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Uncertainty modelling and conditioning with convex imprecise previsions

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Abstract

Two classes of imprecise previsions, which we termed convex and centered convex previsions, are studied in this paper in a framework close to Walley's and Williams' theory of imprecise previsions. We show that convex previsions are related with a concept of convex natural extension, which is useful in correcting a large class of inconsistent imprecise probability assessments, characterised by a condition of avoiding unbounded sure loss. Convexity further provides a conceptual framework for some uncertainty models and devices, like unnormalised supremum preserving functions. Centered convex previsions are intermediate between coherent previsions and previsions avoiding sure loss, and their not requiring positive homogeneity is a relevant feature for potential applications. We discuss in particular their usage in (financial) risk measurement. In a final part we introduce convex imprecise previsions in a conditional environment and investigate their basic properties, showing how several of the preceding notions may be extended and the way the generalised Bayes rule applies.

Keywords: Convex imprecise previsions; Convex natural extension; Risk measures; Convex conditional imprecise previsions; Generalised Bayes rule

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1. Introduction

Imprecise probability theory is developed in [16] in terms of two major classes of (unconditional) imprecise previsions, relying upon reasonable consistency requirements: *avoiding sure loss* and *coherent* previsions. The condition of avoiding sure loss is less restrictive than coherence but is often too weak.

Coherent imprecise previsions have been studied more extensively, while imprecise previsions that avoid sure loss received less attention, and it is an interesting problem to state whether some special class of previsions avoiding sure loss can be identified, which is such that

(a) its properties are not too far from those of coherent previsions;
(b) it gives further insight into the theory of imprecise previsions or generalises some of its basic aspects;
(c) it may express beliefs which do not match with coherence but which are useful in formalising and dependably modelling certain kinds of problems.

This paper deals with two classes of imprecise previsions, convex and centered convex previsions, which let us provide some answers to points (a)–(c).

Convex and centered convex previsions were first studied in [12] and then in [14]. This paper is an extended version of [14] in a first part, then introduces convex conditional previsions in its Section 5. In the first part, Section 3, we quote without proof results proved in [12] or [14]; some of them are also generalised and proved in a conditional framework in Section 5.

After recalling some basic notions in Section 2, we study the larger class of convex lower previsions in Section 3.1. Although our conclusion is that convexity is an unsatisfactory consistency requirement—for instance, convex previsions do not necessarily avoid sure loss—it is however important as far as (b) is concerned. That is seen in Section 3.2, where a notion of convex natural extension is discussed which formally parallels the basic concept of natural extension in [16]. We characterise lower previsions whose convex natural extension is finite as those complying with the (mild) requirement of avoiding unbounded sure loss. In this case the convex natural extension indicates a canonical (least-committal) way of correcting them into a convex assessment. As discussed in Section 3.2.1, it is then easy to make a further correction to achieve the stronger (and more satisfactory) property of centered convexity.

Centered convex previsions are discussed in Section 3.3, together with generalisations of the important envelope theorem. They are a special class of previsions avoiding sure loss, retaining several properties of coherent imprecise previsions, and hence they appear to fulfil requirement (a).

Convex previsions are studied in the paper following a behavioural approach, close to those in [16,19]. Section 3.4 contains a brief discussion of alternative approaches, and makes some comparisons with notions developed in the literature [11,16] which are close to convex previsions.

Section 4 gives some answers to point (c). Here convex previsions provide a conceptual framework for certain kinds of uncertainty models, as shown in Examples 1 (overly prudential assessments) and 2 (supremum preserving functions). These models are sometimes employed in practice, although they cannot usually be regarded as satisfactory. Centered convex previsions do not require the positive homogeneity condition $\underline{P}(\lambda X) = \lambda\underline{P}(X)$, $\forall \lambda > 0$, and hence seem appropriate to capture risk aversion. In Section 4.1 we focus in particular on risk measurement problems, showing that the results in Section 3 may be used to define convex risk measures (centered or not) for an arbitrary set of random variables $\mathscr{D}$. In particular, the definition of convex risk measure coincides, when $\mathscr{D}$ is a linear space, with the concept of convex risk measure recently introduced in the literature to consider liquidity risks [6,7].

In Section 5 we introduce convex conditional imprecise previsions and explore their essential properties. It turns out that a meaningful part of the results for the unconditional case can be generalised. This applies in particular to the convex natural extension, when the convex conditional prevision is centered (Section 5.1). Further, convex conditional previsions are characterised by a set of axioms when their domain has a special structure, and the generalised Bayes rule, an important inferential device, holds for them too (Section 5.2). Conditional convexity can be exploited in particular to introduce convex measures for conditional risks (Section 5.3). Section 6 concludes the paper.

2. Preliminaries

Unless otherwise specified, in the sequel we shall denote with $\mathscr{D}$ an *arbitrary* set of *bounded* random variables (or gambles, in the notation of [16]). A *lower prevision* $\underline{P}$ (an *upper prevision* $\overline{P}$, a *prevision* P) on $\mathscr{D}$ is a real-valued function with domain $\mathscr{D}$. In particular, if $\mathscr{D}$ contains only indicator functions of events, $\underline{P}$ ($\overline{P}, P$) is termed lower probability (upper probability, probability).

Lower (and upper) previsions should satisfy some consistency requirements: the commonest are the condition of *avoiding sure loss* and the stronger *coherence* condition [16].

Definition 1. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a lower prevision on $\mathscr{D}$ that *avoids sure loss* iff, for all $n \in \mathbb{N}^+$, $\forall X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i))$, $\sup \underline{G} \geqslant 0$.

Definition 2. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a *coherent lower prevision* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_0, X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_0, s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0))$, $\sup \underline{G} \geqslant 0$.

The condition of avoiding sure loss is too weak under many respects: for instance, it does not require that $\underline{P}(X) \geqslant \inf X$, nor does it impose monotonicity. On the other hand, it is simpler to assess and to check than coherence.

Behaviourally, a lower prevision $\underline{P}(X)$ may be viewed as a supremum buying price for X [16], and $s(X - \underline{P}(X))$ is an *elementary gain* from a bet on X, with stake s. We shall say that the bet is *in favour* of X if $s \geqslant 0$, whilst $-s(X - \underline{P}(X))$ ($s \geqslant 0$) is an elementary gain from a bet *against* X. Definitions 1 and 2 both require that no admissible linear combination $\underline{G}$ of elementary gains originates a sure loss bounded away from zero. The difference is that the concept of avoiding sure loss considers only bets in favour of the X_i, while coherence considers also (at most) one bet against a random variable in $\mathscr{D}$.

We recall the following properties of coherent lower previsions, which hold whenever the random variables involved are in $\mathscr{D}$:

(a) $\underline{P}(\lambda X) = \lambda \underline{P}(X)$, $\forall \lambda > 0$ (positive homogeneity),
(b) $\inf X \leqslant \underline{P}(X) \leqslant \sup X$ (internality),
(c) $\underline{P}(X + Y) \geqslant \underline{P}(X) + \underline{P}(Y)$ (superlinearity).

Coherent precise previsions may be defined by modifying Definition 2 to allow $n \geqslant 0$ bets in favour of and $m \geqslant 0$ bets against random variables in $\mathscr{D}$ ($m, n \in \mathbb{N}$). A coherent precise prevision P is necessarily *linear* and *homogeneous*: $P(aX + bY) = aP(X) + bP(Y)$, $\forall a, b \in \mathbb{R}$. In particular $P(0) = 0$.

Coherent lower previsions may be characterised using precise previsions [16]:

Theorem 1 (*Lower envelope theorem*). *A lower prevision* $\underline{P}$ *on* $\mathscr{D}$ *is coherent iff* $\underline{P}$ *is the lower envelope of some set* $\mathscr{M}$ *of coherent precise previsions on* $\mathscr{D}$, *i.e. iff* $\underline{P}(X) = \inf_{P \in \mathscr{M}} P(X), \forall X \in \mathscr{D}$ (inf *is attained*).

Upper and *lower* previsions are customarily related by the *conjugacy* relation $\overline{P}(X) = -\underline{P}(-X)$. An upper prevision $\overline{P}(X)$ may be viewed as an infimum selling price for X and an *elementary gain* from a bet concerning X is written as $s(\overline{P}(X) - X)$. The definitions of coherence and of the condition of avoiding sure loss are modified accordingly.

When $\mathscr{D}$ is a set of bounded *conditional* random variables, the consistency requirements recalled above may be generalised as follows: firstly, we associate the elementary gain $g_i = s_i B_i (X_i - \underline{P}(X_i|B_i))$ to a bet on $X_i|B_i \in \mathscr{D}$, where the same symbol B_i is employed for both event B_i and its indicator function (de Finetti's convention). If $B_i = 0$ (i.e. if event B_i does not occur) the bet on $X_i|B_i$ is called off and nothing is won or lost, since then $g_i = 0$.

Secondly, when considering the elementary gains $g_1, \ldots, g_n$, we call *support* of $\underline{s} = (s_1, \ldots, s_n)$ the event $S(\underline{s}) = \vee\{B_i : s_i \neq 0, i = 1, \ldots, n\}$. Conditioning $\underline{G} = \sum_{i=1}^{n} g_i$ on $S(\underline{s})$ enables us to evaluate $\sup \underline{G}$ in the next definition only when at least one of the bets on $X_1|B_1, \ldots, X_n|B_n$ is effective.

Definition 3. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a conditional lower prevision *avoiding uniform loss* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_1|B_1, \ldots, X_n|B_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^{n} s_i B_i (X_i - \underline{P}(X_i|B_i))$, $\sup\{\underline{G}|S(\underline{s})\} \geqslant 0$.

We recall that the notion of avoiding uniform loss is the proper generalisation of the concept of avoiding sure loss to the conditional case (cf. [16,17]). It is less clear, however, how coherence should be generalised. In our opinion, Williams' definition [16,19], which we report in an equivalent form in Definition 4, is preferable under many respects, especially generality and simplicity, but other meaningful notions have also been considered in the literature [15,16,18].

