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Circular embeddings of planar graphs in nonspherical surfaces

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Abstract

We show that every 3-connected planar graph has a circular embedding in some nonspherical surface. More generally, we characterize those planar graphs that have a 2-representative embedding in some nonspherical surface.

1. Introduction

An embedding ϕ of a graph G in a surface Σ that is not the sphere is *ρ-representative* if, for every noncontractible cycle Γ in Σ, $|\Gamma \cap \phi(G)| \geqslant \rho$. A basic result in the theory of representativity is the following.

Theorem 1.1. *Any embedding of a planar graph in a surface other than the sphere is not 3-representative.*

This result is proved by Robertson and Vitray [1]. See also [2]. A natural question, posed in [1] is: Which planar graphs have a 2-representative embedding in some surface other than the sphere? One of the main purposes of this article is to answer completely this question. The main part of this answer is the following theorem.

Theorem 1.2. *Every 3-connected planar graph has a 2-representative embedding in some surface other than the sphere.*

This is proved in Section 3.

Standard results in the theory show that it suffices to consider the question for 2-connected graphs. For a 2-connected graph G, an embedding of G in some

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nonspherical surface is 2-representative if and only if every face is bounded by a cycle of G. An embedding is *circular* if every face is bounded by a cycle. Thus, we are going to show, in Section 3, that every 3-connected planar graph has a circular embedding in some surface other than the sphere. (It is well-known that a circular embedding into the sphere exists.)

Some of the interest in the Robertson–Vitray question arises from the connection with cycle double covers. A *cycle double cover* in a graph G is a list of cycles $\mathscr{C}$ of G such that each edge of G lies in exactly 2 members of $\mathscr{C}$.

Conjecture 1.3. Every 2-edge-connected graph has a cycle double cover.

The face boundaries of a circular embedding form a cycle double cover, so a conjecture that implies Conjecture 1.3 is the following conjecture.

Conjecture 1.4. Every 2-connected graph has a circular embedding in some surface.

Robertson and Vitray had hope of gaining insight into Conjecture 1.4 by considering embeddings of planar graphs into surfaces other than the sphere. Unfortunately, our techniques do not shed light on Conjecture 1.4.

The two known proofs of Theorem 1.1 rely on the structure of the embedding of the planar graph in the surface that is not the sphere. One can ask to what extent this is necessary. In Section 2, we shall prove the following result.

Theorem 1.5. *Let G be a 2-connected planar graph and let $\mathscr{C}$ be a cycle double cover of G such that no proper nonempty subset of $\mathscr{C}$ forms a cycle double cover of a subgraph of G. If there is a cycle C in $\mathscr{C}$ and an embedding ϕ of G in the sphere such that C is not a face boundary of ϕ, then some other cycle C' of $\mathscr{C}$ is such that $C \cap C'$ has two nonadjacent vertices.*

Theorem 1.5 is a generalization of Theorem 1.1, in that the face boundaries of a 2-representative embedding of a 2-connected graph form a cycle double cover satisfying the hypotheses of Theorem 1.5. If G is 3-connected (which is enough to prove Theorem 1.1), then the two cycles C and C' quickly yield the necessary noncontractible cycle that meets G in only 2 points.

2. Proof of Theorem 1.5

Proof of Theorem 1.5. Let $G, \mathscr{C}, C$ and ϕ be as in the hypothesis of Theorem 1.5. In the embedding ϕ of G, the curve $\phi(C)$ partitions $\phi(G)$ into the part inside $\phi(C)$, $\phi(C)$ itself and the part outside $\phi(C)$. Suppose, first, that no cycle of $\mathscr{C}$ has an edge inside $\phi(C)$ and an edge outside $\phi(C)$. Then each cycle in $\mathscr{C}$ either is inside or on $\phi(C)$ or is outside or on $\phi(C)$. Let $\mathscr{C}_I$ denote the subset of $\mathscr{C}$ consisting of the inside cycles and let $\mathscr{C}_O$ contain the outside cycles. Thus, $\mathscr{C}$ is the disjoint union of $\mathscr{C}_I$, $\mathscr{C}_O$ and $\{C\}$.

