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On contact tops and integrable tops

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ABSTRACT

In this paper, we introduce a geometric structure called top, which is a trivialized bundle of plane pencils over a Riemannian 3-manifold, defined as the set of kernels of a circle of 1-forms (e.g., of contact and integrable forms) with particular properties with respect to the metric. We classify the manifolds which admit tops and we describe the associated metrics.

1. DEFINITIONS AND MAIN RESULTS

The central object of this paper are tops.[1] These new structures are defined as trivialized plane pencil bundles over a 3-dimensional Riemannian manifold with some regularity properties with respect to the metric. More precisely, we define them as the set of kernels of a circle of 1-forms.

There are two different motivations for this work. First, trivialized plane pencil bundles appear as the geometric structures associated to contact circles, that is, to circles of contact forms, in the same way as contact structures are associated to contact forms. We present these structures in the context of Riemannian manifolds and characterize certain families of contact circles which have special metric properties.

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[1] Here a top is not the integrable system from Hamiltonian mechanics, as it is in the usual sense. We use the same terminology, because this word describes so well the structure we introduce.

The second aim is to answer the following question: In which extent can the properties of a particular example on $\mathbb{S}^3$ (see Examples 13 and 15) with the usual metric be found on other manifolds and in relation with other metrics?

Under some additional coorientability assumptions, we associate a circle of differential 1-forms to a trivialized plane pencil bundle on a 3-manifold M, that is, a family $\mathcal{S}^1\{\omega_1, \omega_2\} = \{\cos\theta\omega_1 + \sin\theta\omega_2, \theta \in [0, 2\pi[\}$ generated by two 1-forms ω_1 and ω_2 which are linearly independent in every point of M. Given a two-dimensional subbundle Ω of the space of differential 1-forms on M, the kernels of the non-zero elements of Ω define a plane pencil in the tangent space at any point $x \in M$. Moreover, all multiples of a given form by a positive function have the same kernel. So it is enough to consider the unit circle of Ω to realize all the planes of the plane pencil as kernels of elements of this family of 1-forms.

For example, a trivialized pencil bundle can be defined by a contact circle (see Section 2 for a precise definition). In that case, the pencil bundle is given by the set of the contact structures associated to the elements of the contact circle.

All objects in this paper are supposed to be smooth.

Let $\mathcal{F}$ be a plane pencil bundle obtained as the set of kernels of a circle of 1-forms $\mathcal{S}^1\{\omega_1, \omega_2\} = \{\omega^\theta := \cos\theta\omega_1 + \sin\theta\omega_2, \theta \in [0, 2\pi[\}$. Then the map

$$\begin{aligned} \tau: \quad \mathcal{F} \quad &\longrightarrow M \times \mathbb{P}^1(\mathbb{R}), \\ \ker(\omega_x^\theta) &\longmapsto (x, \theta \bmod \pi) \end{aligned}$$

is a trivialization of $\mathcal{F}$. The plane bundles $\xi_\theta = \ker(\omega^\theta)$ are obtained by the "inverse map"

$$\begin{aligned} \mathbb{P}^1(\mathbb{R}) &\longrightarrow \mathcal{F}, \\ \theta &\longmapsto \xi_\theta \end{aligned}$$

such that $\pi_2(\tau(\xi_\theta)) = \theta$. This family is double-covered by the $\mathbb{S}^1$-family of plane bundles obtained as the kernels of the defining circle of 1-forms.

Definition 1. A trivialized plane pencil bundle which is defined by the set of kernels of the elements of a circle of 1-forms $\mathcal{S}^1\{\omega_1, \omega_2\}$ will be called an *indexed pencil bundle*.

An indexed pencil bundle is called *contact pencil bundle* if for all $\theta \in \mathbb{P}^1(\mathbb{R})$, the plane bundle ξ_θ is a contact structure, and *integrable pencil bundle* if all these plane bundles are integrable.

The line bundle $L_x = \bigcap_{\theta\in\mathbb{P}^1(\mathbb{R})}(\xi_\theta)_x$ is called the *axis bundle* of the indexed pencil bundle.

Remark. If $\mathcal{F}$ is an indexed pencil bundle, then every plane bundle which is obtained as the kernel of some element of the defining circle of 1-forms is coorientable. On the other hand, a trivialized plane pencil bundle with the property that all plane bundles ξ_θ with $\pi_2(\tau(\xi_\theta)) = \theta$ are coorientable can be obtained via a circle of 1-forms, given by $\omega^\theta = g(X_\theta, \cdot)$, where X_θ is a differentiable family of vector fields such that $(X_\theta)_x \perp (\xi_\theta)_x$, $\forall x \in M$, $\forall \theta \in [0, \pi]$, and where g is a metric on M.

A trivialized plane pencil bundle on a Riemannian manifold can have a certain number of geometric regularity properties. In particular, we can consider the way the plane bundles ξ_θ rotate about the axis bundle with respect to the parallel transport along some particular curves. To make this more precise, we first need to define a appropriate compatibility condition between a trivialized pencil bundle and a metric.

Definition 2. A metric on a manifold M is called *compatible* with an indexed pencil bundle if the angle between two given plane bundles ξ_{θ_1} and ξ_{θ_2} is constant on M.

A moving frame (X_1, X_2, X_3) is said to be *adapted* to an indexed pencil bundle if X_3 is parallel to the axis of the pencil and if there are forms ω_1 and ω_2 in the defining family of 1-forms with $\omega_1(X_2) = \omega_2(X_1) = 0$ everywhere on M.

Remark. A metric for which a moving frame (X_1, X_2, X_3) adapted to some indexed pencil bundle is orthonormal is compatible with the indexed pencil bundle whose trivialization is induced by the family $S^1\{\omega_1, \omega_2\}$, with $\omega_1(X_2) = \omega_2(X_1) = 0$.

The rotation speed of a vector field about some other vector field along a given curve is defined as follows:

Definition 3. Let (M, g) be an oriented Riemannian manifold, Y and Z unit vector fields and γ a curve on M. We will call the quantity

$$R_\gamma(Y, Z) = g(\nabla_{\dot\gamma} Y, Z \times Y)$$

the *rotation speed* of Y about Z along γ with respect to the parallel transport.

We choose this terminology, because $g(\nabla_{\dot\gamma} Y, Z \times Y)$ can be seen as the component of the projection of the covariant derivative of Y along γ onto the plane orthogonal to Z which is orthogonal to the projection of Y onto this same plane.

Definition 4. Let $\mathcal{F}$ be an indexed pencil bundle on an oriented Riemannian manifold (M, g). The *spinning direction* of $\mathcal{F}$ is defined to be an orientation of the axis bundle such that the rotation speed of X_1 about X_3 along every integral curve of this axis bundle is non-negative for a positively oriented adapted moving frame (X_1, X_2, X_3) such that X_3 is positively oriented on the axis bundle.