Definition 4. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a *coherent conditional lower prevision* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_0|B_0, \ldots, X_n|B_n \in \mathscr{D}$, $\forall s_0, \ldots, s_n \geqslant 0$, defining $S^*(\underline{s}) = \vee\{B_i : s_i \neq 0, i = 0, \ldots, n\}$ and $\underline{G} = \sum_{i=1}^{n} s_i B_i (X_i - \underline{P}(X_i|B_i)) - s_0 B_0 (X_0 - \underline{P}(X_0|B_0))$, $\sup\{\underline{G}|S^*(\underline{s})\} \geqslant 0$.

Lastly, in the computations of Section 5 we will often make use of the fact that $\sup X|B = \sup_{\omega \Rightarrow B} X(\omega)$, where all ω belong to a sufficiently large underlying possibility space, and will also employ the equality $f(X_1, \ldots, X_n)|B = f(X_1|B, \ldots, X_n|B)$, where f is any real function (here, often $f = \underline{G}$).

3. Convex lower previsions

3.1. Convex previsions

Definition 5. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a *convex* [1] *lower prevision* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_0, X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n \geqslant 0$ such that $\sum_{i=1}^{n} s_i = 1$ (*convexity condition*), defining $\underline{G} = \sum_{i=1}^{n} s_i (X_i - \underline{P}(X_i)) - (X_0 - \underline{P}(X_0))$, $\sup \underline{G} \geqslant 0$.

Any coherent lower prevision is convex, since Definition 5 is obtained from Definition 2 adding the constraint $\sum_{i=1}^{n} s_i = s_0 = 1$ (note that we would get a definition equivalent to Definition 5 requiring only $\sum_{i=1}^{n} s_i = s_0 > 0$). Conversely, a convex lower prevision does not even necessarily avoid sure loss:

Proposition 1. *Let $\underline{P}$ be a convex lower prevision on $\mathscr{D}$ and let $0 \in \mathscr{D}$. Then $\underline{P}$ avoids sure loss iff $\underline{P}(0) \leqslant 0$.*

Convexity is characterised by a set of axioms if $\mathscr{D}$ has a special structure [12]:

[1] The term 'convex' refers to the convexity condition $\sum_{i=1}^{n} s_i = 1$ ($s_i \geqslant 0$), which distinguishes convex lower (upper) previsions from coherent lower (upper) previsions (cf. Definitions 2, 5 and 9) and convex natural extensions from natural extensions (cf. Definition 6 and Section 3.2.1). The term 'convex prevision' is therefore unrelated with convexity or concavity properties of previsions as real functions.

Theorem 2. *Let* $\underline{P} : \mathscr{D} \to \mathbb{R}$.

(*a*) *If* $\mathscr{D}$ *is a linear space containing real constants,* $\underline{P}$ *is a convex lower prevision iff it satisfies the following axioms:* [2]
  ($A1$) $\underline{P}(X + c) = \underline{P}(X) + c, \forall X \in \mathscr{D}, \forall c \in \mathbb{R}$ (*translation invariance*),
  ($A2$) $\forall X, Y \in \mathscr{D}$, *if* $Y \leqslant X$ *then* $\underline{P}(Y) \leqslant \underline{P}(X)$ (*monotonicity*),
  ($A3$) $\underline{P}(\lambda X + (1-\lambda)Y) \geqslant \lambda\underline{P}(X) + (1-\lambda)\underline{P}(Y), \forall X, Y \in \mathscr{D}, \quad \forall \lambda \in [0,1]$ (*concavity*).

(*b*) *If* $\mathscr{D}$ *is a convex cone,* $\underline{P}$ *is a convex lower prevision iff it satisfies* ($A3$) *and*
  ($A4$) $\forall \mu \in \mathbb{R}, \forall X, Y \in \mathscr{D}$, *if* $X \geqslant Y + \mu$ *then* $\underline{P}(X) \geqslant \underline{P}(Y) + \mu$.

Proposition 2 (*Some properties of convex lower previsions* [3]). *Let* $\underline{P}$ *be a convex lower prevision on* $\mathscr{D}$. *The following properties hold* (*whenever all random variables involved are in* $\mathscr{D}$):

(*a*) *If* $\underline{P}(0) \geqslant 0$, $\underline{P}(\lambda X) \geqslant \lambda\underline{P}(X), \forall \lambda \in [0,1]$ *and* $\underline{P}(\lambda X) \leqslant \lambda\underline{P}(X), \forall \lambda > 1$
(*b*) $\underline{P}(0) + \inf X \leqslant \underline{P}(X) \leqslant \underline{P}(0) + \sup X$
(*c*) $\forall \mu \in \mathbb{R}$, $\underline{P}^*(X) = \underline{P}(X) + \mu$ *is convex on* $\mathscr{D}$.

Property (a) shows that convexity is compatible with lack of positive homogeneity, but requires $\underline{P}(0) \geqslant 0$. If $\underline{P}(0) < 0$, the inequalities in (a) may or may not hold. For instance, they do not with the constant lower prevision $\underline{P}^* = -r < 0$ on $\mathscr{D} = \{0, X, \lambda X\}$ ($0 \in [\inf X, \sup X]$), which is convex $\forall \lambda$ (by property (c) with $\mu = -r$, $\underline{P} = 0$, since then $\underline{P}$ is coherent and hence convex), but $\underline{P}^*(\lambda X) < \lambda\underline{P}^*(X)$, $\forall \lambda \in [0,1[$, $\underline{P}^*(\lambda X) > \lambda\underline{P}^*(X)$, $\forall \lambda > 1$.

Property (b) highlights a sore point of convexity: $\underline{P}(X)$ need not belong to $[\inf X, \sup X]$ (*internality* may fail). [4] It also suggests that internality could be restored imposing $\underline{P}(0) = 0$, if $0 \notin \mathscr{D}$; by (c), if $0 \in \mathscr{D}$ and $\underline{P}(0) \neq 0$, then $\underline{P}^*(X) = \underline{P}(X) - \underline{P}(0)$ is convex and $\underline{P}^*(0) = 0$. Requiring $\underline{P}(0) = 0$ is also the only choice to make $\underline{P}$ avoid sure loss (Proposition 1), while assuring that (a) holds.

Thinking of the meaning of a lower prevision, it appears extremely reasonable to add condition $\underline{P}(0) = 0$ to convexity: it would be at least weird to give a non-zero estimate (even imprecise) of the non-random variable 0.

3.2. Convex natural extension

Before considering the stronger class of centered convex previsions, we introduce the notion of convex natural extension, which is strictly related to convexity.

[2] (A1) and (A2) can be replaced by $\underline{P}(X) - \underline{P}(Y) \leqslant \sup(X - Y), \forall X, Y \in \mathscr{D}$.

[3] Further properties are given in [12].

[4] Non-internality cannot anyway be two-sided: if there exists $X \in \mathscr{D}$ such that $\underline{P}(X) > \sup X$ ($\underline{P}(X) < \inf X$), then $\underline{P}(Y) > \inf Y$ ($\underline{P}(Y) < \sup Y$), $\forall Y \in \mathscr{D}$. This is easily seen applying Definition 5, with $n = 2$, $\{X_0, X_1\} = \{X, Y\}$.

Definition 6. Let $\underline{P} : \mathscr{D} \to \mathbb{R}$ be a lower prevision, Z an arbitrary bounded random variable. Define $g_i = s_i(X_i - \underline{P}(X_i))$, $L(Z) = \{\alpha : Z - \alpha \geqslant \sum_{i=1}^{n} g_i$, for some $n \geqslant 1$, $X_i \in \mathscr{D}, s_i \geqslant 0$, with $\sum_{i=1}^{n} s_i = 1\}$. $\underline{E}_c(Z) = \sup L(Z)$ is termed *convex natural extension* [5] of $\underline{P}$ on Z.

Clearly, $L(Z)$ is always non-empty (putting $n = 1$, $s_1 = 1$, $X_1 = X \in \mathscr{D}$ in its definition, $\alpha \in L(Z)$ for $\alpha \leqslant \inf Z - \sup X + \underline{P}(X)$), while $\underline{E}_c(Z)$ can in general be infinite. This situation is characterised in the next Proposition 3.

Definition 7. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a lower prevision that *avoids unbounded sure loss* on $\mathscr{D}$ iff there exists $k \in \mathbb{R}$ such that, for all $n \in \mathbb{N}^+$, $\forall X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n \geqslant 0$ with $\sum_{i=1}^{n} s_i = 1$, $\sup \sum_{i=1}^{n} s_i (X_i - \underline{P}(X_i)) \geqslant k$.

Remark 1. Definition 7 generalises Definition 1: $\underline{P}$ avoids unbounded sure loss if and only if $\underline{P} + k$ avoids sure loss for some $k \in \mathbb{R}$, since the last inequality in Definition 7 may be written as $\sup \sum_{i=1}^{n} s_i (X_i - (\underline{P}(X_i) + k)) \geqslant 0$ and the constraint $\sum_{i=1}^{n} s_i = 1$ is not restrictive for Definition 1. Note also that if $\underline{P} + k$ avoids sure loss, then so does $\underline{P} + h$, $\forall h \leqslant k$. Therefore, when $\underline{P}$ avoids unbounded sure loss, defining $\overline{k} = \sup\{k \in \mathbb{R} : \underline{P} + k \text{ avoids sure loss}\}$, $\underline{P}$ avoids sure loss too whenever $\overline{k} \geqslant 0$. As a further remark, it can be seen that the constraint $\sum_{i=1}^{n} s_i = 1$ is essential in Definition 7: wiping it out would make Definition 7 equivalent to Definition 1.

Proposition 3. *$\underline{E}_c(Z)$ is finite, $\forall Z$, iff $\underline{P}$ avoids unbounded sure loss.*

The condition of avoiding unbounded sure loss is rather mild. For instance, it clearly holds whenever $\mathscr{D}$ is finite. It is also implied by convexity [14], while the converse implication is generally not true. We state now some properties of the convex natural extension; an indirect characterisation will be given in Theorem 5.