Consider the symmetric difference of the cycles in $\mathscr{C}_I$. This must be a subset of C, and, therefore, is either C or empty. The same is true for the symmetric difference of the cycles in $\mathscr{C}_O$. Since $\mathscr{C}$ is a cycle double cover containing C, exactly one of the symmetric differences is C and one is empty. Without loss of generality, we can assume that the symmetric difference of the cycles in $\mathscr{C}_I$ is empty. But this is a contradiction: $\mathscr{C}_I$ is a cycle double cover of a proper nonempty subgraph of G.

It follows that some cycle C' of $\mathscr{C}$ has an edge $e_I = v_I w_I$ inside $\phi(C)$ and an edge $e_O = v_O w_O$ outside. Let the labelling be chosen so that $C' - \{e_I, e_O\}$ consists of two paths, one joining v_I to v_O and the other joining the w's. Each of these paths, travelling from the inside vertex, meets C for the first time, say at v and w. If vw is an edge of $C \cap C'$, then the subgraph of C' consisting of the edges vw and e_I, together with the two subpaths used to locate v and w is a cycle that does not contain e_O. This is impossible. □

For a cycle double cover $\mathscr{C}$, the *dual graph* is the graph with vertex set $\mathscr{C}$ and two cycles in $\mathscr{C}$ are joined if they have a common edge. The hypothesis of Theorem 1.5 that no subset of $\mathscr{C}$ be a double cover of a proper nonempty subgraph of G is equivalent to the connection of the dual of $\mathscr{C}$.

It is not clear to what extent this hypothesis in Theorem 1.5 is necessary. Suppose four 3-connected graphs are pieced together in a 'K_4-like' way as illustrated in Fig. 1. A cycle double cover for the resulting graph can be obtained from the face boundaries of each of the four graphs separately. These are not the face boundaries of the planar embedding of their union, yet no two cycles of the cover intersect in nonadjacent vertices. In this case, the dual has four components. We know of no 3-connected planar graph G with a cycle double cover $\mathscr{C}$ such that the dual of $\mathscr{C}$ has fewer than four components and no two cycles in $\mathscr{C}$ intersect in nonadjacent vertices.

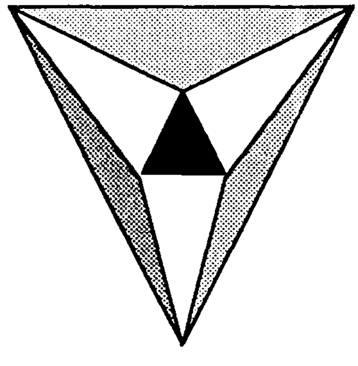


Fig. 1.

3. Proof of Theorem 1.2

Theorem 1.2. *Every 3-connected planar graph has a circular embedding in some surface other than the sphere.*

Proof. The following result is standard and forms the core of our arguments.

Lemma 3.1. *If C and C' are two face boundaries of an embedding of a 3-connected graph in the sphere, then $C \cap C'$ is empty, a single vertex or an edge with its two ends.*

Let G be a 3-connected planar graph. By standard arguments, some face of G in an embedding in the sphere has length at most 5. We consider two possibilities.

Case 1: Some face has length 3. Let e_i, $i=1,2,3$, be the edges and let v_i, $i=1,2,3$, be the vertices incident with a face of length 3, so that, modulo 3, e_i is incident with v_i and v_{i+1}. Let C_i be the other face boundary containing e_i, $i=1,2,3$.

Lemma 3.1 implies that, again taking indices modulo 3, v_{i+2} is not a vertex of C_i. This implies that $C_i' = (C_i - e_i) \cup P_i$ is a cycle, where P_i is the path of length 2 joining v_i and v_{i+1} through v_{i+2}. We exhibit a new embedding of G.

Replace the triangular face and the three faces bounded by the C_i with three faces bounded by the C_i'. This amounts to putting a crosscap in the middle of the triangular face and putting each of the e_i through the crosscap so as to switch their orders in each of the rotations around the v_i. If follows that G has a circular embedding in the real projective plane. *End of case 1.*

Case 2: There is no face of length 3. Then, for k either 4 or 5, there is a face of length k.