The following definition has been modeled on the properties of the fundamental example on $\mathbb{S}^3$ developed below (see Examples 13 and 15).

Definition 5. Let (M, g) be an oriented Riemannian manifold of dimension 3. An indexed pencil bundle on M is called a *top* if there is an orthonormal moving frame (X_1, X_2, X_3) adapted to the pencil bundle such that

(i) Along any geodesic γ, the angle between $\dot{\gamma}$ and X_3 is constant.
(ii) Along any geodesic γ which is transverse to the axis bundle, the rotation speed of X_3 about $\dot{\gamma}$ is constant with respect to the parallel transport.
(iii) Along any geodesic γ, the rotation speed of X_1 about X_3 is constant with respect to the parallel transport.
(iv) Along any pair of geodesics γ_1 and γ_2 such that the angles between $\dot{\gamma}_1$ and X_3 and between $\dot{\gamma}_2$ and X_3 are equal, the rotation speed of X_1 about X_3 is the same.

All geodesics are supposed to be parameterized by arc length.

If every plane bundle defined by the trivialization is a contact structure, the top will be called a *contact top*; if all these plane bundles are integrable, it will be called an *integrable top*.

Remarks.

1. A top is completely determined by a moving frame satisfying the four conditions above.
2. If a top is determined by an orthonormal moving frame (X_1, X_2, X_3), the underlying indexed pencil bundle is defined by the dual forms ω_1 and ω_2 which correspond to X_1 and X_2.
3. Condition (i) implies that the integral curves of the axis bundle are geodesics (see also Lemma 18).

We will see that a top is always either a contact top or an integrable top (see Corollary 19). Contact tops are always defined by taut contact circles (see the definitions in Section 2; some examples will also be given there and in Section 3).

The first main result of this paper is a characterization of tops through the properties of the Lie brackets of an adapted orthonormal moving frame.

Theorem 6. *An orthonormal global moving frame (X_1, X_2, X_3) on an oriented Riemannian manifold M determines a top if and only if the corresponding Lie brackets are of the following type:*

$$\begin{cases} [X_1, X_2] = cX_3, \\ [X_2, X_3] = kX_1, \\ [X_3, X_1] = kX_2, \end{cases} \tag{1}$$

where c and k are real numbers.

As a corollary, we obtain a description of the manifolds which admit tops.

Corollary 7. *An oriented and complete connected manifold M carries a top $\mathcal{T}$ if and only if M is diffeomorphic to the left-quotient of one of the following Lie groups by a discrete subgroup*:

$$\left.\begin{array}{l}\widetilde{E}_2\\ SU(2)\\ \widetilde{SL}(2,\mathbb{R})\end{array}\right\}\ \textit{for a contact top;}\qquad \left.\begin{array}{l}\mathbb{R}^3\\ Nil^3\end{array}\right\}\ \textit{for an integrable top,}$$

where $\widetilde{E}_2$ is the universal cover of the group of direct isometries of the plane and Nil^3 the 3-dimensional Heisenberg group.

We are now interested in describing the metrics on these manifolds for which tops can be constructed. They have to be adapted to the geometric properties of tops.

Definition 8. A metric g on a 3-manifold M is called a *spinning metric* if

- there is a unit vector field Z on M, called *pivot field*, which is geodesic and Killing, such that the sectional curvature of any plane only depends on its angle with Z and if
- the extremal sectional curvatures are constant on M.

These are the only metrics to which tops can be associated, as the following theorem states.

Theorem 9. *Let (M,g) be an oriented Riemannian 3-manifold. If (M,g) admits a top, then g is a spinning metric. If g is a spinning metric and $H_1(M,\mathbb{R})=0$, then there exists a spinning metric on (M,g).*

Sections 2 and 3 contain background information and examples concerning contact tops and integrable tops. In Section 4, we prove Theorem 6 and several corollaries. In Section 5, we analyze the properties of spinning metrics and prove Theorem 9. In Section 6, we determine the metrics for which a given contact pencil bundle defines a top. In Section 7, some partial classification results are discussed, especially the uniqueness problem of a top for a given spinning metric. In the last part of this paper, a connection to Sasakian geometry is made. In particular, we discuss which Sasakian structures define tops and which tops give rise to a Sasakian structure.

To conclude, let us remark that it is possible to consider tops in higher dimensions, but there are considerable additional difficulties. The first one is the problem to find a good definition, because the common subspace bundle of a family of hyperplanes in $\mathbb{R}^n$ can be of different dimensions, depending on the number of generators of the defining family of 1-forms, and as soon as the codimension of the common subspace bundle is greater than 2, one can not use the concept of rotation speed any more. Moreover, the study of contact p-spheres on higher-dimensional manifolds is much less developed than for contact circles on 3-manifolds.

Some examples coming from contact geometry have been an important motivation to introduce contact tops. The study of contact circles has been initiated by H. Geiges and J. Gonzalo in 1995 (see [2]).

Definition 10. If all non-trivial, normalized linear combinations of two contact forms are contact forms, this family is called a *contact circle*. We note $\mathcal{S}^1_c\{\omega_1, \omega_2\} := \{\lambda_1\omega_1 + \lambda_2\omega_2, \lambda_1^2 + \lambda_2^2 = 1\}$.

Similarly, a contact sphere is generated by three contact forms.

H. Geiges and J. Gonzalo distinguish a certain class of contact circles, defined as follows:

Definition 11. A contact circle $\mathcal{S}^1_c\{\omega_1, \omega_2\}$ on a 3-manifold M is said to be *taut* if all its elements define the same volume form, that is, if $\omega \wedge d\omega$ is constant on $\mathcal{S}^1_c\{\omega_1, \omega_2\}$.

Example 12 (*Contact circle on* $\mathbb{T}^3$). On the 3-torus with pseudo-coordinates $(\theta_1, \theta_2, \theta_3)$, the forms

$$\begin{aligned}
\omega_1 &= \cos(n\theta_1)d\theta_2 + \sin(n\theta_1)d\theta_3,\\
\omega_2 &= -\sin(n\theta_1)d\theta_2 + \cos(n\theta_1)d\theta_3
\end{aligned}$$

generate a contact circle, for $n \in \mathbb{N}^*$.

Indeed, for $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$, with $\lambda_1^2 + \lambda_2^2 = 1$, we have $\omega \wedge d\omega = -nd\theta_1 \wedge d\theta_2 \wedge d\theta_3$, so all non-trivial linear combinations are contact forms. They all define the same volume form, so the contact circle is taut.