Theorem 3. *Let $\underline{P} : \mathscr{D} \to \mathbb{R}$ be a lower prevision which avoids unbounded sure loss, $\underline{E}_c$ its convex natural extension, and $\mathscr{L} (\supset \mathscr{D})$ the set of all bounded random variables (on a large enough possibility space). Then*

(*a*) *$\underline{E}_c$ is a convex prevision on $\mathscr{L}$ and $\underline{E}_c(X) \geqslant \underline{P}(X), \forall X \in \mathscr{D}$.*
(*b*) *$\underline{P}$ is convex if and only if $\underline{E}_c = \underline{P}$ on $\mathscr{D}$.*
(*c*) *If $\underline{P}^*$ is a convex prevision on $\mathscr{L}$ such that $\underline{P}^*(X) \geqslant \underline{P}(X)$ $\forall X \in \mathscr{D}$, then $\underline{P}^*(Z) \geqslant \underline{E}_c(Z), \forall Z \in \mathscr{L}$.*
(*d*) *If $\underline{P}$ is convex, $\underline{E}_c$ is the minimal convex extension of $\underline{P}$ to $\mathscr{L}$.*
(*e*) *$\underline{P}$ avoids sure loss on $\mathscr{D}$ if and only if $\underline{E}_c$ avoids sure loss on $\mathscr{L}$.*

[5] The reason why $\underline{E}_c$ is termed 'extension' becomes patent from the later Theorem 3, especially (d).

3.2.1. The role of the convex natural extension

The properties of $\underline{E}_c$ closely resemble those of the *natural extension* $\underline{E}$ [16] of a lower prevision $\underline{P}$, whose definition differs from that of $\underline{E}_c$ only for the lack of the constraint $\sum_{i=1}^{n} s_i = 1$. In particular, as $\underline{E}$ characterises coherence of $\underline{P}$ ($\underline{P}$ is coherent iff $\underline{E}$ coincides with $\underline{P}$ on $\mathscr{D}$), $\underline{E}_c$ characterises convexity of $\underline{P}$.

Property (d) lets us extend $\underline{P}$ to *any* $\mathscr{D}' \supset \mathscr{D}$ (maintaining convexity) by considering the restriction of $\underline{E}_c$ to $\mathscr{D}'$. Moreover, (e) guarantees that $\underline{E}_c$ inherits the condition of avoiding sure loss when $\underline{P}$ satisfies it.

It is well-known that the natural extension $\underline{E}$ is finite iff $\underline{P}$ avoids sure loss, and when finite it can correct $\underline{P}$ into a coherent assessment in a canonical way. Analogously, the convex natural extension $\underline{E}_c$ is finite iff $\underline{P}$ avoids unbounded sure loss, and can be used to correct $\underline{P}$ into a convex assessment, although property (e) warns us that $\underline{E}_c$ will still incur sure loss if $\underline{P}$ does so. This problem can be solved using Proposition 2, (c): $\underline{P}^*(X) = \underline{E}_c(X) - \underline{E}_c(0)$ is a correction of $\underline{P}$ which avoids sure loss by Proposition 1, as $\underline{P}^*(0) = 0$. This also means that $\underline{P}^*$ is a centered convex prevision by the next Definition 8.

Alternatively, $\underline{E}_c$ may be employed to correct an assessment $\underline{P}$ which avoids unbounded sure loss (but not sure loss) into $\underline{P}'$, which avoids sure loss but is not necessarily convex. In fact, $\underline{P} + h$ avoids sure loss $\forall h \leqslant \bar{k} < 0$ (cf. Remark 1). Since it can be shown that $\bar{k} = -\underline{E}_c(0)$, it ensues that $\underline{E}_c(0)$ is the minimum k to be subtracted from $\underline{P}$ to make $\underline{P}' = \underline{P} - k$ avoid sure loss.

Hence, the convex natural extension points out ways of correcting an assessment incurring (bounded) sure loss into one avoiding sure loss, a problem which cannot be answered using the natural extension. These corrections can be applied in several interesting situations, including, as already noted, the case of a finite $\mathscr{D}$.

3.3. Centered convex previsions and envelope theorems

The considerations at the end of Section 3.1 lead us naturally to the following stronger notion of centered convexity:

Definition 8. A lower prevision $\underline{P}$ on domain $\mathscr{D}$ $(0 \in \mathscr{D})$ is *centered convex* (*C*-convex, in short) iff it is convex and $\underline{P}(0) = 0$.

Proposition 4. *Let $\underline{P}$ be a centered convex lower prevision on $\mathscr{D}$. Then*

(*a*) $\underline{P}$ *has a convex natural extension* (*hence at least one centered convex extension*) *on any* $\mathscr{D}' \supset \mathscr{D}$.
(*b*) $\underline{P}(\lambda X) \geqslant \lambda \underline{P}(X)$, $\forall \lambda \in [0,1]$, $\underline{P}(\lambda X) \leqslant \lambda \underline{P}(X)$, $\forall \lambda \in]-\infty, 0[\cup]1, +\infty[$
(*c*) $\inf X \leqslant \underline{P}(X) \leqslant \sup X$, $\forall X \in \mathscr{D}$
(*d*) $\underline{P}$ *avoids sure loss.*

Properties (a)–(d) show that centered convexity is significantly closer to coherence than convexity: *C*-convex lower previsions are a special class of previsions avoiding sure loss, retaining several properties of coherence and the extension property of convexity, but not requiring positive homogeneity.

A convex prevision $\underline{P}$ which is not centered may still be avoiding sure loss, if $\underline{P}(0) < 0$ (Proposition 1), but in general it is only warranted to avoid unbounded sure loss [14], a very weak consistency requirement.

An indirect comparison among convexity, centered convexity and coherence is given by their corresponding envelope theorems. We firstly recall that it was proved in [16] that any lower envelope of coherent lower previsions is coherent. Here is the parallel statement for convex lower previsions, while the generalisation of Theorem 1 (lower envelope theorem) comes next.

Proposition 5. *Let $\mathscr{P}$ be a set of convex lower previsions all defined on $\mathscr{D}$. If $\underline{P}(X) = \inf_{\underline{Q}\in\mathscr{P}}\underline{Q}(X)$ is finite $\forall X \in \mathscr{D}$, $\underline{P}$ is convex on $\mathscr{D}$.*

Theorem 4. (*Generalised envelope theorem*) *$\underline{P}$ is convex on $\mathscr{D}$ iff there exist a set $\mathscr{P}$ of coherent precise previsions on $\mathscr{D}$ and a function $\alpha : \mathscr{P} \to \mathbb{R}$ such that*:

(*a*) $\underline{P}(X) = \inf_{P\in\mathscr{P}}\{P(X) + \alpha(P)\}$, $\forall X \in \mathscr{D}$ (inf *is attained*).

Moreover, $\underline{P}$ *is C-convex iff* ($0 \in \mathscr{D}$ *and*) *both* (*a*) *and the following* (*b*) *hold*:

(*b*) $\inf_{P\in\mathscr{P}}\alpha(P) = 0$ (inf *is attained*).

A similar result was proved in risk measurement theory [6], requiring $\mathscr{D}$ to be a linear space. The proof of Theorem 4, given in [12] in the framework of imprecise prevision theory, is simpler and imposes no structure on $\mathscr{D}$.

Remark 2. In particular, the constructive implication of the theorem (for convex previsions) enables us to obtain convex previsions as lower envelopes of translated precise previsions. Its proof follows easily: every precise prevision P is convex and so is $P + \alpha(P)$, by Proposition 2 (c); $\inf_{P\in\mathscr{P}}\{P(X) + \alpha(P)\}$ is a convex prevision by Proposition 5.

Remark 3. Let $\underline{P}$ be a lower prevision, $\mathscr{L}$ ($\supset \mathscr{D}$) the set of all bounded random variables and $\mathscr{S}$ the set of all coherent precise previsions on $\mathscr{L}$. Define also $\mathscr{M}(\underline{P}) = \{(Q, r) \in \mathscr{S} \times \mathbb{R} : Q(X) + r \geqslant \underline{P}(X), \forall X \in \mathscr{D}\}$. It ensues from Theorem 4 that convexity of $\underline{P}$ can be equivalently characterised by the condition $\underline{P}(X) = \inf\{Q(X) + r : (Q, r) \in \mathscr{M}(\underline{P})\}$, $\forall X \in \mathscr{D}$; C-convexity can be characterised by adding the constraint $\inf\{r : \exists Q \in \mathscr{S} : (Q, r) \in \mathscr{M}(\underline{P})\} = 0$ (cf. also the following Theorem 5, where the lower envelope concerns all $X \in \mathscr{L}$).

The characterisations of convexity, centered convexity and coherence given by the envelope theorems differ about the role of function α, that is unconstrained with convexity, such that $\min\alpha = 0$ with centered convexity, identically null with coherence (in this case Theorem 4 reduces to Theorem 1). See also Example 1 in Section 4 for an interpretation of α in a practical problem.

The next theorem characterises the convex natural extension as the lower envelope of a set of translated coherent precise previsions and can be proved in a way similar to the natural extension theorem in [16], Section 3.4.

Theorem 5. *Let* $\underline{P}$ *be a lower prevision on* $\mathscr{D}$ *which avoids unbounded sure loss, and define* $\mathscr{L}$, $\mathscr{S}$ *and* $\mathscr{M}(\underline{P})$ *as in Remark* 3. *Then,* $\mathscr{M}(\underline{P}) = \mathscr{M}(\underline{E}_c)$ *and* $\underline{E}_c(X) = \inf\{Q(X) + r : (Q, r) \in \mathscr{M}(\underline{P})\}$, $\forall X \in \mathscr{L}$.

3.4. Convex previsions and bounded rationality

Convex previsions were so far discussed following a behavioural approach which parallels Walley's approach to imprecise probabilities. Under appropriate conditions, there are some alternative ways of modelling the same beliefs, in terms of acceptable random variables or of partial preference orderings.

In particular, *if* $\mathscr{D}$ *is a linear space* any convex prevision $\underline{P}$ on $\mathscr{D}$ can be alternatively described by means of a convex set of *acceptable* random variables $\mathscr{A} = \{X \in \mathscr{D} : \underline{P}(X) \geqslant 0\}$ satisfying the property:

(B1) if $X \in \mathscr{A},\ Y \in \mathscr{D},\ Y \geqslant X$ then $Y \in \mathscr{A}$

If, in addition, $\underline{P}$ avoids sure loss then $\sup X \geqslant 0$, $\forall X \in \mathscr{A}$. Conversely, if $\mathscr{A} \subset \mathscr{D}$ is non-empty, convex and satisfies (B1), $\underline{P}(X) = \sup\{c \in \mathbb{R} : X - c \in \mathscr{A}\}$, when finite, is a convex lower prevision on $\mathscr{D}$. If further $\sup X \geqslant 0$, $\forall X \in \mathscr{A}$, $\underline{P}$ is finite and avoids sure loss. This characterisation in terms of acceptance sets is similar to those given in [16] for coherent lower previsions and in [6,7] for convex risk measures (cf. Section 4.1) and can be proved likewise.