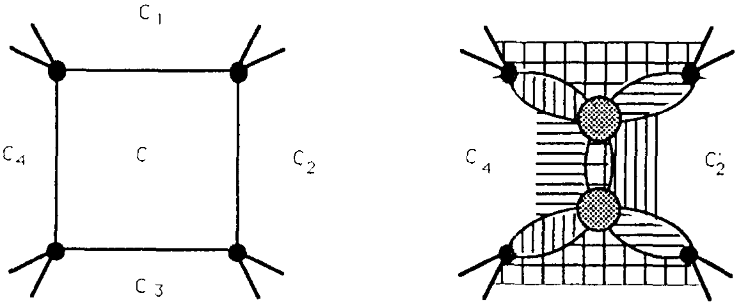
Let C be a cycle of length k bounding a face and let its edges in order be $e_1, \ldots, e_k$, with e_i again being incident with v_i and v_{i+1}. (Of course the indices are to be read modulo k.) Let C_i be the cycle bounding the other face incident with e_i. We now prove that at most one of $C_i \cap C_{i+2}$ and $C_{i+1} \cap C_{i+3}$ is nonempty.

For ease of notation, assume $i=1$ and that both $C_1 \cap C_3$ and $C_2 \cap C_4$ are nonempty. If $C_1 \cap C_3$ consists of an edge with its ends v and w, choose the labelling so that, in $C_1 - e_1$, v is nearer v_1 than w. This implies that v is nearer v_4 than w is in $C_3 - e_3$. If $C_1 \cap C_3$ is a single vertex, let this vertex be labelled with both v and w.

Let P_1 be the path in $C_1 - e_1$ joining v_1 to v and let P_2 be the path joining v_2 to w. Similarly, in $C_3 - e_3$, P_3 joins v_3 to w and P_4 joins v_4 to v. Let $\bar{C}_1 = P_2 \cup P_3 + e_2$ and let $\bar{C}_2 = P_1 \cup P_4 \cup P$, where P is the path (v_1, v_2, v_3, v_4). Evidently, both $\bar{C}_1$ and $\bar{C}_2$ separate the faces bounded by C_2 and C_4.

Let x be a vertex in $C_2 \cap C_4$. Then x must be in both $\bar{C}_1$ and $\bar{C}_2$, and is not one of the v_i. This implies that $v = w = x$. Therefore, v is common to C_1 and C_2, so that, by Lemma 3.1, vv_2 is an edge of C_2. Similarly, vv_3 is an edge of C_2, so that C_2 is a triangle, the required contradiction.

In the case $k=4$, we can conclude that at least one of $C_1 \cap C_3$ and $C_2 \cap C_4$ is empty. We assume, without loss of generality, that it is the former. It follows that



(a) (b)

Fig. 2.

$\hat{C}=(C_1-e_1)\cup(C_3-e_3)+\{e_2,e_4\}$ is a cycle. Also a cycle is C'_2 obtained from C_2 by replacing e_2 with the path (v_2,v_1,v_4,v_3). Similarly, replacing e_4 with the path (v_4,v_3,v_2,v_1) turns C_4 into a cycle C'_4. Replacing C and the C_i with three faces bounded by $\hat{C}$, C'_2 and C'_4 produces a circular embedding of G in the Klein bottle. This is illustrated in Fig. 2, where Fig. 2(a) is the neighbourhood of the face bounded by C in the plane and Fig. 2(b) has modified this by the addition of two crosscaps and a redrawing of the four edges e_i.

In the case $k=5$, suppose there are two $C_i\cap C_{i+2}$ that are nonempty. Then they must have one index in common, so they are, say, $C_3\cap C_5$ and $C_5\cap C_2$. In this case, we can conclude that $C_2\cap C_4$ and $C_1\cap C_3$ are both empty. If at most one $C_i\cap C_{i+2}$ is nonempty, then we can assume without loss of generality that any such involves C_5. Thus, again we have that $C_2\cap C_4$ and $C_1\cap C_3$ are empty. Thus, in every case we can assume that $C_2\cap C_4$ and $C_1\cap C_3$ are empty.