Example 13 (*Contact sphere on* $\mathbb{S}^3$). We consider the 3-sphere as the unit sphere of the quaternionic space $\mathbb{H}$. On this sphere, we have 3 independent contact forms, induced by the following forms on $\mathbb{H}$:

$$\begin{aligned}
\alpha_q &= \langle qi, dq\rangle = q_1dq_2 - q_2dq_1 + q_4dq_3 - q_3dq_4,\\
\beta_q &= \langle qj, dq\rangle = q_1dq_3 - q_3dq_1 + q_2dq_4 - q_4dq_2,\\
\gamma_q &= \langle qk, dq\rangle = q_3dq_2 - q_2dq_3 + q_1dq_4 - q_4dq_1.
\end{aligned}$$

The induced forms generate a contact sphere. Indeed, any normalized linear combination $\omega := \lambda_1\alpha + \lambda_2\beta + \lambda_3\gamma$ satisfies:

$$\omega \wedge d\omega \wedge (q_1dq_1 + q_2dq_2 + q_3dq_3 + q_4dq_4) = dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4,$$

which is a volume form on $\mathbb{H}$. This volume form is independent of the parameters $\lambda_1, \lambda_2, \lambda_3$, so α, β and γ induce a taut contact sphere on $\mathbb{S}^3$.

We have another regularity condition for contact circles, with a more obvious geometric meaning than tautness:

Definition 14. A contact circle $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ is said to be *round* if the Reeb vector field of any element $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ of the contact circle is given by $R = \lambda_1 R_1 + \lambda_2 R_2$, where R_1 and R_2 are the Reeb vector fields of ω_1 and ω_2.

On 3-manifolds, roundness and tautness are equivalent, but on higher-dimensional manifolds, these properties are independent (see [9]).

The first important question is about the existence of contact circles. For closed manifolds, H. Geiges and J. Gonzalo give a general answer:

Theorem (See [3]). *Every closed and orientable 3-manifold admits a contact circle.*

For taut contact circles, which have richer geometric properties, the existence problem is more complex:

Theorem (See [2]). *A closed and orientable 3-manifold M admits a taut contact circle if and only if M is diffeomorphic to the left-quotient of a Lie group G by the action of a discrete subgroup, where G is either $SU(2)$ or $\widetilde{SL}(2,\mathbb{R})$, the universal cover of $SL(2,\mathbb{R})$, or $\widetilde{E}_2$, the universal cover of the Euclidean group.*

A contact circle defines a contact pencil bundle in an obvious way. If we write the contact circle as $\{\omega^\theta = \cos\theta\omega_1 + \sin\theta\omega_2, \theta \in [0, 2\pi[\}$, the corresponding trivialization is given by $\tau(\ker(\omega^\theta)) = \theta \bmod \pi$.

A round contact circle defines a privileged plane bundle, which is transverse to the axis bundle $\ker\omega_1 \cap \ker\omega_2$. It is spanned by the Reeb vector fields R_1 and R_2 of any two generating forms ω_1 and ω_2.

Example 15. The contact circles in Example 12 and the contact circles generated by any two of the three forms of Example 13 define contact tops, for the flat metric on $\mathbb{T}^3$ and the usual metric on $\mathbb{S}^3$.

To see this in the case of the 3-sphere, consider the standard metric and an orthonormal frame (X_1, X_2, X_3) dual to the induced contact forms $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. This frame satisfies

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1 \quad \text{and} \quad [X_3, X_1] = 2X_2.$$

On $\mathbb{S}^3$, a vector field $Z = a_1X_1 + a_2X_2 + a_3X_3$ is geodesic if and only if its coefficients are constant on $\mathbb{S}^3$. So X_3 is a geodesic vector field. Let ρ with $\dot{\rho} = a_1X_1 + a_2X_2 + a_3X_3$ be a geodesic.

Then the rotation speed of X_1 about X_3 along ρ is

$$R_\rho(X_1, X_3) = g(\nabla_{\dot{\rho}}X_1, X_2) = 2a_3,$$

so it is constant and only depends on the angle between X_3 and $\dot{\rho}$, that is, on a_3, which is constant.

The rotation speed of X_3 about $\dot{\rho}$ along ρ is

$$R_\rho(X_3, \dot{\rho}) = a_1^2 + a_2^2,$$

so it is constant, too.

Thus, the contact circle $S_c^1\{\tilde{\alpha}, \tilde{\beta}\}$ defines a contact top.

3. INTEGRABLE TOPS

An integrable top on a Riemannian manifold M defines a circle of foliations of M which are everywhere transverse to each other. A trivial example is the following:

Example 16. A pencil of planes at the origin of $\mathbb{R}^3$ which is parallel transported to all other points of $\mathbb{R}^3$ defines an integrable top with respect to the Euclidean metric. In this example, all rotations speeds are zero and all angles are constant, so the conditions of Definition 5 are trivially satisfied. All plane bundles defined by this top are integrable.

Another example is the following indexed pencil bundle on the Heisenberg group.

Example 17. We represent the Heisenberg group as $\mathbb{R}^3$ with the group operation $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$, and with the metric for which the system $(X_1, X_2, X_3) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial z} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$ is orthonormal. The tangent vector field $X = a_1X_1 + a_2X_2 + a_3X_3$ of a geodesic γ in this space satisfies $\nabla_X X = 0$, thus

$$X(a_1) = 0, \quad X(a_2) = 0, \quad X(a_3) = 0.$$

So a geodesic γ with unit speed is defined by $\dot{\gamma}(t) = a_1X_1 + a_2X_2 + a_3X_3$, for $a_1, a_2, a_3 \in \mathbb{R}$.

Thus, the vector field X_3 is geodesic and the rotation speed of X_1 about X_3 along γ is

$$R_\gamma(X_1, X_3) = -\frac{a_3}{2},$$

so it is constant and it only depends on the angle between X_3 and $\dot{\gamma}$, that is, on a_3, which is constant, too.

The rotation speed of X_3 about $\dot{\gamma}$ along γ is

$$R_\gamma(X_3, \dot{\gamma}) = \frac{1}{2}(a_1^2 + a_2^2),$$

so it is constant, as $a_1^2 + a_2^2 = 1 - a_3^2$.

Thus, (X_1, X_2, X_3) defines a top on the Heisenberg group.

We are now going to have a closer look at indexed plane pencil bundles.

Definition 5 describes tops in a geometric way. We will now characterize them using Lie brackets, in order to have some additional working tools. This is done by Theorem 6, which implicitly gives us much important information about the manifolds which carry tops and about the plane bundles which are associated to tops, as a number of corollaries will show.

We first need a lemma which relates several geometric properties of a vector field on a Riemannian manifold.

Lemma 18. *Let Z be a unitary vector field on a Riemannian manifold. Then the following conditions are equivalent*:

(i) *Z is a Killing vector field and its integral curves are geodesics.*
(ii) *Along any geodesic γ, the angle between $\dot\gamma$ and Z is constant.*
(iii) *For every orthonormal frame (X_1, X_2, Z), there is a real-valued function k, such that $[Z, X_1] = kX_2$ and $[X_2, Z] = kX_1$.*

Proof. (i) $\Leftrightarrow$ (iii). Let $(X_1, X_2, Z = X_3)$ be an orthonormal frame on M. We write

$$[X_i, X_j] = \sum_{k=1}^{3} c_{ij}^k X_k.$$

The integral curves of Z are geodesics if and only if $\nabla_Z Z = 0$, which is equivalent to $c_{31}^3 = c_{23}^3 = 0$. For all calculations of this type, we write $\Gamma_{ij}^k = g(\nabla_{X_i} X_j, X_k)$ and we use the relation $\Gamma_{ij}^k = \frac{1}{2}(c_{ij}^k + c_{ki}^j + c_{kj}^i)$ (see, e.g., [5, p. 48]).