Preference orderings represent an alternative way of expressing beliefs and they are linked to acceptance sets by means of the canonical correspondence $X \succeq Y$ iff $X - Y \in \mathscr{A}$. By exploiting this relation, it can be proved that, given a centered convex lower prevision $\underline{P}$, the corresponding preference ordering $X \succeq Y$ iff $\underline{P}(X - Y) \geqslant 0$ satisfies a set of five axioms proposed in [11], Section 2, where a representation of beliefs in terms of confidence-weighted probabilities is derived from such axioms, in a finite setting. Conversely, every preference ordering $X \succeq Y$ satisfying such axioms lets us define the centered convex lower prevision $\underline{P}(X) = \sup\{c \in \mathbb{R} : X - c \succeq 0\}$.

The behavioral approach looks nimbler than the alternatives mentioned above, because it does not require operating on linear spaces and appears easier to be generalised in a conditional environment, which we do in Section 5.

As for centered convex previsions, they are closely related with the consistency notion of *n-coherence* discussed in [16], Appendix B. It is illustrated there how *n*-coherence can be appropriate for certain 'bounded rationality' models. If the model does not require positive homogeneity, *n*-coherence alone is inadequate: 1-coherence is too weak, being equivalent to the internality condition (c) in Proposition 4, 2-coherence is too strong, as on linear spaces it is equivalent to two axioms, one of which is positive homogeneity. As a matter of fact, C-convex previsions are a special class of 1-coherent (but not necessarily 2-coherent) previsions.

Hence centered convex previsions are especially adequate to model lack of positive homogeneity, which may express a type of risk aversion. It has to be specified however that often this is an essentially *practical* need, in models like the convex risk measures discussed in Section 4.1. In a *foundational* view, another kind of risk aversion concerns the lower unboundedness of the gains in Definition 2. This should not be an issue as for the elicitation and assessment of coherent lower previsions. In fact, coherence of lower previsions is equivalent to their *constrained* coherence, whose definition is obtained from Definition 2 adding the constraint $\sup|\underline{G}| \leqslant k$, $k \in \mathbb{R}^+$. [6] Hence a risk-averse agent can assess coherent imprecise previsions, without his/her assessments being conditioned by fearing that the related gains might actually turn into too large losses.

4. Some applications

We have seen so far that convexity may help in correcting several inconsistent assessments. As noted in Section 3.2.1, its usefulness in this problem is essentially instrumental: we may easily go further and arrive at a centered convex correction, which guarantees a more satisfactory degree of consistency.

Some uncertainty modelisations also give rise to convex previsions, as in the examples which follow. We emphasise that we do not maintain that these models are reasonable, but simply that they are sometimes adopted in practice, and that convexity supplies a conceptual framework for them.

Example 1 (*Overly prudential assessments*). Persons or institutions which have to evaluate the random variables in a set $\mathscr{D}$ are often unfamiliar with uncertainty theories. In this case, a solution is to gather n experts and ask each of them to formulate a precise prevision (or an expectation) for all $X \in \mathscr{D}$. Choosing $\underline{P}(X) = \min_{i=1,\dots,n} P_i(X), \forall X$ (where P_i is expert i's evaluation) as one's own opinion is an already prudential way of pooling the experts' opinions, and originates a coherent lower prevision. Some more caution or lack of confidence toward some experts may lead to replacing every P_i with $P_i^* = P_i - \alpha_i$ before performing the minimum, where $\alpha_i \geqslant 0$ measures in some way the final assessor's personal caution or his/her (partially) distrusting expert i. By Theorem 4, $\underline{P}^* = \min_{i=1,\dots,n} P_i^*$ is convex (cf. Remark 2). More generally, $\underline{P}^*$ is of course convex also when the sign of the α_i is unconstrained ($\alpha_i < 0$ if, for instance, expert i's opinion is believed to be biased and

[6] The proof that constrained coherence implies coherence (the vice versa is trivial) relies on the fact that whenever $\underline{G}$ in Definition 2 is such that $\sup|\underline{G}| > k$, then, defining $\underline{G}' = (k/\sup|\underline{G}|)\underline{G}$, $\sup|\underline{G}'| = k$ and $\sup \underline{G} \geqslant 0$ iff $\sup \underline{G}' \geqslant 0$ (cf. also [5]).

below the 'real' prevision). It is interesting to observe that if $\alpha_i \geqslant 0$ for at least one i, $\underline{P}^*$ avoids sure loss too (since then $\underline{E}_c(0) \leqslant 0$ by Theorem 5, hence $\underline{E}_c$ avoids sure loss by Proposition 1, and so does $\underline{P}^*$ by Theorem 3 (e)). In particular, the following situation may be not unusual with an unexperienced assessor: $\alpha_i > 0$ for some i, and $0 \notin \mathscr{D}$, because the assessor thinks that no expert is needed to evaluate 0, he himself can assign, of course, $\underline{P}^*(0) = 0$. If such is the case, the extension of $\underline{P}^*$ on $\mathscr{D} \cup \{0\}$ keeps on avoiding sure loss, as is easily seen, but is generally not convex (to see this with a simple example, suppose $X \in \mathscr{D}$, $\underline{P}^*(X) < \inf X$ and use the result in footnote 4 to obtain that $\underline{P}^*(0) < 0$ is then necessary for convexity).

In the following example and in Section 4.1 we shall refer to upper previsions, to which the theory developed so far extends with mirror-image modifications. We report the conjugates of Definition 5 and Theorem 4.

Definition 9. $\overline{P} : \mathscr{D} \to \mathbb{R}$ is a *convex upper prevision* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_0, X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n \geqslant 0$ such that $\sum_{i=1}^n s_i = 1$ (*convexity condition*), defining $\overline{G} = \sum_{i=1}^n s_i(\overline{P}(X_i) - X_i) - (\overline{P}(X_0) - X_0)$, $\sup \overline{G} \geqslant 0$.

Theorem 6. *$\overline{P}$ is convex on its domain $\mathscr{D}$ iff there exist a set $\mathscr{P}$ of coherent precise previsions* (*all defined on $\mathscr{D}$*) *and a function $\alpha : \mathscr{P} \to \mathbb{R}$ such that*:

(*a*) $\overline{P}(X) = \sup_{P \in \mathscr{P}}\{P(X) + \alpha(P)\}$, $\forall X \in \mathscr{D}$ (sup *is attained*).
Moreover, $\overline{P}$ *is C-convex iff* ($0 \in \mathscr{D}$ *and*) *both* (*a*) *and the following* (*b*) *hold*:
(*b*) $\sup_{P \in \mathscr{P}} \alpha(P) = 0$ (sup *is attained*).

Example 2 (*Supremum preserving functions*). Let $\mathbb{P} = \{\omega_i\}_{i \in I}$ be a (not necessarily finite) set of exhaustive non-impossible elementary events or *atoms*, i.e. $\omega_i \neq \emptyset \, \forall i \in I$, $\vee_{i \in I} \omega_i = \Omega$, $\omega_i \wedge \omega_j = \emptyset$ if $i \neq j$. Given a function $\pi : \mathbb{P} \to [0, 1]$, define $\Pi : 2^{\mathbb{P}} - \{\emptyset\} \to [0, 1]$ ($2^{\mathbb{P}}$ is the powerset of $\mathbb{P}$) by

$$\Pi(A) = \sup_{\omega_i \Rightarrow A} \pi(\omega_i), \quad \forall A \in 2^{\mathbb{P}} - \{\emptyset\}. \tag{1}$$

As well-known, if π is normalised (i.e., $\sup \pi = 1$) and extended to $\emptyset$ putting $\pi(\emptyset)(= \Pi(\emptyset)) = 0$, Π is a normalised possibility measure, a special case of coherent upper probability [3]. Without these additional assumptions, Π is a convex upper prevision (probability). To see this, define for $i \in I$, $P_i(\omega_i) = 1$, $P_i(\omega_j) = 0 \, \forall j \neq i$, $\alpha_i = \pi(\omega_i) - 1$, and extend (trivially) each P_i to $2^{\mathbb{P}}$. It is not difficult to see that $\Pi(A) = \sup_{i \in I}\{P_i(A) + \alpha_i\}$, $\forall A \in 2^{\mathbb{P}}$ and therefore Π is convex by Theorem 6. If $\sup \pi < 1$, Π has the unpleasant property that $\Pi(\Omega) < 1$, and also $\Pi(\emptyset) < 0$ (this means that Π incurs sure loss and is not C-convex). Functions similar to these kinds of unnormalised possibilities were considered in the literature relating possibility and fuzzy set theories, and their unsatisfactory properties were already pointed out (see e.g. [9], Section 2.6 and the references quoted therein).

4.1. Convex risk measures

Further applications of convex imprecise previsions are suggested by their not requiring positive homogeneity, cf. Proposition 4 (b). Considering the well-known behavioural interpretation of lower (and upper) previsions [16], applications could evidently be related to situations of risk aversion, because of which an agent's supremum buying price for the random quantity λX might be less than λ times his/her supremum buying price for X, when $\lambda > 1$.

We discuss here an application to (financial) risk measurement. The literature on risk measures is quite large, as this topic is very important in many financial, banking or insurance applications. Formally, a risk measure is a map ρ from a set $\mathscr{D}$ of random variables into $\mathbb{R}$. Therefore ρ associates a real number $\rho(X)$ to every $X \in \mathscr{D}$, which should measure how 'risky' X is, and whether it is acceptable to buy or hold X. Intuitively, X is acceptable (not acceptable) if $\rho(X) \leqslant 0$ (if $\rho(X) > 0$), and $\rho(X)$ should determine the maximum amount of money which could be subtracted from X, keeping it acceptable (the minimum amount of money to be added to X to make it acceptable).