For $1\leqslant i<j\leqslant 5$, let $P(i,j)$ denote the path $(v_i, v_{i+1},\ldots,v_j)$ and let $P(j,i)$ denote the path $(v_j,v_{j+1},\ldots,v_5,v_1,\ldots,v_i)$. Let $C'_1=(C_1-e_1)\cup(C_3-e_3)\cup P(4,1)\cup P(2,3)$, $C'_2=(C_2-e_2)\cup(C_4-e_4)\cup P(5,2)\cup P(3,4)$ and $C'_3=(C_5-e_5)\cup P(1,5)$. It is verified readily that these are all cycles in G and, replacing C and C_i with the three C'_i yields a new embedding of G in the sphere with 3 crosscaps. This embedding is obtained from the original planar embedding by adding three crosscaps and redrawing the edges $e_1,\ldots,e_5$ as illustrated in Fig. 3. □

4. Planar graphs with 2-representative non-spherical embeddings

In this section we characterize completely those planar graphs that have a 2-representative embedding in some surface other than the sphere.

We begin by noting that standard arguments about the representativity of an embedding (as given in [1]) show that if ϕ is a 2-representative embedding of a graph

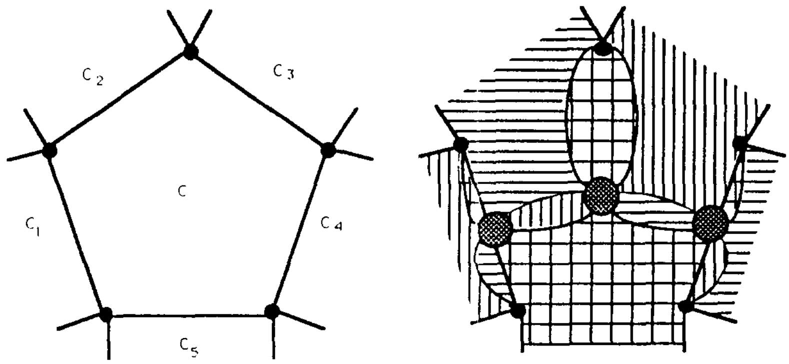


Fig. 3.

G in some surface Σ, then there is a block B of G such that the induced embedding $\phi(B)$ in Σ is also 2-representative. The other blocks are drawn in a planar way in the faces of $\phi(B)$. The consequence is that if G is planar and not 2-connected, then G has a 2-representative embedding in some surface if and only if some block of G has such an embedding. Thus, we may assume G is 2-connected.

The structure of 2-connected graphs has been described by Tutte [3]. Every 2-connected graph has a unique decomposition into what we shall call 3-blocks. A 3-block is either a 3-connected graph, or a cycle of length at least 3 or a bond with at least 3 edges. (A *bond* is a graph on 2 vertices such that every edge has distinct ends.) The idea behind the decomposition is to find a 2-separation (H, K) in the graph, i.e. a pair of subgraphs H and K such that $G=H\cup K$, $H\cap K$ consists of two isolated vertices v and w and each of H and K have at least two edges. We then split G into the two graphs $H+vw$ and $K+vw$. Repeat with these new graphs until there is no 2-separation.

The new edge vw must be added each time; it is the *virtual edge* of the 2-separation. After we have decomposed, we wish to avoid the situation where two cycles (or two bonds) share a virtual edge. If C_1 and C_2 are two cycles among the resulting graphs, and they have a virtual edge e in common, replace these two cycles with the single cycle $(C_1\cup C_2)-e$. Similarly for two bonds sharing a common virtual edge. Eliminating all such occurrences produces the desired list of 3-blocks.

Suppose (H, K) is a 2-separation of a graph G, with H and K having the vertices v and w in common. If $H+vw$ and $K+vw$ each have circular embeddings, then so does G. Such an embedding of G can be obtained by gluing the two embeddings together along the virtual edge vw and then deleting vw. If either of the surfaces involved is not the sphere, then the embedding of G will not be in the sphere.

The following fact is immediate from Theorem 1.2 and the above remarks.