Z is a Killing vector field if and only if for every pair of vector fields (X,Y) on M, we have $g(\nabla_X Z, Y) + g(X, \nabla_Y Z) = 0$, which means that $c_{31}^1 = c_{23}^2 = 0$ (for $X = Y = X_1$ and for $X = Y = X_2$) and $c_{23}^1 = c_{31}^2$ (for $X = X_1$ and $Y = X_2$).

(ii) $\Leftrightarrow$ (iii). For a geodesic γ with $\dot\gamma(s) = a_1(s)X_1 + a_2(s)X_2 + a_3(s)X_3$, we have

$$\frac{\partial}{\partial s}\big(g(X_3, \dot\gamma)\big) = g(\nabla_{\dot\gamma} X_3, \dot\gamma) = a_1^2 c_{13}^1 + a_2^2 c_{23}^2 + a_1 a_2\big(c_{13}^2 + c_{23}^1\big) + a_1 a_3 c_{13}^3 + a_2 a_3 c_{23}^3.$$

So the constancy of the angle between Z and $\dot\gamma$ along any geodesic γ is equivalent to the property (iii). $\square$

Remark. It follows from this lemma that a unitary vector field which defines the axis bundle of a top is a Killing vector field.

Proof of Theorem 6. Let (X_1, X_2, X_3) be a positively oriented orthonormal moving frame which defines a top. Again, we write the corresponding Lie brackets as

$$[X_i, X_j] = \sum_{k=1}^{3} c_{ij}^k X_k.$$

Furthermore, let γ be a geodesic such that $\dot\gamma$ has unit length. We write $\dot\gamma = a_1 X_1 + a_2 X_2 + a_3 X_3$, where the a_i are functions on M. We will express the geometric conditions of Definition 5 in terms of the functions c_{ij}^k:

(i) By Lemma 18, the property that the angle between X_3 and $\dot\gamma$ is constant along any geodesic γ is equivalent to the identities $c_{31}^1 = c_{23}^2 = c_{31}^3 = c_{23}^3 = 0$ and $c_{23}^1 = c_{31}^2$ on M. In that case, it also follows that a_3 is constant along γ.

(iv) Under the assumption that property (i) is satisfied, we have

$$\begin{aligned} R_\gamma(X_1, X_3) &= a_1 \Gamma_{11}^2 + a_2 \Gamma_{21}^2 + a_3 \Gamma_{31}^2 \\ &= a_1 c_{21}^1 + a_2 c_{21}^2 + a_3 \left(c_{23}^1 - \frac{1}{2} c_{12}^3 \right). \end{aligned}$$

Thus, the rotation speed of X_1 about X_3 along γ does not depend on the angles between $\dot\gamma$ and X_1 or X_2, if and only if $c_{21}^1 = c_{21}^2 = 0$.

(ii) Suppose now that γ is transverse to the axis bundle. If properties (i) and (iv) are satisfied, we have

$$\begin{aligned} R_\gamma(X_3, \dot\gamma) &= a_2 g(\nabla_{\dot\gamma} X_3, X_1) - a_1 g(\nabla_{\dot\gamma} X_3, X_2) \\ &= c_{12}^3 (a_1^2 + a_2^2). \end{aligned}$$

Thus, the constancy of the rotation speed of X_3 about $\dot\gamma$ along γ is equivalent to the constancy of c_{12}^3 along γ. Indeed, $a_1^2 + a_2^2$ is constant along γ, since a_3 is, by (i). As this is true for any choice of γ (transverse to X_3), it follows that c_{12}^3 is constant on M if and only if the rotation speed of X_3 about $\dot\gamma$ is constant along any geodesic γ.

(iii) Now we assume that properties (i), (ii) and (iv) are satisfied. So we have

$$R_\gamma(X_1, X_3) = a_3 \left(c_{23}^1 - \frac{1}{2} c_{12}^3 \right).$$

Thus, the rotation speed of X_1 about X_3 is constant along γ if and only if c_{23}^1 is constant along γ. Again, it follows that c_{23}^1 is constant on M if and only if the rotation speed of X_1 about X_3 is constant along any geodesic γ. □

Theorem 6 allows us to describe the nature of the indexed plane bundles associated to a top as well as the manifolds where tops can be constructed.

We first prove Corollary 7, stated in the introduction.

Proof of Corollary 7. Let (X_1, X_2, X_3) be a moving frame on M which defines $\mathcal{T}$ and so satisfies the relations (1). This defines on M a locally free action of the simply connected Lie group $\mathcal{G}$ whose Lie algebra is defined by these relations. According to the classification of Lie algebras of dimension 3, these Lie algebras are $\mathfrak{so}(3, \mathbb{R})$ (for $ck > 0$), $\mathfrak{sl}(2, \mathbb{R})$ (for $ck < 0$), the Lie algebra associated to E_2 (for $c = 0, k \neq 0$), the Lie algebra associated to Nil^3 (for $c \neq 0$, $k = 0$), or $\mathbb{R}^3$ (for $c = k = 0$). Any orbit of this action is diffeomorphic to a quotient of $\mathcal{G}$ by the stabilizer, which is a discrete subgroup. So every orbit is an open submanifold of M and its complement is a union of orbits, so it is open also. Hence, the orbit is a connected component of M and the action is transitive, M being connected. Thus, M is diffeomorphic to a quotient of one of the corresponding Lie groups by a discrete subgroup given by the isotropy subgroup of a point of M.

On the other hand, given one of these Lie groups, which we call $\mathcal{G}$, and a discrete subgroup $\mathcal{S}$, let (X_1, X_2, X_3) be a global moving frame of $\mathcal{G}$, where each X_i is a left-invariant vector field coming from a generator of the associated Lie algebra, such that (1) is satisfied for some constants c and k. In particular, this moving frame is invariant under the left-action of $\mathcal{S}$. Hence, it induces a moving frame on the quotient $\mathcal{S}\backslash\mathcal{G}$ which satisfies the relations (1). According to Theorem 6, there exists a top on $\mathcal{S}\backslash\mathcal{G}$. □

Corollary 19. *All plane bundles defined by a given top are of the same nature. They are either integrable or contact structures.*

Proof. Let ω_1, ω_2 and ω_3 be the dual forms associated to the vector fields X_1, X_2 and X_3 of an orthonormal moving frame of M satisfying the relations (1), that is,

$$\text{(2)} \qquad \begin{cases} d\omega_3 &= -c\omega_1 \wedge \omega_2, \\ d\omega_2 &= -k\omega_3 \wedge \omega_1, \\ d\omega_1 &= -k\omega_2 \wedge \omega_3. \end{cases}$$

Then ω_1 and ω_2 are the generating forms of the top which is determined by (X_1, X_2, X_3). The plane bundles defined by the trivialization of this top are the kernels of the forms $\omega^\theta = \cos\theta\omega_1 + \sin\theta\omega_2$.