Traditional risk measures, like Value-at-Risk (VaR)—probably the most widespread—require assessing (at least) a distribution function for each $X \in \mathscr{D}$. Quite recently, other risk measures were introduced, which do not require this and are therefore appropriate also when conflicting or insufficient information is available. Precisely, coherent risk measures were defined in a series of papers (including [1,2]) using a set of axioms (among these positive homogeneity), and assuming that $\mathscr{D}$ is a linear space. In these papers, coherent risk measures were not related with imprecise previsions theory, while this was done in [13]; see also [10] for a general approach to these and other theories. Convex risk measures were introduced in [6,7] as a generalisation of coherent risk measures which does not require the positive homogeneity axiom. We report the definition in [7]:

Definition 10. Let $\mathscr{V}$ be a linear space of random variables which contains real constants. $\rho : \mathscr{V} \rightarrow \mathbb{R}$ is a *convex risk measure* iff

(C1) $\forall X \in \mathscr{V}$, $\forall \alpha \in \mathbb{R}$, $\rho(X + \alpha) = \rho(X) - \alpha$ (translation invariance),
(C2) $\forall X, Y \in \mathscr{V}$, if $X \leqslant Y$ then $\rho(Y) \leqslant \rho(X)$ (monotonicity),
(C3) $\rho(\lambda X + (1 - \lambda)Y) \leqslant \lambda\rho(X) + (1 - \lambda)\rho(Y)$ $\forall X, Y \in \mathscr{V}$, $\forall \lambda \in [0, 1]$ (convexity).

The potential capability of convex risk measures of capturing risk aversion in a decision theoretic framework is pointed out in [7]. In a risk measurement environment, a motivation for not assuming positive homogeneity is that $\rho(\lambda X)$ may be larger than $\lambda\rho(X)$ for $\lambda > 1$ also because of *liquidity risks*: if we were to sell immediately a large amount λX of a financial investment, we might be forced to accept a smaller reward than λ times the current selling price for X.

It was shown in [13] that risk measures can be encompassed into the theory of imprecise previsions, because a risk measure for X can be interpreted as an upper prevision for $-X$: [7]

$$\rho(X) = \overline{P}(-X). \tag{2}$$

This fact was used in [13] to generalise the notion of coherent risk measures to an arbitrary domain $\mathscr{D}$. An analogue generalisation can be done for convex risk measures [12], as we shall now illustrate.

Definition 11. $\rho : \mathscr{D} \to \mathbb{R}$ is a *convex risk measure* on $\mathscr{D}$ if and only if for all $n \in \mathbb{N}^+$, $\forall X_0, X_1, \ldots, X_n \in \mathscr{D}$, $\forall s_1, \ldots, s_n$ real and non-negative such that $\sum_{i=1}^n s_i = 1$, defining $\overline{G} = \sum_{i=1}^n s_i(X_i + \rho(X_i)) - (X_0 + \rho(X_0))$, $\sup \overline{G} \geqslant 0$.

Note that Definition 11 may be obtained from Definition 9 referring to $-X$ rather than X, for all $X \in \mathscr{D}$.

If $\mathscr{D}$ is a linear space containing real constants, the notion in Definition 11 reduces to that in [6,7], by the next theorem (cf. also Theorem 2 (a)):

Theorem 7. *Let $\mathscr{V}$ be a linear space of bounded random variables containing real constants. A mapping ρ from $\mathscr{V}$ into $\mathbb{R}$ is a convex risk measure according to Definition* 11 *iff it is a convex risk measure according to Definition* 10.

Definition 11 applies to *any* set $\mathscr{D}$ of random variables, unlike Definition 10, which, if $\mathscr{D}$ is arbitrary, requires embedding it into a larger linear space.

Results specular to those presented in Section 3 apply to convex risk measures. For instance, they can be extended on any $\mathscr{D}' \supset \mathscr{D}$, preserving convexity, and avoid sure loss iff $\rho(0) \geqslant 0$ (we say that ρ avoids sure loss on $\mathscr{D}$ iff $\overline{P}(-X) = \rho(X)$ avoids sure loss on $\mathscr{D}^- = \{-X : X \in \mathscr{D}\}$).

Like the general case in Section 3, it appears quite appropriate to add the requirement $\rho(0) = 0$ to convexity, and hence to use *centered convex* risk measures: 0 is the unquestionably reasonable selling or buying price for $X = 0$.

Centered convex risk measures have further nice additional properties, corresponding to those of centered convex lower previsions: they always avoid sure loss, and are such that $-\sup X \leqslant \rho(X) \leqslant -\inf X, \forall X \in \mathscr{D}$. This condition corresponds to internality ((c) of Proposition 4), and is a rationality requirement for risk measures: for instance, $\rho(X) > -\inf X$ would mean that to make X acceptable we require adding to it a sure number ($\rho(X)$) higher than the maximum loss X may cause. Besides, a

[7] We assume that the time gap between the buying and selling time of X is negligible (if not, we should introduce a discounting factor in (2)). This simplifies the sequel, without substantially altering the conclusions.

centered convex risk measure ρ is not necessarily positively homogeneous: $\rho(\lambda X) \geqslant \lambda\rho(X), \forall\lambda \geqslant 1$.

A notion of convex natural extension may also be given for centered convex (or convex) risk measures and its properties correspond to those listed in Theorem 3. When finite, it gives in particular a standard way of 'correcting' other kinds of risk measures into convex risk measures. [8]

The generalised envelope theorem is obtained from the statement of Theorem 6 replacing $\overline{P}(X)$ and $\underline{P}(X)$ with, respectively, $\rho(X)$ and $P(-X)$. Examples of convex risk measures may be found in [6,7,12].

5. Convex conditional previsions

The following definition is a natural generalisation of Definition 5, and a weaker consistency requirement than coherence in Definition 4. We recall from Section 2 that $S(\underline{s}) = \vee\{B_i : s_i \neq 0, i = 1,\ldots,n\}$.

Definition 12. Let $\mathscr{D}$ be a set of conditional random variables. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a *convex conditional lower prevision* on $\mathscr{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_0|B_0,\ldots,X_n|B_n \in \mathscr{D}$, $\forall s_1,\ldots,s_n$ real and non-negative such that $\sum_{i=1}^n s_i = 1$, defining $\underline{G} = \sum_{i=1}^n s_i B_i(X_i - \underline{P}(X_i|B_i)) - B_0(X_0 - \underline{P}(X_0|B_0))$, $\sup\{\underline{G}|S(\underline{s}) \vee B_0\} \geqslant 0$.

Theorem 8. *Let $\mathscr{X}$ be a linear space of bounded random variables, $\mathscr{E} \subset \mathscr{X}$ the set of all indicator functions of events in $\mathscr{X}$. Let also $1 \in \mathscr{E}$ and $BX \in \mathscr{X}$, $\forall B \in \mathscr{E}$, $\forall X \in \mathscr{X}$.* [9] *Define $\mathscr{E}^{\emptyset} = \mathscr{E} - \{\emptyset\}$, $\mathscr{D}_{\rm LIN} = \{X|B : X \in \mathscr{X}, B \in \mathscr{E}^{\emptyset}\}$. $\underline{P} : \mathscr{D}_{\rm LIN} \to \mathbb{R}$ is a convex conditional lower prevision if and only if*:

(*D*1) $\underline{P}(X|B) - \underline{P}(Y|B) \leqslant \sup\{X - Y|B\}, \forall X, Y \in \mathscr{X}, \ \forall B \in \mathscr{E}^{\emptyset}$
(*D*2) $\underline{P}(\lambda X + (1-\lambda)Y|B) \geqslant \lambda\underline{P}(X|B) + (1-\lambda)\underline{P}(Y|B), \forall X, Y \in \mathscr{X}, \ \forall B \in \mathscr{E}^{\emptyset}, \ \forall\lambda \in [0,1]$
(*D*3) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0, \forall X \in \mathscr{X}, \forall A, B \in \mathscr{E}^{\emptyset} : A \wedge B \neq \emptyset$.

Proof. We prove first that if (D1), (D2), (D3) hold $\underline{P}$ is convex. In fact, letting $g_i^* = B_i(X_i - \underline{P}(X_i|B_i))$ $(i = 0,\ldots,n)$, $X = \sum_{i=1}^n s_i g_i^*$, $Y = g_0^*$, $\underline{G} = X - Y$ in Definition 12 and using (D1) at the first inequality, (D2) and (D3) subsequently, we get $\sup\{X - Y|S(\underline{s}) \vee B_0\} \geqslant \underline{P}(X|S(\underline{s}) \vee B_0) - \underline{P}(Y|S(\underline{s}) \vee B_0) \geqslant \sum_{i=1}^n s_i \underline{P}(g_i^*|S(\underline{s}) \vee B_0) - \underline{P}(g_0^*|S(\underline{s}) \vee B_0) = \sum_{s_i \neq 0} s_i \underline{P}(B_i(X_i - \underline{P}(X_i|B_i \wedge (S(\underline{s}) \vee B_0)))|S(\underline{s}) \vee B_0) - \underline{P}(B_0(X_0 - \underline{P}(X_0|B_0 \wedge (S(\underline{s}) \vee B_0)))|S(\underline{s}) \vee B_0) = 0$.

Let now $\underline{P}$ be convex on $\mathscr{D}_{\rm LIN}$. We prove that (D1), (D2), (D3) hold.

[8] Note that this is always possible if $\mathscr{D}$ is finite (cf. Section 3.2.1).
[9] The assumptions imply that if A and $B \in \mathscr{E}$ then $A \wedge B$ and $A \vee B \in \mathscr{E}$.

As for (D1), letting $X_1|B_1 = X|B$, $X_0|B_0 = Y|B$ in Definition 12, we get $\sup\{B(X - \underline{P}(X|B)) - B(Y - \underline{P}(Y|B))|B\} = \sup\{B(X - Y) - B(\underline{P}(X|B) - \underline{P}(Y|B))|B\} \geqslant 0$, which implies $\sup\{X - Y|B\} \geqslant \underline{P}(X|B) - \underline{P}(Y|B)$.

To prove (D2), apply Definition 12 with $n = 2$, $s_1 = \lambda$, $s_2 = 1 - \lambda$, $X_1|B_1 = X|B$, $X_2|B_2 = Y|B$, $X_0|B_0 = \lambda X + (1 - \lambda)Y|B$.

