Gelfand–Ponomarev and Herrmann constructions for quadruples and sextuples

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Abstract

The notions of a perfect element and an admissible element of the free modular lattice $D^r$ generated by $r \geq 1$ elements are introduced by Gelfand and Ponomarev in [I.M. Gelfand, V.A. Ponomarev, Free modular lattices and their representations, Collection of articles dedicated to the memory of Ivan Georgievic Petrovskii (1901–1973), IV. Uspehi Mat. Nauk 29 (6(180)) (1974) 3–58 (Russian); English translation: Russian Math. Surv. 29 (6) (1974) 1–56]. We recall that an element $a \in D$ of a modular lattice $L$ is perfect, if for each finite-dimension indecomposable $K$-linear representation $\rho_X : L \to \mathcal{L}(X)$ over any field $K$, the image $\rho_X(a) \subseteq X$ of $a$ is either zero, or $\rho_X(a) = X$, where $\mathcal{L}(X)$ is the lattice of all vector $K$-subspaces of $X$.

A complete classification of such elements in the lattice $D^4$ associated to the extended Dynkin diagram $\tilde{D}_4$ (and also in $D^r$, where $r > 4$) is given in [I.M. Gelfand, V.A. Ponomarev, Free modular lattices and their representations, Collection of articles dedicated to the memory of Ivan Georgievic Petrovskii (1901–1973), IV. Uspehi Mat. Nauk 29 (6(180)) (1974) 3–58 (Russian); English translation: Russian Math. Surv. 29 (6) (1974) 1–56; I.M. Gelfand, V.A. Ponomarev, Lattices, representations, and their related algebras, I. Uspehi Mat. Nauk 31 (5(191)) (1976) 67–85; I.M. Gelfand, V.A. Ponomarev, Lattices, representations, and their related algebras, II. Uspehi Mat. Nauk 32 (1(193)) (1977) 85–106 (Russian); English translation: Russian Math. Surv. 32 (1) (1977) 91–114]. The main aim of the present paper is to classify all the admissible elements and all the perfect elements in the Dedekind lattice $D^{2,2,2}$ generated by six elements that are associated to the extended Dynkin diagram $\tilde{E}_6$. We recall that in [I.M. Gelfand, V.A. Ponomarev, Free modular lattices and their representations, Collection of articles dedicated to the memory of Ivan Georgievic Petrovskii (1901–1973), IV. Uspehi Mat. Nauk 29 (6(180)) (1974) 3–58 (Russian); English translation: Russian Math. Surv. 29 (6) (1974) 1–56], Gelfand and Ponomarev construct admissible elements of the lattice $D^r$ recurrently. We suggest a direct method for creating admissible elements. Using this method we also construct admissible elements for $D^4$ and show that these elements coincide modulo linear equivalence with admissible elements constructed by Gelfand and Ponomarev. Admissible sequences and admissible elements for $D^{2,2,2}$ (resp. $D^4$) form 14 classes (resp. 8 classes) and possess some periodicity.

Our classification of perfect elements for $D^{2,2,2}$ is based on the description of admissible elements. The constructed set $H^+$ of perfect elements is the union of 64-element distributive lattices $H^+(n)$, and $H^+$ is the distributive lattice itself. The lattice of perfect elements $B^+$ obtained by Gelfand and Ponomarev for $D^4$ can be imbedded into the lattice of perfect elements $H^+$, associated with $D^{2,2,2}$.
Herrmann in [C. Herrmann, Rahmen und erzeugende Quadrupel in modularen Verbänden. (German) [Frames and generating quadruples in modular lattices], Algebra Universalis 14 (3) (1982) 357–387] constructed perfect elements $s_n, l_n, p_{i,n}$ in $D^4$ by means of some endomorphisms $\gamma_{i,j}$ and showed that these perfect elements coincide with the Gelfand–Ponomarev perfect elements modulo linear equivalence. We show that the admissible elements in $D^4$ are also obtained by means of Herrmann’s endomorphisms $\gamma_{i,j}$, Herrmann’s endomorphism $\gamma_{i,j}$ and the elementary map of Gelfand–Ponomarev $\varphi_i$ act, in a sense, in opposite directions, namely the endomorphism $\gamma_{i,j}$ adds the index to the beginning of the admissible sequence, and the elementary map $\varphi_i$ adds the index to the end of the admissible sequence.