If $k = 0$, then ω_1 and ω_2 are integrable, as well as their linear combinations, so the pencil bundle is integrable.

If $k \neq 0$, then ω_1 and ω_2 are contact forms, as well as their linear combinations. So these two forms generate a contact circle $\mathcal{S}^1_c\{\omega_1, \omega_2\}$ and the corresponding pencil bundle is a contact pencil bundle. □

Definition 20 (See [4]). A contact circle $\mathcal{S}^1_c\{\omega_1, \omega_2\}$ is called a K-Cartan structure if there is a 1-form ω_3 such that

$$d\omega_1 = \omega_2 \wedge \omega_3, \quad d\omega_2 = \omega_3 \wedge \omega_1 \quad \text{and} \quad d\omega_3 = K\omega_1 \wedge \omega_2,$$

for some real number K. In particular, a K-Cartan structure is taut.

Corollary 21. *Any contact top is defined by a K-Cartan structure, for $K \in \{-1, 0, 1\}$.*

Proof. In the proof of the preceding Corollary, we show that a contact top is always defined by a taut contact circle. This contact circle is in fact a K-Cartan structure, by (2), where K can be chosen as 0, 1 or -1. □

These results about contact tops can be related to Geiges' and Gonzalo's theorem about the classification of closed and orientable 3-manifolds which carry taut contact circles (see [2] or Section 1), in the following way.

Corollary 7 implies that on all manifolds listed in this theorem taut contact circles exist, since any contact top is defined by a taut contact circle (see Corollary 21).

On the other hand, Proposition 3.5 of [4] states that every conformal class of taut contact circles on a compact left-quotient of $\widetilde{SL}_2$ (resp. $\tilde{E}_2$) contains a K-Cartan structure, for $K = -1$ (resp. 0). Such a conformal class is defined as the set of multiples of a given contact circle, that is, where both generators are multiplied by the same positive function. This does not change the associated contact structures, so a conformal class of taut contact circles defines a trivialized plane pencil bundle, which is a top if this conformal class contains a K-Cartan structure. Thus, on these manifolds, every taut contact circle defines a contact top.

In the following sections, we are going to have a closer look on the metrics which can be associated to tops. The problems are the following: *For which metrics can tops be constructed? How far does a given metric or a given top determine the tops and metrics we can associate to it?* We will partially answer these questions in the following sections.

5. SPINNING METRICS

In the relations (1), X_1 and X_2 play symmetric parts, so a metric for which a moving frame satisfying (1) is orthonormal has to be, in some sense, homogeneous around the vector field X_3, that is, invariant by rotation about the axis given by X_3.

Let us see some examples of spinning metrics (see Definition 8):

(i) A metric with curvature zero is a spinning metric, if there is a geodesic Killing vector field. An example is given by the flat metric on $\mathbb{T}^3$.

(ii) Let (Y_1, Y_2, Y_3) be a moving frame on $\mathbb{S}^3$ which defines the usual metric and where Y_3 is a geodesic Killing vector field. Then any metric for which the moving frame (aY_1, aY_2, bY_3), with $a, b \in \mathbb{R}^*$, is orthonormal, is a spinning metric.

(iii) Another interesting case is the situation where the sectional curvature of the planes which contain Z is positive and where the plane orthogonal to Z has negative sectional curvature. An example of this type is given on the Heisenberg group (see Example 17).

Remark. Metrics with negative constant curvature are not spinning metrics, because they do not admit non-trivial Killing vector fields. According to the calculations in the proof of the following lemma, any plane which contains a pivot field has non-negative sectional curvature.

Lemma 22. *Let g be a spinning metric on a manifold M and let Z be a pivot field of g. If (X_1, X_2, Z) is an orthonormal moving frame, the Lie brackets of these vector fields are of the following type*:

$$
\left\{
\begin{aligned}
[X_1, X_2] &= c_{12}^1 X_1 + c_{12}^2 X_2 + c_{12}^3 Z,\\
[X_2, Z] &= c_{23}^1 X_1,\\
[Z, X_1] &= c_{23}^1 X_2,
\end{aligned}
\right. \tag{3}
$$

where c_{12}^3 is a constant.

Proof. Lemma 18 implies that the Lie brackets satisfy (3) and it only remains to show that c_{12}^3 is a constant.

In the following, we will use the curvature identities (see, for example, [5, p. 69])

$$
\begin{aligned}
&g(R(X,Y)Z,T) = g(R(Z,T)X,Y),\\
&g(R(X,Y)Z,T) = -g(R(Y,X)Z,T),\\
&g(R(X,Y)Z,T) = -g(R(X,Y)T,Z),
\end{aligned}
$$

where $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, and the relation (see [5, p. 45])

$$
\begin{aligned}
g\big(R(X_i, X_j)X_k, X_s\big) = \sum_l \big(\Gamma_{jk}^l \Gamma_{il}^s - \Gamma_{ik}^l \Gamma_{jl}^s\big) + X_i\big(\Gamma_{jk}^s\big) - X_j\big(\Gamma_{ik}^s\big)&\\
- \sum_l c_{ij}^l \Gamma_{lk}^s.&
\end{aligned}
$$

A plane spanned by two vectors

$$
\begin{aligned}
&X = \cos\theta X_1 + \sin\theta X_2 \quad \text{and}\\
&Y = -\sin\varphi \sin\theta X_1 + \sin\varphi\cos\theta X_2 + \cos\varphi Z
\end{aligned}
$$

has sectional curvature

$$
\begin{aligned}
&-g\big(R(X,Y)X,Y\big)\\
&\quad = -\sin^2\varphi g\big(R(X_1,X_2)X_1,X_2\big)\\
&\qquad - \cos^2\varphi\big(\cos^2\theta g\big(R(X_1,Z)X_1,Z\big) + \sin^2\theta g\big(R(X_2,Z)X_2,Z\big)\big)\\
&\qquad - 2\cos\theta\sin\varphi\cos\varphi g\big(R(X_1,X_2)X_1,Z\big)\\
&\qquad - 2\sin\theta\sin\varphi\cos\varphi g\big(R(X_1,X_2)X_2,Z\big)\\
&\qquad - 2\cos\theta\sin\theta\cos^2\varphi g\big(R(X_1,Z)X_2,Z\big)
\end{aligned}
$$

$$
\begin{aligned}
(4) \qquad & = \sin^2\varphi\Big(-\tfrac{3}{4}\big(c_{12}^3\big)^2 - \big(c_{12}^1\big)^2 - \big(c_{12}^2\big)^2 + c_{12}^3 c_{23}^1 - X_2\big(c_{12}^1\big) + X_1\big(c_{12}^2\big)\Big) \\
& + \frac{1}{4}\cos^2\varphi\big(c_{12}^3\big)^2 \\
& + \cos\theta\sin\varphi\cos\varphi X_1\big(c_{12}^3\big) \\
& + \sin\theta\sin\varphi\cos\varphi X_2\big(c_{12}^3\big) \\
& + \cos\theta\sin\theta\cos^2\varphi Z\big(c_{12}^3\big).
\end{aligned}
$$