For (D3), use again Definition 12 with $n = 1$. If $X_1|B_1 = X|A \wedge B$ and $X_0|B_0 = A(X - \underline{P}(X|A \wedge B))|B$, then $\sup\{[AB(X - \underline{P}(X|A \wedge B)) - B(A(X - \underline{P}(X|A \wedge B)) - \underline{P}(A(X - \underline{P}(X|A \wedge B))|B))]|B\} = \underline{P}(A(X - \underline{P}(X|A \wedge B))|B) \geqslant 0$; if $X_1|B_1$ and $X_0|B_0$ are interchanged, we get the reverse inequality $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) \leqslant 0$. $\square$

Theorem 8 generalises Theorem 2 (a) (cf. also footnote 2) and will be used later. Property (D3) is discussed in Section 5.2.

5.1. Convex natural extension in a conditional environment

Definition 13. Let $\underline{P}: \mathscr{D} \to \mathbb{R}$ be a conditional lower prevision, $Z|B$ an arbitrary bounded conditional random variable. Define $g_i = s_i B_i(X_i - \underline{P}(X_i|B_i))$, $L(Z|B) = \{\alpha : \sup\{\sum_{i=1}^n g_i - B(Z - \alpha)|S(\underline{s}) \vee B\} < 0,$ for some $n \geqslant 1, X_i|B_i \in \mathscr{D}, s_i \geqslant 0,$ with $\sum_{i=1}^n s_i = 1\}$. The *convex natural extension* of $\underline{P}$ to $Z|B$ is $\underline{E}_c(Z|B) = \sup L(Z|B)$. [10]

It is not difficult to see that Definition 13 generalises Definition 6. When dropping the constraint $\sum_{i=1}^n s_i = 1$, it reduces to the definition of natural extension in [17]. We show in the sequel that general results established for the unconditional case may be extended to the conditional one. However, there are some differences, and this becomes patent from the very beginning, when we investigate whether, given $Z|B$, the set $L(Z|B)$ is empty or not.

In the unconditional case, $L(Z) = L(Z|\Omega)$ is always non-empty (see Section 3.2), but $L(Z|B)$ is not so in general, without any additional assumption. For instance, let $\underline{P} : \mathscr{D} \to \mathbb{R}$, $\mathscr{D} = \{X|B^C\}$, $B \neq \emptyset$, $B^C \neq \emptyset$, $\underline{P}(X|B^C) \in [\inf X|B^C, \sup X|B^C]$. Then $\forall Z$, $L(Z|B) = \emptyset$, because, $\forall \alpha \in \mathbb{R}$, letting $\underline{G}_\alpha = B^C(X - \underline{P}(X|B^C)) - B(Z - \alpha)$, $\sup \underline{G}_\alpha \geqslant \sup\{\underline{G}_\alpha|B^C\} \geqslant 0$. This example shows also that $L(Z|B)$ may be empty even when $\underline{P}$ is coherent.

The following proposition is useful in suggesting what assumptions should guarantee that $L(Z|B)$ is non-empty.

Proposition 6. *Given $\underline{P} : \mathscr{D} \to \mathbb{R}$, $L(Z|B)$ is non-empty if there is $Y|C \in \mathscr{D}$ such that $C \Rightarrow B$.*

[10] It is easily seen that $L(Z|B) =]-\infty, \underline{E}_c(Z|B)[$. This fact will be used later.

Proof. Let $n = 1$, $s_1 = 1$, $X_1|B_1 = Y|C$ (hence $S(\underline{s}) \vee B = C \vee B = B$) in the supremum in Definition 13, so that $g_1 - B(Z - \alpha) = C(Y - \underline{P}(Y|C)) - B(Z - \alpha) = W$. For $\omega \Rightarrow C$, $W(\omega) = Y(\omega) - \underline{P}(Y|C) - Z(\omega) + \alpha$, while if $\omega \Rightarrow B \wedge C^C$, $W(\omega) = -Z(\omega) + \alpha$. It ensues that $\sup W|B \leqslant \max\{0, \sup Y - \underline{P}(Y|C)\} - \inf Z + \alpha$ and thus $\alpha \in L(Z|B)$ $\forall \alpha < \inf Z - \max\{0, \sup Y - \underline{P}(Y|C)\}$. $\square$

The sufficient condition in Proposition 6 is essentially not restrictive: a natural way to comply with it is to include $0|B$ into $\mathscr{D}$, if we wish to compute $L(X|B)$. It is also natural to put $\underline{P}(0|B) = 0$. The next proposition lets us better clarify this and related questions, when $\underline{P}$ is convex.

Proposition 7. *Let $\mathscr{D}$ be such that $0|B \in \mathscr{D}$, $\forall X|B \in \mathscr{D}$, and let $\underline{P} : \mathscr{D} \to \mathbb{R}$ be a convex conditional lower prevision on $\mathscr{D}$.*

(*a*) *$\underline{P}$ avoids uniform loss iff $\underline{P}(0|B) \leqslant 0, \forall\ 0|B \in \mathscr{D}$.*
Assume now that $\underline{P}(0|B) = 0, \forall\ 0|B \in \mathscr{D}$. Then

(*b*) *if $0|C \notin \mathscr{D}$, the extension of $\underline{P}$ on $\{0|C\}$ such that $\underline{P}(0|C) = 0$ is a convex conditional lower prevision on $\mathscr{D} \cup \{0|C\}$*

(*c*) *if $0|B \in \mathscr{D}$, and $B \Rightarrow C$, there is a unique convex conditional lower prevision which extends $\underline{P}$ on $\{0|C\}$, and necessarily $\underline{P}(0|C) = 0$.*

Proof. Define firstly $g_i^* = B_i(X_i - \underline{P}(X_i|B_i))$ $(i = 0, \dots, n)$.

(a) If $\underline{P}$ avoids uniform loss, $\underline{P}(0|B) \leqslant \sup(0|B) = 0$ as a straightforward consequence of Definition 3.
Conversely, let $B = B_j \in \{B_1, \dots, B_n\}$ such that $s_j \neq 0$ in Definition 12 (this implies $S(\underline{s}) \vee B = S(\underline{s})$). Since $\underline{P}$ is convex and recalling that $\underline{P}(0|B) \leqslant 0$, $0 \leqslant \sup\{\sum_{i=1}^n s_i g_i^* - B(0 - \underline{P}(0|B))\,|\,S(\underline{s})\} \leqslant \sup\{\sum_{i=1}^n s_i g_i^* | S(\underline{s})\}$.
This implies that $\underline{P}$ avoids uniform loss, since the constraint $\sum_{i=1}^n s_i = 1$ is immaterial in checking Definition 3.

(b) To check using Definition 12 that $\underline{P}$ is convex on $\mathscr{D} \cup \{0|C\}$ we have to evaluate the supremum of those gains $\underline{G}$ in Definition 12 where $0|C \in \{X_0|B_0, \dots, X_n|B_n\}$. Noting that whenever $0|C \in \{X_1|B_1, \dots, X_n|B_n\}$ it is not restrictive to assume $0|C = X_n|B_n$, and defining $S'(\underline{s}) = \vee\{B_i : s_i > 0,\ i = 1, \dots, n-1\}$, the following situations are to be considered:
  (i) $0|C = X_0|B_0$, $0|C \neq X_i|B_i$ $(i = 1, \dots, n)$.
  Then $\sup\{\underline{G}\,|\,S(\underline{s}) \vee C\} = \sup\{\sum_{i=1}^n s_i g_i^* - C(0 - 0)|S(\underline{s}) \vee C\} \geqslant \sup\{\sum_{i=1}^n s_i g_i^* | S(\underline{s})\} \geqslant 0$, where the last inequality holds because $\underline{P}$ avoids uniform loss on $\mathscr{D}$, by (a).
  (ii) $0|C = X_n|B_n = X_0|B_0$.
  If $n = 1$, it is trivially $\underline{G} = 0$. For $n > 1$, we get $\sup\{\underline{G}|S(\underline{s}) \vee C\} = \sup\{\sum_{i=1}^{n-1} s_i g_i^* + s_n C(0-0) - C(0-0)|S(\underline{s}) \vee C\} \geqslant \sup\{\sum_{i=1}^{n-1} s_i g_i^* |\, S'(\underline{s})\} \geqslant 0$, the last inequality holding again because $\underline{P}$ avoids uniform loss on $\mathscr{D}$.
  (iii) $0|C = X_n|B_n \neq X_0|B_0$.
  If $s_n = 0$, we easily get $\sup\{\underline{G}\,|\,S(\underline{s}) \vee B_0\} = \sup\{\sum_{i=1}^{n-1} g_i^* - g_0^* | S'(\underline{s}) \vee B_0\} \geqslant 0$ because $\underline{P}$ is convex on $\mathscr{D}$.

If $s_n > 0$, (assuming that $\sum_{i=1}^{n-1} s_i g_i^* = 0$ if $n = 1$) use the equality $\underline{G} = \sum_{i=1}^{n-1} g_i^* + s_n C(0-0) - g_0^* = \sum_{i=1}^{n-1} g_i^* + s_n B_0(0-0) - g_0^* = \underline{G}'$ to write $\sup \underline{G}|S(\underline{s}) \vee B_0 = \sup \underline{G}'|S'(\underline{s}) \vee C \vee B_0 \geqslant \sup \underline{G}'|S'(\underline{s}) \vee B_0 \geqslant 0$, where convexity of $\underline{P}$ on $\mathscr{D}$ is exploited at the last inequality.

(c) Convexity of $\underline{P}$ on $\mathscr{D} \cup \{0|C\}$ implies that when betting in favour of $0|B$ and against $0|C$, $\sup\{B(0-0) - C(0 - \underline{P}(0|C))|C\} = \underline{P}(0|C) \geqslant 0$. Betting in favour of $0|C$ and against $0|B$ gives the reverse inequality. □

Definition 14. $\underline{P} : \mathscr{D} \to \mathbb{R}$ is a *centered* (conditional) lower prevision if $0|B \in \mathscr{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathscr{D}$.