Theorem 4.1. *Let G by a 2-connected planar graph such that some 3-block is a 3-connected graph. Then G has a 2-representative embedding in some surface other than the sphere.*

There are some 2-connected graphs which have no 2-representative embeddings. The cycles and bonds are examples. (Every cycle double cover comes from the face boundaries of some planar embedding.) On the other hand, we have the following fact.

Lemma 4.2. *If G is obtained from a cycle of length at least* 3 *by doubling every edge, then G has a* 2*-representative embedding.*

Proof. Let G be obtained from the cycle of length n by doubling every edge, so G has n vertices and $2n$ edges. Let the cycle have, in order, the edges $e_1, \ldots, e_n$ and let the two copies of e_i in G be denoted e_i^1 and e_i^2. We distinguish two cases.

Case 1: n is even. Let C_1 be the cycle consisting of the edges e_i^1 and let C_2 be made up of the e_i^2. Let C_3 be the cycle with edges e_i^1 if i is odd and e_i^2 if i is even. Finally, let C_4 have the e_i^1 if i is even and the e_i^2 if i is odd. We claim these four cycles are the face boundaries of an embedding of G.

To see this, let each bound a disc and identify the disc boundaries as indicated by the edges of G. At the vertex v_i of G incident with the four edges e_i^j and e_{i+1}^j, $j = 1, 2$, these discs give the rotation $(e_i^1, e_{i+1}^1, e_i^2, e_{i+1}^2)$, so that the topological space made up from the identified discs is a surface, as claimed. (The surface in this case is orientable with $(n-2)/2$ handles.)

Case 2: n is odd. Let C_1 be the cycle consisting of the edges e_i^1 and let C_2 be the cycle made up of e_1^1 and the edges e_i^2, $i = 2, \ldots, n$. Let C_3 contain the edges e_i^2 if i is odd and e_i^1 when i is even. Finally, C_4 has e_1^2, e_i^1 if i is odd and $i > 1$ and e_i^2 if i is even. As in case 1, these cycles form the face boundaries of an embedding in some surface. (In this case, the surface is nonorientable with $n-2$ crosscaps.) □

Observe that the 3-blocks of the graph obtained by doubling every edge of the n-cycle are an n-cycle and n bonds, each bond having 3 edges. Every edge of the n-cycle is a virtual edge. We are now prepared for the final characterization.

Theorem 4.3. *A* 2*-connected planar graph G has a* 2*-representative embedding if and only if either some* 3*-block is* 3*-connected or some* 3*-block is a cycle of virtual edges.*

Proof. The 'if' direction follows immediately from Theorem 4.1, Lemma 4.2 and the observation that if no 3-block is 3-connected then every virtual edge joins a cycle and a bond.

For the 'only if', let G be a 2-connected planar graph such that no 3-block is 3-connected and no 3-block is a cycle with only virtual edges. We shall show that G has no 2-representative embedding.

Suppose, to the contrary, that some such G has a 2-representative embedding. Choose G to have the minimum number of edges. Then every 3-block is either a bond or a cycle. As G cannot be either a bond or a cycle, G has more than one 3-block.

Let T be the graph whose vertices are the 3-blocks of G and the edges of T are the virtual edges, each of which joins the two 3-blocks containing it. Then T is a tree. Let B be a 3-block that is a leaf vertex of T, containing the virtual edge e.

If B were a cycle, then the graph obtained from G by replacing the path $B-e$ with e would have fewer edges ($B-e$ must have length at least 2). It would also have a 2-representative embedding, and so be a smaller counterexample. Hence, B is a bond.

Let C be the other 3-block containing the virtual edge e. Then C is a cycle and so some edge f of C is not a virtual edge. Consider the 2-representative embedding of G. There are two face boundaries, M and N, which contain the edge f. These are cycles of G. As every cycle of G through f must contain an edge of $B-e$, this is true in particular of M and N. Also, any cycle of G through an edge of $B-e$ either is a digon in $B-e$ or contains f. It follows that the cycles M and N use different edges in $B-e$ and that the other boundary cycles through $B-e$ are all digons. Therefore, we can obtain a 2-representative embedding (in the same surface) of the graph obtained from G by deleting all but one edge of $B-e$ from G. This would be a smaller counterexample, which is the desired contradiction. □

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