As g is a spinning metric, this expression does not depend on θ, hence

$$X_1\big(c_{12}^3\big) = 0, \qquad X_2\big(c_{12}^3\big) = 0, \qquad Z\big(c_{12}^3\big) = 0.$$

Thus, c_{12}^3 is constant on M. □

Lemma 23. *Let g be a spinning metric on a 3-manifold M. Let Z be a pivot field of g, let α be the sectional curvature of any plane containing Z and let β be the sectional curvature of the plane which is orthogonal to Z. Then α and β are the extremal sectional curvatures of g and the sectional curvature of a plane which forms an angle φ with Z is*

$$\alpha\cos^2\varphi + \beta\sin^2\varphi.$$

Proof. This is another consequence of the calculation of the sectional curvature of an arbitrary plane for a spinning metric in the proof of Lemma 22. □

From now on, when we talk about the extremal sectional curvatures α and β of a spinning metric, α will be the sectional curvature of any plane which contains a pivot field Z, and β will be the sectional curvature of the plane orthogonal to Z. This does not depend on the choice of Z, by Lemma 24 below.

Theorem 9, stated in the introduction, specifies the close relationship between tops and spinning metrics.

To prove it, we need the following lemma:

Lemma 24. *If g is a spinning metric on a 3-manifold M and if (M, g) is not a space of constant curvature, then the pivot field of g is unique up to the sign.*

Proof. This is a consequence of Lemma 23. □

Proof of Theorem 9. Let us see why g is a spinning metric if there is a top $\mathcal{T}$ on (M, g). Let $\mathcal{T}$ be defined by an orthonormal moving frame (X_1, X_2, X_3) satisfying (1). So X_3 is geodesic and Killing. According to the calculations in the proof of Lemma 22, the sectional curvature of the plane spanned by the vectors

$$
\begin{aligned}
X &= \cos\theta X_1 + \sin\theta X_2 \quad \text{and} \\
Y &= -\sin\varphi\sin\theta X_1 + \sin\varphi\cos\theta X_2 + \cos\varphi X_3,
\end{aligned}
$$

for arbitrary angles θ and φ, is $\cos^2\varphi\frac{c^2}{4} + \sin^2\varphi(ck - \frac{3}{4}c^2)$. This value does not depend on θ. The extremal sectional curvatures are

$$(5) \qquad \alpha = \frac{c^2}{4} \quad \text{and} \quad \beta = ck - \frac{3}{4}c^2,$$

so they are constant. Hence, g is a spinning metric.

Let now g be a spinning metric on M and Z an associated pivot field and let (X_1, X_2, Z) be a positively oriented orthonormal moving frame. By Lemma 22, these vector fields satisfy (3), where c_{12}^3 is a constant.

The Jacobi condition yields

$$(6) \qquad \begin{cases} X_1(c_{23}^1) - c_{12}^2 c_{23}^1 + Z(c_{12}^1) &= 0, \\ X_2(c_{23}^1) + c_{12}^1 c_{23}^1 + Z(c_{12}^2) &= 0, \\ Z(c_{12}^3) &= 0. \end{cases}$$

Moreover, the extremal sectional curvatures of a spinning metric are constant, so

$$(7) \qquad \beta := -\frac{3}{4}\left(c_{12}^3\right)^2 - \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2 + c_{12}^3 c_{23}^1 - X_2\left(c_{12}^1\right) + X_1\left(c_{12}^2\right) = const,$$

by (4). We modify (X_1, X_2, Z) by a rotation about Z. We obtain (Y_1, Y_2, Z), where

$$Y_1 = \cos\psi X_1 + \sin\psi X_2 \quad \text{and} \quad Y_2 = -\sin\psi X_1 + \cos\psi X_2.$$

These vector fields satisfy

$$\begin{cases} [Y_1, Y_2] = \left(c_{12}^1 - X_1(\psi)\right)X_1 + \left(c_{12}^2 - X_2(\psi)\right)X_2 + c_{12}^3 Z, \\ [Y_2, Z] = \left(Z(\psi) + c_{23}^1\right)Y_1, \\ [Z, Y_1] = \left(Z(\psi) + c_{23}^1\right)Y_2. \end{cases}$$

To construct a top, we need to find a function ψ on M such that (Y_1, Y_2, Z) satisfies (1), that is, we have to integrate the following system:

$$\begin{cases} X_1(\psi) = c_{12}^1, \\ X_2(\psi) = c_{12}^2, \\ Z(\psi) = -c_{23}^1 + h, \end{cases}$$

where h is a real number such that $\beta = c_{12}^3 h - \frac{3}{4}(c_{12}^3)^2$.

Let α be the differential 1-form defined by $\alpha(X_1) = c_{12}^1$, $\alpha(X_2) = c_{12}^2$ and $\alpha(X_3) = -c_{23}^1 + h$. Then the above system of PDEs can be written as

$$(8) \qquad d\psi = \alpha$$

and there is a local solution if and only if $d\alpha = 0$. This condition is equivalent to the first two equations of (6) and to (7). Thus, there is a local solution, hence a local top. Moreover, two local solutions ψ_1 and ψ_2 differ only by a constant rotation, since we have $d(\psi_1 - \psi_2) = \alpha - \alpha = 0$. So this local top is unique up to a reparametrization of its defining family of 1-forms.

If $h = 0$, this local top is integrable; if $h \neq 0$, it is a contact top. If $c_{12}^3 = 0$, we can choose h arbitrarily. In that case, we can locally construct both integrable tops and contact tops.