Convex conditional previsions which are centered generalise unconditional centered convex previsions. Let $\underline{P}$ be a centered convex conditional lower prevision. Then, if $0 \notin \mathscr{D}$, Proposition 7 (c), with $C = \Omega$, implies $\underline{P}(0) = 0$. Moreover, $\underline{P}$ avoids uniform loss by Proposition 7 (a) (which generalises Proposition 1) and $L(X|B) \neq \emptyset$ whenever $0|B \in \mathscr{D}$, by Proposition 6. If $0|B \notin \mathscr{D}$, we may extend $\underline{P}$ on $\mathscr{D} \cup \{0|B\}$ putting $\underline{P}(0|B) = 0$, before computing $L(X|B)$. By Proposition 7 (b), $\underline{P}$ remains centered convex on $\mathscr{D} \cup \{0|B\}$.

Therefore centered convex previsions appear to be an appropriate and easy choice to ensure both the minimal consistency requirement of avoiding uniform loss and the existence of the convex natural extension (which is necessarily finite, cf. the later Proposition 8), whatever is $X|B$. However, various results in the sequel hold and will be proved under more general assumptions.

Remark 4. If $\underline{P}$ is a centered convex conditional prevision, the term $(1 - \sum_{i=1}^{n} s_i) B_0(0 - \underline{P}(0|B_0))$ can be added to every gain $\underline{G} = \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i|B_i)) - B_0(X_0 - \underline{P}(X_0|B_0))$ such that $\sum_{i=1}^{n} s_i < 1$, without modifying either it or its conditioning event in Definition 12. This implies that Definition 12 can be equivalently stated relaxing the condition $\sum_{i=1}^{n} s_i = 1$ to $\sum_{i=1}^{n} s_i \leqslant 1$, when $\underline{P}$ is centered.

Proposition 8. *Let $\underline{P} : \mathscr{D} \to \mathbb{R}$ be a conditional lower prevision avoiding uniform loss. Then $\underline{E}_c(X|B) \leqslant \sup X|B$, $\forall X|B$.* [11]

Proof. Let $c = \sup X|B$, $X_i|B_i \in \mathscr{D}$, $s_i \geqslant 0$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^{n} s_i = 1$. Then, exploiting the fact that $\underline{P}$ avoids uniform loss in the last inequality, we can write $\sup\{\sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i)) - B(X - c)|S(\underline{s}) \vee B\} \geqslant \sup\{\sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i))| S(\underline{s})\} \geqslant 0$. Hence $c \notin L(X|B)$. Recalling that $L(X|B) =]-\infty, \underline{E}_c(X|B)[$, the inequality $\underline{E}_c(X|B) \leqslant \sup X|B$ follows. □

[11] Possibly $\underline{E}_c(X|B) = -\infty$.

The convex natural extension has several properties, extending those of the unconditional case.

Theorem 9. *Let* $\underline{P} : \mathscr{D} \to \mathbb{R}$ *be a conditional lower prevision and* $\mathscr{D}_{\mathrm{LIN}} \supset \mathscr{D}$ *be defined as in Theorem* 8. *If* $\underline{E}_c$ *is finite on* $\mathscr{D}_{\mathrm{LIN}}$, *then*:

(*a*) $\underline{E}_c(X|B) \geqslant \underline{P}(X|B), \forall X|B \in \mathscr{D}$
(*b*) $\underline{E}_c$ *is a convex lower prevision on* $\mathscr{D}_{\mathrm{LIN}}$
(*c*) *If* $\underline{P}^*$ *is a convex lower prevision on* $\mathscr{D}_{\mathrm{LIN}}$ *such that* $\underline{P}^*(X|B) \geqslant \underline{P}(X|B), \forall X|B \in \mathscr{D}$, *then* $\underline{P}^*(X|B) \geqslant \underline{E}_c(X|B), \forall X|B \in \mathscr{D}_{\mathrm{LIN}}$
(*d*) $\underline{P}$ *is a convex lower prevision on* $\mathscr{D}$ *iff* $\underline{E}_c = \underline{P}$ *on* $\mathscr{D}$
(*e*) *If* $\underline{P}$ *is convex,* $\underline{E}_c$ *is its minimal convex extension to* $\mathscr{D}_{\mathrm{LIN}}$.

Proof

(a) If $X|B \in \mathscr{D}$, taking $n = 1$, $X_1|B_1 = X|B$ in the definition of $\underline{E}_c(X|B)$, we obtain $\sup\{B(X - \underline{P}(X|B)) - B(X - \alpha)|B\} = \alpha - \underline{P}(X|B) < 0, \quad \forall \alpha < \underline{P}(X|B)$. Hence $\underline{E}_c(X|B) \geqslant \underline{P}(X|B)$.

(b) To prove convexity of $\underline{E}_c$ we shall show that it satisfies properties (D1), (D2), (D3) in Theorem 8.
To prove (D1), let $X|B, Y|B \in \mathscr{D}_{\mathrm{LIN}}$. If $\alpha \in L(X|B)$, there exist $X_i|B_i \in \mathscr{D}$, $s_i \geqslant 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} s_i = 1$ such that, letting $\underline{G}_1 = \sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i))$, $W_1 = \underline{G}_1 - B(X - \alpha)$, $\sup\{W_1|S(\underline{s}) \vee B\} < 0$. Using $BX - BY \leqslant B\sup\{X - Y|B\}$, we obtain $\sup\{\underline{G}_1 - B(Y - \alpha + \sup\{X - Y|B\})|S(\underline{s}) \vee B)\} \leqslant \sup\{W_1|S(\underline{s}) \vee B)\} < 0$. It follows $\alpha - \sup(X - Y|B) \in L(Y|B)$, $\forall \alpha \in L(X|B)$, hence $\underline{E}_c(X|B) - \sup\{X - Y|B\} \leqslant \underline{E}_c(Y|B)$.
To prove (D2), let $\beta \in L(Y|B)$ and $Y_j|C_j \in \mathscr{D}$, $t_j \geqslant 0$ $(j = 1, \dots, m)$, with $\sum_{j=1}^{m} t_j = 1$ such that, defining $W_2 = \sum_{j=1}^{m} t_j C_j(Y_j - \underline{P}(Y_j|C_j)) - B(Y - \beta)$, $\sup\{W_2|S(\underline{t}) \vee B\} < 0$. Given $\lambda \in]0, 1[$, we get $\sup\{\lambda W_1 + (1 - \lambda)W_2|S(\underline{s}) \vee S(\underline{t}) \vee B\} \leqslant \max\{\lambda \sup\{W_1|S(\underline{s}) \vee B\}, (1 - \lambda)\sup\{W_2|S(\underline{t}) \vee B\}\} < 0$. (To prove the weak inequality, use $S(\underline{s}) \vee S(\underline{t}) \vee B = [S(\underline{s}) \vee B] \vee [(S(\underline{s}) \vee B)^C \wedge S(\underline{t})] = [S(\underline{t}) \vee B] \vee [(S(\underline{t}) \vee B)^C \wedge S(\underline{s})]$; the left-hand supremum is the maximum of the suprema obtained alternatively conditioning on each of the four bracketed events. These suprema are negative, recalling also that $W_1|(S(\underline{s}) \vee B)^C \wedge S(\underline{t}) = W_2|(S(\underline{t}) \vee B)^C \wedge S(\underline{s}) = 0$ and that $W_1(\omega)$ and $W_2(\omega)$ cannot be both null at any $\omega \Rightarrow S(\underline{s}) \vee S(\underline{t}) \vee B$.)
This implies $\lambda\alpha + (1 - \lambda)\beta \in L(\lambda X + (1 - \lambda)Y|B)$, $\forall \alpha \in L(X|B)$, $\forall \beta \in L(Y|B)$, $\forall \lambda \in]0, 1[$, from which $\lambda\underline{E}_c(X|B) + (1 - \lambda)\underline{E}_c(Y|B) \leqslant \underline{E}_c(\lambda X + (1 - \lambda)Y|B)$ follows.
As for (D3), let $X|A \wedge B \in \mathscr{D}_{\mathrm{LIN}}$, $W = A(X - \underline{E}_c(X|A \wedge B))$. To prove that $\sup L(W|B) = 0$, we show that $L(W|B) =]-\infty, 0[$. Given $\delta > 0$, it ensues from the definition of $\underline{E}_c(X|A \wedge B)$ that there exist $X_i|B_i \in \mathscr{D}$, $s_i \geqslant 0\,(i = 1, \dots, n)$ with $\sum_{i=1}^{n} s_i = 1$ such that, defining $\underline{G} = \sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i))$ and

$Z_1 = \underline{G} - AB(X - \underline{E}_c(X|A \wedge B) + \delta)$, $\sup\{Z_1|S(\underline{s}) \vee (A \wedge B)\} < 0$. Hence $Z_2 = \underline{G} - B(W + \delta) = Z_1 - BA^C\delta$ $(\leqslant Z_1)$ is such that $\sup\{Z_2|S(\underline{s}) \vee B\} = \max\{\sup\{Z_2|S(\underline{s}) \vee (A \wedge B)\}, \sup\{Z_2|S(\underline{s})^C \wedge A^C \wedge B\}\} \leqslant \max\{\sup\{Z_1|S(\underline{s}) \vee (A \wedge B)\}, -\delta\} < 0$ (omit the second argument in the maxima if $S(\underline{s})^C \wedge A^C \wedge B = \emptyset$).

This implies $-\delta \in L(W|B), \forall \delta > 0$. Further, $0 \notin L(W|B)$: by contradiction, assuming $0 \in L(W|B)$ would imply, as can be easily seen, $\underline{E}_c(X|A \wedge B) \in L(X|A \wedge B) =]-\infty, \underline{E}_c(X|A \wedge B)[$.

From Theorem 8, $\underline{E}_c$ is therefore a convex lower prevision on $\mathscr{D}_{\mathrm{LIN}}$.

