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INTERVAL MATROIDS AND GRAPHS

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A base of the cycle space of a binary matroid M on E is said to be convex if its elements can be totally ordered in such a way that for every $e \in E$ the set of elements of the base containing e is an interval. We show that a binary matroid is cographic iff it has a convex base of cycles; equivalently, graphic matroids can be represented as "interval matroids" (matroids associated in a natural way to interval systems). As a consequence, we obtain characterizations of planar graphs and cubic cyclically-4-edge-connected planar graphs in terms of convex bases of cycles.

1. Definitions

The definitions not given here will be found in [1] or [2]. Let E be a finite non-empty set; $\mathcal{P}(E)$ will be considered as a vector space over $GF(2)$ (the addition being the symmetric difference of subsets). A *binary matroid on* E will be defined here as a pair $M=(E,\mathcal{C})$ where $\mathcal{C}$ is a subspace of $\mathcal{P}(E)$; the elements of $\mathcal{C}$ are the *cycles* of M and $\mathcal{C}$ is called the *cycle space of* M; a *base of cycles of* M is a base of $\mathcal{C}$; a *base of* M is a maximal (with respect to inclusion) subset of E containing no non-empty cycle of M. Let B be a base of M; $\forall e \in E-B$, $B\cup\{e\}$ contains an unique non-empty cycle C_e of M; $\{C_e \mid e\in E-B\}$ is a base of cycles of M; any base of cycles of M which can be constructed in this way will be called a *fundamental base of cycles of* M.

Two elements of $\mathcal{P}(E)$ will be said to be *orthogonal* if the cardinality of their intersection is even. Given a subspace $\mathcal{C}$ of $\mathcal{P}(E)$, the set of elements of $\mathcal{P}(E)$ orthogonal to every element of $\mathcal{C}$ is a subspace of $\mathcal{P}(E)$ denoted by $\mathcal{C}^\perp$; let us recall that $(\mathcal{C}^\perp)^\perp=\mathcal{C}$ and that $\mathcal{C}$ and $\mathcal{C}^\perp$ are called *orthogonal subspaces* of $\mathcal{P}(E)$; the associated binary matroids $M=(E,\mathcal{C})$ and $M^*=(E,\mathcal{C}^\perp)$ are said to be *dual matroids* (M is the *dual of* M^* and conversely); the cycles of M^* are called *cocycles* of M.

Let H be a graph with vertex-set V and edge set E; $\forall S\subsetneq V$, let $\omega_H(S)$ denote the set of edges of H with exactly one end in S. A *cycle of* H is a subset C of E such that:

$$\forall v\in V \quad |C\cap\omega_H(\{v\})| \equiv 0 \pmod 2).$$

A *cocycle of H* is any subset of E of the form $\omega_H(S)$ $(S \subseteq V)$.
(Note that our definition of a cocycle is different of that given in [2] since we consider as cocycles not only the minimal cutsets, but all edge-disjoint unions of minimal cutsets).

The set $\mathscr{C}$ of cycles of H and the set $\mathscr{K}$ of cocycles of H are orthogonal subspaces of $\mathscr{P}(E)$; $M = (E, \mathscr{C})$ is the *cycle matroid of H*, and its dual $M^* = (E, \mathscr{K})$ is the *cocycle matroid* of H.

A binary matroid will be said *graphic* (respectively: *cographic*) if it is isomorphic to the cycle matroid (respectively: cocycle matroid) of some graph.

A graph is *planar* if and only if its cycle matroid is cographic (Whitney's Duality Theorem).

2. Cographic matroids, interval matroids and convex bases of cycles

2.1. *Interval matroids*

$\mathbf{N}$ denotes the set of integers.

An interval of $\mathbf{N}$ is a set of the form: $\{n \in \mathbf{N} \mid p \leq n \leq q\}$ for some $p, q \in \mathbf{N}$.

An *interval system* is a finite non-empty family $(I_e, e \in E)$ of intervals of $\mathbf{N}$.

A *cycle* of the interval system $(I_e, e \in E)$ is a subset C of E such that: $\forall n \in \mathbf{N}$, $|\{e \in C \mid n \in I_e\}| \equiv 0 \pmod 2$.

The set $\mathscr{C}$ of cycles of $(I_e, e \in E)$ is a subspace of $\mathscr{P}(E)$, and the associated binary matroid $M = (E, \mathscr{C})$ is the *cycle matroid* of $(I_e, e \in E)$.

A binary matroid will be called an *interval matroid* if it is isomorphic to the cycle matroid of some interval system.