There is a global solution of Eq. (8) if and only if these local tops can be connected in a unique way, that is, if the integral of α over every closed cycle in M is an integer multiple of 2π. This is true under our assumption that $H^1_{de\ Rham}(M) = 0$. □

6. METRICS COMPATIBLE WITH A GIVEN CONTACT TOP

Let us now see how much freedom we have to choose a metric for a given top. *On a manifold (M, g) with a given top, are there other metrics for which the same indexed pencil bundle defines a top?*

We are going to answer this question for contact tops only, because they are defined by round contact circles. This determines a privileged plane bundle transverse to the axis bundle of the top, generated by the Reeb vector fields associated to the generating forms of the contact circle. The roundness property implies that this plane does not depend on the choice of these forms. We have the following result:

Proposition 25. *Let $\mathcal{T}$ be a contact top on a Riemannian 3-manifold (M, g). Then $\mathcal{T}$ is a contact top for another metric g' if and only if the transition matrix between an orthonormal frame for g and an orthonormal frame for g' is of the following form:*

$$\begin{pmatrix} \lambda\alpha & \mu\gamma & 0 \\ \lambda\beta & \mu\delta & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \textit{with} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in O(2, \mathbb{R}) \textit{ and } \lambda, \mu, \nu \in \mathbb{R}.$$

Proof. Let (X_1, X_2, X_3) be an orthonormal moving frame for g which defines $\mathcal{T}$ and thus satisfies (1) for some constants c and k. As $\mathcal{T}$ is a contact top, we have $k \neq 0$. Then another moving frame (Y_1, Y_2, Y_3) which defines the same indexed pencil bundle, can be written as

$$Y_1 = \lambda(\alpha X_1 + \beta X_2), \quad Y_2 = \mu(\gamma X_1 + \delta X_2), \quad Y_3 = \nu X_3,$$

where α, β, γ and δ are real numbers such that $\Delta := \alpha\delta - \beta\gamma \neq 0$, and where λ, μ and ν are non-vanishing functions on M. Indeed, as $\mathcal{T}$ is defined by a taut, thus round, contact circle (by Corollary 21), Y_1 and Y_2 span the same plane as X_1 and X_2. As (X_1, X_2, X_3) satisfies (1), we have

$$[Y_1, Y_2] = -\frac{Y_2(\lambda)}{\lambda} Y_1 + \frac{Y_1(\mu)}{\mu} Y_2 + c\Delta \frac{\lambda\mu}{\nu} Y_3,$$

$$[Y_2, Y_3] = \frac{\mu\nu(\gamma^2+\delta^2)}{\lambda\Delta} k Y_1 - \left(\frac{Y_3(\mu)}{\mu} + \frac{\alpha\gamma+\beta\delta}{\Delta}\nu k\right) Y_2 + \frac{Y_2(\nu)}{\nu} Y_3,$$

$$[Y_3, Y_1] = \left(\frac{Y_3(\lambda)}{\lambda} - \frac{\alpha\gamma+\beta\delta}{\Delta}\nu k\right) Y_1 + \frac{\lambda\nu(\alpha^2+\beta^2)}{\mu\Delta} k Y_2 - \frac{Y_1(\nu)}{\nu} Y_3.$$

Thus, (Y_1, Y_2, Y_3) satisfies (1) for some constants $\tilde{c}$ and $\tilde{k}$ if and only if we have

$$\begin{aligned}
&Y_1(\mu) = Y_2(\lambda) = Y_1(\nu) = Y_2(\nu) = 0, \quad c\frac{\lambda\mu}{\nu} = const,\\
&\frac{\mu}{\lambda}(\gamma^2+\delta^2) = \frac{\lambda}{\mu}(\alpha^2+\beta^2),\\
&\frac{\mu\nu}{\lambda} = const, \quad \frac{\lambda\nu}{\mu} = const, \quad \mu\nu k(\alpha\gamma+\beta\delta) + \Delta Y_3(\mu) = 0,\\
&\lambda\nu k(\alpha\gamma+\beta\delta) - \Delta Y_3(\lambda) = 0.
\end{aligned}$$

So λ, μ and ν are constant on M. Up to multiplying λ and μ by constant factors, we can assume that $\alpha^2+\beta^2 = \gamma^2+\delta^2 = 1$. The above relations show that the matrix $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ is an orthogonal matrix. Thus, (Y_1, Y_2, Y_3) satisfies (1) if and only if the transition matrix of (Y_1, Y_2, Y_3) with respect to (X_1, X_2, X_3) corresponds to an endomorphism of the announced type. If g' is the metric for which (Y_1, Y_2, Y_3) is orthonormal, then $\mathcal{T}$ is a top for g', by Theorem 6. □

7. CLASSIFICATION PROBLEMS

Another natural problem is the classification of tops on a given Riemannian manifold. We will consider the question of uniqueness:

Can two orthonormal global moving frames on a Riemannian 3-manifold (M, g) determine two different tops, that is, tops which are not related by an isometry or a reparametrization of the defining family of 1-forms?

Lemma 26. *Two tops which have the same axis bundle and the same spinning direction are related by a reparametrization of the defining family of* 1*-forms.*

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tops determined by two positively oriented orthonormal moving frames (X_1, X_2, Z) and (Y_1, Y_2, Z), respectively, and let the associated constants be c, k and $\tilde{c}, \tilde{k}$. By (5), the metric determines these constants up to simultaneous multiplication by -1, so we have $c = \varepsilon\tilde{c}$ and $k = \varepsilon\tilde{k}$, with $\varepsilon = \pm 1$. The rotation speed of $\mathcal{T}_1$ and of $\mathcal{T}_2$ about their axis bundle along a geodesic γ which forms an angle φ with the common axis bundle is given by

$$R_\gamma(X_1, Z) = \left(k - \frac{c}{2}\right)\cos\varphi \quad \text{and} \quad R_\gamma(Y_1, Z) = \varepsilon\left(k - \frac{c}{2}\right)\cos\varphi.$$

As the spinning direction is the same, it follows that $\varepsilon = 1$.

So the rotation speeds along all geodesics are equal, and the two tops can only differ by a fixed rotation around the axis bundle. This means that the second factors of the corresponding trivializations $\tau_1, \tau_2 : \mathcal{F} \to M \times \mathbb{P}^1(\mathbb{R})$ differ by a constant translation in $\mathbb{P}^1(\mathbb{R})$, which can be expressed by a reparametrization of the defining family of 1-forms. $\square$

Remark. It follows from this proof that two moving frames which determine two tops with the same axis bundle define the same orientation if and only if their associated constants c and k coincide.

We will consider different situations with respect to the above uniqueness question:

- If (M, g) is not a space of constant curvature, Lemma 24 implies the uniqueness of the axis bundle of a top on M. In that case, Lemma 26 grants the uniqueness of a top on M, up to reparametrization and spinning direction.
- On $\mathbb{S}^3$ with the usual metric, any pivot field defines a contact top, by Theorem 9 and the construction done in its proof. Furthermore, as two tops which are given by the same pivot field have the same axis bundle, they coincide up to reparametrization and spinning direction. So we have to determine whether there are geodesic Killing vector fields on $\mathbb{S}^3$ which are not isometric to each other.
  Let $\mathbb{S}^3$ be the unit sphere of the space of quaternions $\mathbb{H}$. Its algebra of Killing fields is of dimension 6 and it is generated by the vector fields which to a point q associate the elements qi, qj, qk, iq, jq and kq of $T_q\mathbb{S}^3$, respectively.
  Let $v = a_1qi + a_2qj + a_3qk + b_1iq + b_2jq + b_3kq$ be a Killing vector field. Its integral curves are geodesics if and only if either $a_1 = a_2 = a_3 = 0$ or $b_1 = b_2 = b_3 = 0$. Indeed, as the first three vector fields qi, qj and qk define a moving frame, any vector field can be written as $w = aqi + bqj + cqk$ with functions a, b and c on M. w is geodesic if and only if a, b and c are constants. The same argument holds for the last three vector fields.
  Hence, a pivot field Z of the usual metric on $\mathbb{S}^3$ is given either by $a_1qi + a_2qj + a_3qk$ or by $b_1iq + b_2jq + b_3kq$, where the a_i and b_i are real numbers. Two unitary vector fields of one of these families are of course isometric. To see whether two unitary vector fields of different families are isometric, it is enough to consider the question for the vector fields $X_q = qi$ and $Y_q = iq$. A map f exchanges X and Y if $Tf_q(X_q) = Y_{f(q)}$ for any point q of $\mathbb{S}^3$. For $f(q_1, q_2, q_3, q_4) = (q_1, q_2, -q_3, -q_4)$, the tangent map Tf satisfies this condition.
  We have proved the following uniqueness result:

