = R_{34}(7) = -R_{45}(7) = -\frac{(q^{11}; q^{11})_\infty}{[q^3; q^{11}]_\infty} \tag{17}$$

$$\begin{aligned} R_{01}(8) &= -R_{12}(8) = R_{23}(8) = -R_{45}(8) \\ &= -q \frac{[q; q^{11}]_\infty (q^{11}; q^{11})_\infty}{[q^4; q^{11}]_\infty [q^5; q^{11}]_\infty} \end{aligned} \tag{18}$$

$$R_{01}(9) = -R_{34}(9) = R_{45}(9) = -\frac{(q^{11}; q^{11})_\infty}{[q^4; q^{11}]_\infty} \tag{19}$$

$$R_{23}(10) = -R_{34}(10) = \frac{(q^{11}; q^{11})_\infty}{[q^5; q^{11}]_\infty} \tag{20}$$

and all other $R_{b,b+1}(d)$, $0 \leqslant b \leqslant 4$, *are zero.*

This is the simpler form of Theorem (6.7) of [G1]. We prove the following inequalities, for $n \geqslant 0$,

$$M(0, 11, 11n) \geqslant M(1, 11, 11n) \tag{21}$$

$$M(1, 11, 11n+1) \geqslant M(0, 11, 11n+1) \tag{22}$$

$$M(0, 11, 11n+3) \geqslant M(1, 11, 11n+3) \tag{23}$$

$$M(0, 11, 11n+4) \geqslant M(1, 11, 11n+4) \tag{24}$$

$$M(1, 11, 11n+5) \geqslant M(2, 11, 11n+5) \tag{25}$$

$$M(1, 11, 11n+7) \geqslant M(0, 11, 11n+7) \tag{26}$$

$$M(0, 11, 11n+9) \geqslant M(1, 11, 11n+9) \tag{27}$$

$$M(2, 11, 11n+10) \geqslant M(3, 11, 11, +10) \tag{28}$$

$$M(0, 7, 7n) \geqslant M(1, 7, 7n) \tag{29}$$

$$M(2, 7, 7n+2) \geqslant M(1, 7, 7n+2) \tag{30}$$

(29) and (30) are conjectures of Garvan, [G1]

2. THE HIRSCHHORN'S RESULT

Our method is completely analogous to that of Garvan [G1]. We need the Jacobi's triple product identity (Thm. 2.8 in [A])

$$[z; q]_\infty (q; q)_\infty = \sum_{m=-\infty}^{\infty} (-1)^m z^m q^{m(m-1)/2}. \tag{31}$$

Setting $m = 11n + t$, $-5 \leqslant t \leqslant 5$ in (31), we find

$$[s; q]_\infty (q, q)_\infty = \sum_{t=-5}^{5} (-1)^t z^t q^{t(t-1)/2} [z^{11} y^{5+t}; y^{11}]_\infty (y^{11}; y^{11})_\infty. \tag{32}$$

From (32) with the help of (6), we find

$$\begin{aligned} &[\zeta; q^3]_\infty (q^3; q^3)_\infty \\ &\quad = (1-\zeta)\, Q(15) + (\zeta^2 - \zeta^{-1})\, Q(12)\, q^3 + (\zeta^4 - \zeta^{-3})\, Q(6)\, yq^7 \\ &\qquad + (\zeta^{-4} - \zeta^5)\, Q(3)\, y^2 q^8 + (\zeta^{-2} - \zeta^3)\, Q(9)\, q^9 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
&[\zeta q; q^3]_\infty (q^3; q^3)_\infty \\
&\quad = (Q(16) + \zeta^4 y^2 Q(5)) - \zeta(Q(14) + \zeta^2 y Q(8))\, q \\
&\qquad - \zeta^{-1}(Q(13) + \zeta^6 y^3 Q(2))\, q^2 - \zeta^{-3} y(Q(7) - \zeta^{-1} y Q(4))\, q^4 \\
&\qquad + \zeta^2 P(0)\, q^5 + \zeta^{-2}(Q(10) - \zeta^{-3} y^3 Q(1))\, q^7.
\end{aligned} \tag{34}$$

Winquist's identity [W, Thm. 1] can be written as

$$\begin{aligned}
&[a; q]_\infty [b; q]_\infty [ab; q]_\infty [ab^{-1}; q]_\infty (q; q)_\infty^2 \\
&\quad = [a^3; q^3]_\infty (q^3; q^3)_\infty^2 \{[b^3 q; q^3]_\infty - b[b^3 q^2; q^3]_\infty\} \\
&\qquad - ab^{-1}[b^3; q^3]_\infty (q^3; q^3)_\infty^2 \{[a^3 q; q^3]_\infty - a[a^3 q^2; q^3]_\infty\}.
\end{aligned} \tag{35}$$