1. Introduction

... He took me aside and explained to me why it is important to study the structure consisting of four subspaces of a vector space $V$... Thus, all invariants of matrices... are encoded in the invariant theory of four subspaces. It must be added that the invariant theory of four subspaces is more general than the invariant theory of operators. The problem is that of explicitly describing the lattice of subspaces generated by four subspaces in general position... The lattice generated by four subspaces in general position is infinite, but nevertheless it should be explicitly described in some sense or other.

G.-C. Rota, Ten mathematics problems I will never solve, [44, p. 10, Problem nine]

1.1. Motivation

This text is a continuation of the Gelfand–Ponomarev theorems from [17,18] on quadruples of subspaces. In 1970s Gelfand was very enthusiastic about this topic: “Many tame problems of Linear Algebra can be reduced to the classification of systems”[17, p. 163], see epigraph. I believe that the main pathos of Gelfand’s idea is correct. The classification problem of sextuples of subspaces, or rather triples of pairs of subspaces $X_i \subseteq Y_i$ for $i = 1, 2, 3$, considered in this work, is not really encoded by quadruples of subspaces, but we see many common concepts.

The starting point for the study of sextuples of subspaces, was my observation that the collection of subspaces encoded by the quiver $\tilde{E}_6$ is the simplest one after $\tilde{D}_4$ among the extended Dynkin diagrams, and with a symmetry similar to that of $\tilde{D}_4$. I had a hunch that the Gelfand–Ponomarev theorems on quadruples of subspaces (i.e., on representations of $\tilde{D}_4$) and perfect elements should also work for the representations of $\tilde{E}_6$. This is, indeed, the case and while studying the works of Gelfand and Ponomarev and, in particular, thinking about $\tilde{E}_6$, I have understood that there are finitely many constructing blocks. These are series of admissible elements, which are given in this work first for sextuples and next for quadruples of subspaces.

Gelfand and Ponomarev constructed admissible elements for quadruples $\tilde{D}_4$ recurrently [18], and this obscured the fact that they form only a finite supply of series. Gelfand and Ponomarev observed that the admissible elements (for quadruples) are enumerated by so-called admissible sequences — a numerical code which resembles, to an extent, Young diagrams in representation theory. Here, I construct admissible sequences for sextuples and demonstrate that they can be expressed in terms of a finite number of periodic series.

In a sense, admissible sequences describe the complexity of the corresponding lattice polynomial. The greater the complexity of the admissible polynomial (the more additions and intersections it contains), the longer is its admissible sequence. At a certain moment the admissible sequence starts to repeat itself but with an extra complication: we get a near periodicity of admissible sequences. To observe this periodicity on the admissible polynomials themselves is an incomparably more difficult problem. Essentially, the periodicity and the fact that there are only finitely many distinct series of admissible sequences follow from the very simple fundamental rules as follows.

1 Here, G.-C. Rota narrates about his meeting with Gelfand in the seventies, when he (Gelfand) received an honorary degree from Harvard.

2 The first version of this work was put in arXiv under the title “Admissible and perfect elements in modular lattices”. The definitions of notions admissible and perfect will be given shortly.

3 Meaning systems of subspaces in a finite-dimensional space.
Sextuples. Any admissible sequence is constructed from three symbols \{1, 2, 3\}. Any symbol of the admissible sequence lying between two identical symbols maybe changed into the third lacking symbol, namely:

\[ i ji = iki \quad \text{for all } \{i, j, k\} = \{1, 2, 3\}, \]

see Section 2.10.