Proposition 27. *On $\mathbb{S}^3$ with the usual metric, all tops are isometric, up to spinning direction and reparametrization of the defining family of* 1*-forms.*

- On a space M of constant zero curvature, there might be non-isometric tops. In particular, an isometry transforms a closed curve into a closed curve. On

$\mathbb{T}^3$, there are contact tops with closed integral curves and others whose integral curves are not closed.

Example 28 (*Contact top on $\mathbb{T}^3$ with closed integral curves*). On $\mathbb{T}^3$ with the pseudo-coordinates $(\theta_1, \theta_2, \theta_3)$, we consider the contact pencil bundle associated to the contact circle of Example 12, for $n = 1$, generated by

$$\omega_1 = \cos\theta_1 d\theta_2 + \sin\theta_1 d\theta_3 \quad \text{and} \quad \omega_2 = -\sin\theta_1 d\theta_2 + \cos\theta_1 d\theta_3.$$

An adapted moving frame is given by the Reeb vector fields of ω_1 and ω_2 and by a third vector field which defines the axis bundle:

$$R_1 = \cos\theta_1 \frac{\partial}{\partial\theta_2} + \sin\theta_1 \frac{\partial}{\partial\theta_3}, \qquad R_2 = -\sin\theta_1 \frac{\partial}{\partial\theta_2} + \cos\theta_1 \frac{\partial}{\partial\theta_3}, \qquad X = \frac{\partial}{\partial\theta_1}.$$

The Lie brackets are $[R_1, R_2] = 0$, $[R_2, X] = R_1$ and $[X, R_1] = R_2$, so this moving frame determines a contact top on $\mathbb{T}^3$. The integral curves of its axis bundle are the integral curves of X, that is, the curves defined by $\theta_2 = \theta_3 = const$, which are closed.

Example 29 (*Contact top on $\mathbb{T}^3$ with non-closed integral curves*). We modify a little the above example to get a moving frame of $\mathbb{T}^3$ which defines the same metric, but determines a different top, whose integral curves are not closed. Let us consider the following moving frame:

$$\begin{aligned}
R_1 &= \cos(\theta_1 + \varepsilon\theta_2)\left(\frac{\partial}{\partial\theta_2} - \varepsilon\frac{\partial}{\partial\theta_1}\right) + \sin(\theta_1 + \varepsilon\theta_2)\frac{\partial}{\partial\theta_3}, \\
R_2 &= -\sin(\theta_1 + \varepsilon\theta_2)\left(\frac{\partial}{\partial\theta_2} - \varepsilon\frac{\partial}{\partial\theta_1}\right) + \cos(\theta_1 + \varepsilon\theta_2)\frac{\partial}{\partial\theta_3}, \\
X &= \frac{\partial}{\partial\theta_1} + \varepsilon\frac{\partial}{\partial\theta_2}.
\end{aligned}$$

If ε is not a rational number, the integral curves of X are not closed, but each one is dense on a 2-torus contained in $\mathbb{T}^3$. The Lie brackets are the following:

$$[R_1, R_2] = 0, \qquad [R_2, X] = \left(1 + \varepsilon^2\right)R_1, \qquad [X, R_1] = \left(1 + \varepsilon^2\right)R_2,$$

so this moving frame determines a contact top. It is defined by the round contact circle generated by the forms

$$\begin{aligned}
&\omega_1 = \cos(\theta_1 + \varepsilon\theta_2)d(\theta_2 - \varepsilon\theta_1) + \sin(\theta_1 + \varepsilon\theta_2)d\theta_3 \quad \text{and} \\
&\omega_2 = -\sin(\theta_1 + \varepsilon\theta_2)d(\theta_2 - \varepsilon\theta_1) + \cos(\theta_1 + \varepsilon\theta_2)d\theta_3.
\end{aligned}$$

8. TOPS AND SASAKIAN GEOMETRY

The study of contact tops can be placed in the context of metric contact geometry. Therefore, it is suitable to examine the relations to Sasakian geometry.

Let (X_1, X_2, X_3) be a moving frame which determines a top on (M, g) and thus satisfies (1). Let $(\omega_1, \omega_2, \omega_3)$ be the dual 1-forms. Then for $c \neq 0$, ω_3 defines a K-contact structure, that is, the Reeb vector field X_3 is Killing. In dimension 3, a manifold is Sasakian if and only if it is K-contact ([8]; see also [1, Corollaries 6.3 and 6.5]), so in this case (M, g) is Sasakian. On the other hand, according to Itoh (see [6]), no torus can carry a K-contact structure. But we have examples of contact tops on $\mathbb{T}^3$ (see Examples 28 and 29), which correspond to the case $c = 0$.

So some tops define Sasakian structures and some do not, and the other way round a Sasakian structure (Φ, ξ, η, g) (see [1] for the notations) on a 3-manifold M defines a top in some cases and in others does not. Indeed, according to R. Lutz [7, §1.8], we can choose two vector fields X_1 and X_2 on M such that (X_1, X_2, ξ) is an orthonormal moving frame satisfying the Lie bracket relations:

$$[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 + 2\xi, \qquad [X_2, \xi] = c_{23}^1 X_1,$$
$$[\xi, X_1] = c_{23}^1 X_2,$$

for some functions c_{12}^1, c_{12}^2 and c_{23}^1.

In view of expression (4), g is then a spinning metric if and only if

$$-\left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2 + 2c_{23}^1 - X_2\left(c_{12}^1\right) + X_1\left(c_{12}^2\right)$$

is constant on M, that is, if the Sasakian structure has constant sectional curvature. This is a necessary condition for the existence of a top on M, according to Theorem 9. Furthermore, the Lie bracket relations imply that ξ is a pivot field for this metric. Thus, locally there are vector fields $\tilde{X}_1$ and $\tilde{X}_2$, such that $(\tilde{X}_1, \tilde{X}_2, \xi)$ determines a top, as shown in the proof of Theorem 9.

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