If we take (y^m, y^n, y^{11}) for (a, b, q) in (35), we have

$$\begin{aligned}
&P(m)\, P(n)\, P(m+n)\, P(m-n)\, P(0)^2 \\
&\quad = Q(3m)(Q(3n+11) - y^n Q(3n+22)) \\
&\qquad - y^{m-n} Q(3n)(Q(3m+11) - y^m Q(3m+22)).
\end{aligned} \tag{36}$$

Taking $a = \zeta^9$ and $b = \zeta^5$ in (35), Garvan [G1] finds

$$\begin{aligned}
&\frac{(q; q)_\infty}{(\zeta q; q)_\infty (\zeta^{-1} q; q)_\infty} \\
&\quad = (1-\zeta^3)^{-1} (1-\zeta^4)^{-1} (1-\zeta^5)^{-1} (1-\zeta^9)^{-1} (y; y)_\infty^{-1} \\
&\qquad \times \{[\zeta^5; q^3]_\infty ([\zeta^4 q; q^3]_\infty - \zeta^5 [\zeta^7 q; q^3]_\infty) \\
&\qquad - \zeta^5 [\zeta^4; q^3]_\infty ([\zeta^5 q; q^3]_\infty - \zeta^9 [\zeta^6 q; q^3]_\infty)\} (q^3; q^3)_\infty^2.
\end{aligned} \tag{37}$$

By using (33) and (34) together with the replacements, in (36), $(m, n) = (5, 2)$, $(5, 1)$, $(5, 3)$, $(4, 2)$, $(4, 1)$, $(4, 3)$, $(5, 4)$, $(2, 1)$, $(3, 2)$ and $(3, 1)$ respectively, and considering (8), we find, after some simplification, that the right-hand side of (33) reduces to the right-hand side of (9).

3. INEQUALITIES

By using Jacobi's triple product, we find

$$\frac{(q; q)_\infty}{[z; q]_\infty} = \frac{[-z; q]_\infty (q; q)_\infty}{[z^2; q^2]_\infty} = \frac{1}{[z^2; q^2]_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2}. \tag{38}$$

Now, taking $z=q^a$ and q^{11} for q in (38), we see that

$$\frac{(q^{11};q^{11})_\infty}{[q^a;q^{11}]_\infty} \tag{39}$$

has nonnegative coefficients in q. The equation (5.7) of [G2] also shows that (39) has nonnegative coefficients in q.

For the rest of the inequalities, we need the quintuple product identity [Go]

$$[-z;q]_\infty\,[z^2q;q]_\infty\,(q;q)_\infty=\{[z^3q;q^3]_\infty+z[z^3q^2;q^3]_\infty\}(q^3;q^3)_\infty. \tag{40}$$

If we use the obvious relations

$$\begin{aligned}[-z;q]_\infty\,[z;q]_\infty=[z^2;q^2]_\infty,\qquad [z^2;q^2]_\infty\,[z^2q;q^2]_\infty&=[z^2;q]_\infty\\ [z^3;q]_\infty=[z^3q;q^3]_\infty\,[z^3;q^3]_\infty\,[z^{-3}q;q^3]_\infty&\\ =[z^3q^2;q^3]_\infty\,[z^3;q^3]_\infty\,[z^{-3}q^2;q^3]_\infty&\end{aligned} \tag{41}$$

from (40), we have

$$\frac{[z^2;q]_\infty\,(q;q)_\infty}{[z;q]_\infty\,[z^3;q]_\infty}=\frac{(q^3;q^3)_\infty}{[z^3;q^3]_\infty\,[z^{-3}q;q^3]_\infty}+z\,\frac{(q^3;q^3)_\infty}{[z^3;q^3]_\infty\,[z^{-3}q^2;q^3]_\infty}. \tag{42}$$

From this we see that

$$\frac{[q^{2a};q^{11}]_\infty\,(q^{11};q^{11})_\infty}{[q^a;q^{11}]_\infty\,[q^{3a};q^{11}]_\infty} \tag{43}$$

has nonnegative coefficients in q for the values $a=1, 2, 3$. We also conjecture that the same holds for $a=4$ and $a=5$ (except one).

In the case of modulo 7, we have from Theorem 5.4) of [G1]

$$R_{01}(0)=\frac{[q^3;q^7]_\infty\,(q^7;q^7)_\infty}{[q;q^7]_\infty\,[q^2;q^7]_\infty} \tag{44}$$

$$R_{12}(2)=-R_{23}(2)=-\frac{[q^2;q^7]_\infty\,(q^7;q^7)_\infty}{[q;q^7]_\infty\,[q^3;q^7]_\infty}. \tag{45}$$

(42) also states that

$$\frac{[q^{2a}; q^7]_\infty (q^7; q^7)_\infty}{[q^a; q^7]_\infty [q^{3a}; q^7]_\infty} \tag{46}$$

has nonnegative coefficients in q for the values $a = 1, 2$.

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