2.2. *A characterization of cographic matroids*

Let $M = (E, \mathscr{C})$ be a binary matroid. A subset $\mathscr{S}$ of $\mathscr{C}$ will be said to be convex if there exists an injection σ from $\mathscr{S}$ to $\mathbf{N}$ such that: $\forall e \in E$, $\{\sigma(C) \mid C \in \mathscr{S}, e \in C\}$ is an interval of $\mathbf{N}$; it is clear that every subset of a convex subset of $\mathscr{C}$ is also convex.

Proposition 1. *Let M be a binary matroid. The following conditions are equivalent:*
(a) *M is cographic.*
(b) *M has a convex base of cycles.*
(c) *The dual of M is an interval matroid.*

Proof. (a)$\Rightarrow$(b). Let H be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge-set E; let $M = (E, \mathscr{K})$ be the cocycle matroid of H; we shall construct a convex base of $\mathscr{K}$. $\forall i \in \{1, \ldots, n-1\}$ let $V_i = \{v_1, \ldots, v_i\}$ and $K_i = \omega_H(V_i)$. Let $e \in E$. If e is a loop, e belongs to no K_i. Otherwise let v_k and v_l $(k<l)$ be the two ends of e; then:

$$e\in K_i \Leftrightarrow v_k\in V_i,\ v_l\notin V_i \Leftrightarrow k\leqslant i<l.$$

Hence $\{K_i \mid i\in\{1,\dots,n-1\}$ is a convex subset of $\mathscr{K}$. Moreover:

$$\omega_H(\{v_1\})=K_1;\qquad \omega_H(\{v_i\})=K_{i-1}+K_i\quad \forall i\in\{2,\dots,n-1\},$$

since $\{\omega_H(\{v_i\})\mid i\in\{1,\dots,n-1\}\}$ spans $\mathscr{K}$, it follows that $\{K_i\mid i\in\{1,\dots,n-1\}\}$ spans $\mathscr{K}$. Hence $\{K_i\mid i\in\{1,\dots,n-1\}\}$ contains a base of $\mathscr{K}$, and this is a convex base of cycles of M.

(b)$\Rightarrow$(c). Let $M=(E,\mathscr{C})$ be a binary matroid with a convex base of cycles $\{C_i\mid i\in\{1,\dots,r\}\}$; we may assume that: $\forall e\in E, \{i\in\{1,\dots,r\}\mid e\in C_i\}$ is an interval I_e of $\mathbf{N}$.

Then for a given $i\in\{1,\dots,r\}$ and $K\subseteq E$:

$$K\cap C_i=\{e\in K\mid e\in C_i\}=\{e\in K\mid i\in I_e\}.$$

Hence, for a given $K\subseteq E$:

$$\forall i\in\{1,\dots,r\},\quad |K\cap C_i|\equiv 0 \pmod 2$$

if and only if

$$\forall i\in\{1,\dots,r\},\quad |\{e\in K\mid i\in I_e\}|\equiv 0 \pmod 2.$$

This means that the set of cocycles of M is identical to the set of cycles of the interval system $(I_e, e\in E)$, or, equivalently, that the dual of M is the cycle matroid of the interval system $(I_e, e\in E)$.

(c)$\Rightarrow$(a). Let $(I_e, e\in E)$ be an interval system and $M=(E,\mathscr{C})$ be its cycle matroid; we must show that the dual of M is cographic; clearly we may assume that $\bigcup_{e\in E} I_e$ is of the form $\{1,\dots,r\}$ and that $I_e\neq\emptyset$ $\forall e\in E$. $\forall e\in E$ we shall denote by p_e and q_e the elements of $\{1,\dots,r\}$ such that $I_e=\{n\in\mathbf{N}\mid p_e\leqslant n\leqslant q_e\}$.

Let us construct a graph H as follows. H has $r+1$ vertices $v_1,\dots,v_{r+1}$; for every element e of E we shall define an edge $\phi(e)$ of H with ends v_{p_e}, v_{q_e+1}. $\forall i\in\{1,\dots,r\}$ let $V_i=\{v_1,\dots,v_i\}$. Then:

$$\begin{aligned}\{e\in E\mid i\in I_e\}&=\{e\in E\mid p_e\leqslant i\leqslant q_e\}\\ &=\{e\in E\mid v_{p_e}\in V_i,\ v_{q_e+1}\notin V_i\}\\ &=\{e\in E\mid \phi(e)\in\omega_H(V_i)\}\qquad \forall i\in\{1,\dots,r\}.\end{aligned}$$

or

$$\phi(\{e\in E\mid i\in I_e\})=\omega_H(V_i)\quad \forall i\in\{1,\dots,r\}.$$

We have seen in the first part of the proof that $\{\omega_H(V_i)\mid i\in\{1,\dots,r\}\}$ is a spanning subset of the cocycle space of H.