(c) Let $\underline{P}^*$ be as in the statement of (c) and $X|B \in \mathscr{D}_{\mathrm{LIN}}$. Since $-\underline{P}(X_i|B_i) \geqslant -\underline{P}^*(X_i|B_i)$ $(i = 1, \dots, n)$, we get, $\forall X_i|B_i \in \mathscr{D}$, $\forall s_i \geqslant 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^n s_i = 1$, $\sup\{\sum_{i=1}^n s_i B_i(X_i - \underline{P}(X_i|B_i)) - B(X - \underline{P}^*(X|B))|S(\underline{s}) \vee B\} \geqslant \sup\{\sum_{i=1}^n s_i B_i(X_i - \underline{P}^*(X_i|B_i)) - B(X - \underline{P}^*(X|B))|S(\underline{s}) \vee B\} \geqslant 0$. The last inequality holds by the convexity of $\underline{P}^*$ and implies $\underline{P}^*(X|B) \notin L(X|B) =]-\infty, \underline{E}_c(X|B)[$.

(d) If $\underline{E}_c = \underline{P}$ on $\mathscr{D}$, $\underline{P}$ is convex by (b). Conversely, suppose $\underline{P}$ is convex on $\mathscr{D}$, $X|B \in \mathscr{D}$. Then, $\forall X_i|B_i \in \mathscr{D}$ $(i = 1, \dots, n)$, $\forall s_i \geqslant 0$ with $\sum_{i=1}^n s_i = 1$, $\sup\{\sum_{i=1}^n s_i B_i(X_i - \underline{P}(X_i|B_i)) - B(X - \underline{P}(X|B))|S(\underline{s}) \vee B\} \geqslant 0$. It ensues $\alpha < \underline{P}(X|B), \forall \alpha \in L(X|B)$. Hence, recalling also (a), $\underline{E}_c(X|B) = \underline{P}(X|B)$.

(e) Follows immediately from (c) and (d). □

If $\underline{P}$ is centered and avoids uniform loss, the finiteness condition about $\underline{E}_c$ in Theorem 9 can be easily fulfilled. In fact, extending $\underline{P}$ to $\{0|B : 0|B \in \mathscr{D}_{\mathrm{LIN}}\}$ by setting $P(0|B) = 0$, $\underline{P}$ keeps on avoiding uniform loss. Then $\underline{E}_c(X|B)$ is finite, $\forall X \mid B \in \mathscr{D}_{\mathrm{LIN}}$, by Propositions 6 and 8. Moreover, $\underline{E}_c$ is centered, because $0 = \underline{P}(0|B) \leqslant \underline{E}_c(0|B) \leqslant 0$, $\forall 0 \mid B \in \mathscr{D}_{\mathrm{LIN}}$ by Theorem 9 (a) and Proposition 8 and therefore, being convex, avoids uniform loss too.

5.2. The generalised Bayes rule

Putting $B = \Omega$ in (D3) of Theorem 8, [12] we obtain the following formula, called *generalised Bayes rule* (GBR) in [16]:

$$\underline{P}(A(X - \underline{P}(X|A))) = 0. \tag{3}$$

Clearly, conditions (D1), (D2), (D3) in Theorem 8, hence in particular the GBR, are necessary consistency requirement for convex previsions also when $\mathscr{D} \neq \mathscr{D}_{\mathrm{LIN}}$, provided that the relevant quantities are defined. Further, the GBR supplies us with an updating rule for consistently adding conditional to unconditional assessments (or more generally, from (D3) of Theorem 8, for adding assessments in a more specific conditional environment $A \wedge B$). It is important with respect to this to state whether (3) determines uniquely $\underline{P}(X|A)$, or in other words whether $\underline{P}(X|A)$ is the only solu-

[12] (D3) was considered in [19], referring to coherent conditional imprecise previsions.

tion to $\underline{P}(A(X-r))=0$. If $\underline{P}(A)>0$, this is true for coherent lower previsions and is proved in [16] using their superlinearity. Superlinearity does not necessarily hold with convex previsions, but the next proposition shows that the unicity result applies to them too.

Proposition 9. *If $\underline{P}$ is a convex prevision on $\mathscr{D} \supset \{A, X|A, A(X-\underline{P}(X|A))\}$, $\underline{P}(A)>0$, then $\underline{P}(X|A)$ is the unique solution of $\underline{P}(A(X-r))=0$.*

Proof. The proof relies on the following fact: if $\mathscr{D}_0=\{A, A(X-r), A(X-t)\}$, $A\neq\emptyset$, $r\neq t$ and $\underline{P}:\mathscr{D}_0\to\mathbb{R}$ is such that $\underline{P}(A)>0$, $\underline{P}(A(X-r))=\underline{P}(A(X-t))=0$, then $\underline{P}$ is no convex lower prevision on $\mathscr{D}_0$.

To prove this, suppose $r<t$ and find, using Definition 5, a gain $\underline{G}$ such that $\sup\underline{G}<0$. Define for this $\underline{G}=s_1A(X-t)+s_2(A-\underline{P}(A))-A(X-r)$ with $s_1\geqslant 0$, $s_2>0$, $s_1+s_2=1$. Since $\sup\{\underline{G}|A^C\}=-s_2\underline{P}(A)<0$, to comply with Definition 5 $\sup\{\underline{G}|A\}$ should be non-negative, which means $\sup\{X(s_1-1)-s_1t+s_2(1-\underline{P}(A))+r|A\}=\sup\{-s_2X|A\}+r-t+s_2(t+1-\underline{P}(A))\geqslant 0$, that is $\inf X|A\leqslant((r-t)/s_2)+t+1-\underline{P}(A)$, $\forall s_1\geqslant 0$, $\forall s_2$ such that $0<s_2\leqslant 1$, $s_1+s_2=1$. It is patent that this condition cannot be satisfied for all such s_2, since $\lim_{s_2\to 0^+}((r-t)/s_2)=-\infty$.

The thesis of the proposition follows at once, since we already know that $\underline{P}(X|A)$ is a solution to $\underline{P}(A(X-r))=0$. $\square$

5.3. Discussion

From what we have seen so far, the generalisation of convex imprecise previsions in a conditional environment that we are considering extends several results from the unconditional case, starting from the notion of convex natural extension. Again, using a centered convex prevision $\underline{P}$ seems preferable, because it guarantees both that P avoids uniform loss and that its convex natural extension exists and is finite. As an important consequence, a centered convex (conditional) prevision on $\mathscr{D}$ can be always extended on any $\mathscr{D}'\supset\mathscr{D}$, and this is an essential fact for the well-foundedness of the theory.

It is not difficult to see that some other properties of convex unconditional previsions generalise to conditional ones. For instance, Proposition 5 holds replacing 'convex' with 'convex conditional' in its statement. Also, if $\underline{P}(0|B)\geqslant 0$, condition $\underline{P}(\lambda X|B)\geqslant\lambda\underline{P}(X|B)$, $\forall\lambda\in[0,1]$ is necessary for convexity of $\underline{P}$.

It is not clear, however, what should correspond to the generalised envelope theorem in a conditional environment: further investigation is needed on this. So far we know that the GBR applies, and this already supplies us with a technique for consistently extending an unconditional judgement on X to $X|A$, when $\underline{P}(A)>0$. We have to evaluate $\underline{P}(A(X-r))$ for various $r\in\mathbb{R}$: if $\underline{P}(A(X-r))<0$ (>0), then, by the monotonicity property (A2) in Theorem 2, r will be an upper (a lower) bound for $\underline{P}(X|A)$. When $\underline{P}$ is also centered, property $\underline{P}(X|A)\in I_0=[\inf X|A, \sup X|A]$ holds (use (D1) in Theorem 8 with, alternatively, $X|A = 0|A$ and $Y|A = 0|A$). Because of this and monotonicity, I_0 can be a starting interval to compute $\underline{P}(X|A)$ using a bisection algorithm (cf. also [4] in the framework of coherent previsions). This requires assessing $\underline{P}(A(X - r))$ (for instance, using the convex natural extension) for a sufficiently large number of $r \in I_0$. The GBR suggests also the interpretation of $\underline{P}(X|A)$ as the supremum of the amounts r that can be subtracted from X keeping $X - r$ desirable (or acceptable), assuming that A occurs.

The problem of jointly evaluating unconditional and conditional risks in a consistent way is a natural application of convex conditional previsions, whenever convex risk measures evaluate the unconditional risks. To the best of our knowledge, *convex risk measures for conditional risks* have not been considered in the literature, but it is simple to generalise Definition 11 to cover this case. Equality (2) becomes $\rho(X|B) = \overline{P}(-X|B)$, so that results for upper rather than lower conditional previsions should be preferably employed, and they are easily obtained using the conjugacy relation $\overline{P}(-X|B) = -\underline{P}(X|B)$. In particular, the GBR tells us that to evaluate $\rho(X|B)$ a subject should determine the supremum μ of the amounts r he would accept to subtract from X, keeping $X - r$ acceptable, with the proviso that B is true: then $\rho(X|B) = -\mu$.

Finally, we mention a potential application of convex previsions. Unconditional centered convex previsions are linked with recently introduced models for pricing derivative securities (or *contingent claims*) in markets with frictions. In fact, when the set $\mathscr{D}$ is convex, unconditional centered convex *upper* previsions satisfy the assumptions on pricing rules in [8], Section 2, and their avoiding sure loss implies that a technical condition called *viability* holds. It would be interesting to investigate how conditional convex previsions could be employed in pricing problems for contingent claims evaluated at different time steps (conditioning each time on the past price values). The convex natural extension might be used to identify (conditional or unconditional) prices for additional contingent claims consistent with given ones. In other words, it would solve the problem of how to price a not yet marketed contingent claim without modifying the existing prices.

6. Conclusions

In this paper we studied convex and centered convex previsions. Convex previsions do not necessarily satisfy minimal consistency requirements, but are useful in generalising natural extension-like methods of correcting inconsistent assessments and in providing a conceptual framework for some uncertainty models. Centered convex previsions are in a sense intermediate between the avoiding sure loss condition and coherence: their properties are closer to coherence than those of a generic prevision that avoids sure loss, but are also compatible with lack of positive homogeneity. Because of this, they are potentially useful at least in models which incorporate some forms of risk aversion. We outlined a risk measurement application, where they lead to defining convex risk measures, and believe that several applications of convex imprecise previsions are still to be explored. We showed that the concept of convex imprecise prevision may be consistently generalised to the conditional case, extending many results from the unconditional framework and proving that the generalised Bayes rule may be applied. In our opinion, there is scope for further investigating properties and applications of convex conditional previsions, for instance concerning envelope theorems, risk measures for conditional risks and the pricing rules problems briefly outlined in the preceding subsection.

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