Moreover, $\{\{e\in E\mid i\in I_e\}\mid i\in\{1,\dots,r\}\}$ is a spanning subset of the cocycle space of the interval system $(I_e, e\in E)$.

It follows that ϕ defines an isomorphism from the cocycle space of $(I_e, e \in E)$ to the cocycle space of H; hence the dual of M is isomorphic to the cocycle matroid of H.

This completes the proof.

Remark. The equivalence of (a) and (c) of Proposition 1 can be formulated as follows: A binary matroid is an interval matroid if and only if it is graphic.

3. Some characterizations of planar graphs

3.1. *General planar graphs*

Proposition 2. *A graph is planar if and only if it has a convex base of cycles.*

This is a direct consequence of Proposition 1 together with Whitney's duality theorem.

3.2. *Cubic cyclically-4-edge-connected planar graphs*

A cubic graph H is said to be *cyclically-4-edge-connected* if: $\forall S \subset V(H)$, $S \neq \emptyset$: $|\omega_H(S)| \geq 3$ with equality if and only if $|S| = 1$ or $|S| = |V(H)| - 1$.

Proposition 3. *A cyclically-4-edge-connected cubic graph is planar if and only if it has a convex fundamental base of cycles.*

Proof. Let H be a cyclically-4-edge-connected cubic graph.

(a) If H has a convex fundamental base of cycles, H is planar by Proposition 2.

(b) Conversely, if H is planar, let us assume that H is represented in the plane and that H^* is a geometric dual of H (see [3, Chapter 3]). The cycle matroid of H is isomorphic to the cocycle matroid of H^*. Moreover it is easy to prove that H^* is a triangulation of the plane with the following properties:

- H has no loops and no multiple edges;
- every triangle of H^* is the boundary of some face of H^*.

By a theorem of H. Whitney ([4], extended by W.T. Tutte in [5]), H^* contains an Hamiltonian cycle; hence H^* contains an Hamiltonian chain K. This means that the vertices of H^* can be labelled $v_1, \ldots, v_n$ in such a way that each edge of K has two consecutive ends v_i, v_{i+1} $(i \in \{1, \ldots, n-1\})$.

$$\forall i \in \{1, \ldots, n-1\} \quad \text{let} \quad V_i = \{v_1, \ldots, v_i\} \quad \text{and} \quad K_i = \omega_{H^*}(V_i).$$

It will be easily seen that $\{K_i \mid i \in \{1, \ldots, n-1\}\}$ is a fundamental base of cocycles of H^* and that it is convex (see the first part of the proof of Proposition 1).

Hence, by duality, H has a convex fundamental base of cycles.

3.3. *A remark on the Four-Color Theorem*

It is known that a loopless graph H is vertex-colorable with 4 colors iff $E(H)$ is the union of two cocycles of H (see for instance [6]).

For planar graphs the dual statement is as follows: a planar bridgeless graph H (represented in the plane) is face-colorable with 4 colors iff $E(H)$ is the union of two cycles of H.

Remarks. If H is cubic, $E(H)$ is the union of two cycles of H iff H is edge-colorable with 3 colors ([6]).

It is well-known that the four-Color Theorem ([7]) is equivalent to the following: every cyclically-4-edge-connected planar cubic graph is edge-colorable with 3 colors.

Using Propositions 2 and 3 it is now easy to show that the following statements are equivalent to the Four-Color Theorem:

(a) For every bridgeless graph H with a convex base of cycles, $E(H)$ is the union of two cycles of H.

(b) For every bridgeless graph H with a convex fundamental base of cycles, $E(H)$ is the union of two cycles of H.

(c) Every bridgeless cubic graph with a convex fundamental base of cycles is edge-3-colorable.

Remark. It is shown in [8] and [9] that for every bridgeless graph H, $E(H)$ is the union of 3 cycles of H.

Suppose that these 3 cycles form a convex subset of the cycle space of H (we shall say that they form a *convex* 3-*cycle cover of* H). Assume that the 3 cycles are C_1, C_2, C_3 with:

$$e \in C_1, e \in C_3 \quad \Rightarrow \quad e \in C_2.$$

It is then easy to check that $C_2 \cup (C_1 + C_3) = E(H)$.

Conversely, any set of two cycles the union of which is $E(H)$ becomes by addition of the empty cycle a convex 3-cycle cover of H.

Hence the statement: $E(H)$ is the union of two cycles of H can be replaced in (a) and (b) by: H has a convex 3-cycle cover.

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