Quadruples. Any admissible sequence is constructed from four symbols \{1, 2, 3, 4\}. Any symbol of the admissible sequence lying between two different symbols maybe changed to the fourth lacking symbol, namely:

\[ i jk = ilk \quad \text{for all } \{i, j, k, l\} = \{1, 2, 3, 4\}, \]

see Section 4.6.

Conjecturally, the finite number of basic series of admissible sequences can also be constructed for modular lattices connected with other extended Dynkin diagrams \(\tilde{E}_7\) and \(\tilde{E}_8\), namely for modular lattices \(D_{1,3,3}\) and \(D_{1,2,5}\).

1.2. Modular lattices, linear equivalence, perfect elements

In this section we briefly recall a number of notions.

1.2.1. Linear representations of modular lattices

A lattice \(L\) is a set with two commutative and associative operations: a sum and an intersection. If \(a, b \in L\), then we denote the intersection by \(ab\) and the sum by \(a + b\). Both operations are idempotent

\[ aa = a, \quad a + a = a \]

and satisfy the absorption law

\[ a(a + b) = a, \quad a + ab = a. \]

On the lattice \(L\), an inclusion \(\subseteq\) is defined:

\[ a \subseteq b \iff ab = a \text{ or } a + b = b, \]

see Appendix A.1.

A lattice is said to be modular if, for every \(b, a, c \in L\),

\[ a \subseteq b \implies b(a + c) = a + bc, \quad \text{(1.1)} \]

see Appendix A.1.

Good examples of modular lattices, bringing us some intuition, are the lattices of subspaces of a given vector space, normal subgroups of a given group, ideals of a given ring.

We denote by \(D_r\) the free modular lattice with \(r\) generators \(D_r = \{e_1, \ldots, e_r\}\).

\[ \text{(1.2)} \]

see [18,15]. The word “free” means that no relation exists between generators \(e_i, i = 1, \ldots, r\).

Denote by \(D^{2,2,2}\) the modular lattice with 6 generators \(\{x_1, y_1, x_2, y_2, x_3, y_3\}\) satisfying the following relations:

\[ D^{2,2,2} = \{x_1 \subseteq y_1, x_2 \subseteq y_2, x_3 \subseteq y_3\}. \quad \text{(1.3)} \]

Symbol \(D\) in (1.2) and (1.3) is used in honour of R. Dedekind who derived the main properties of modular lattices, [11]. The modular law (1.1) is also known as Dedekind’s law and modular lattices are sometimes called the Dedekind lattices, see Appendix A.1.

Assume that \(K\) is a field and \(L\) a modular lattice. By a \(K\)-linear representation of \(L\) in a vector \(K\)-space \(X\) of finite \(K\)-dimension we mean a morphism

\[ \rho_X : L \rightarrow \mathcal{L}(X) \]

\[ \text{In Appendix A we give definitions and examples of some notions including modular lattices and their representations.} \]
of modular lattices, where \( \mathcal{L}(X) \) is the lattice of all vector \( K \)-linear subspaces of \( X \). The finite-dimensional \( K \)-linear representations of \( L \) form an additive \( K \)-category \( \text{rep}_K(L) \) with direct sum and morphisms of representations defined in an obvious way, see Appendix A.2.1.

### 1.2.2. Linear equivalence and \( p \)-linear equivalence

We say that the elements \( a, b \in L \) are \( K \)-linearly equivalent, if \( \rho_X(a) = \rho_X(b) \), for each indecomposable \( K \)-linear representation \( \rho_X : L \to \mathcal{L}(X) \). In this case we write

\[
a \equiv b \mod \theta_K, \quad \text{or} \quad a \equiv^K b. \tag{1.4}
\]

Let the variety (see Appendix A.1.4) \( V_K \) be generated by all lattices \( \mathcal{L}(X) \) of subspaces of \( K \)-vector spaces \( X \), with \( X \) finite-dimensional. The variety \( V_K \) depends only on the characteristic \( p \) of the field, namely:

\[
V_K = V_{K'} \quad \text{if and only if} \quad K \text{ and } K' \text{ have the same characteristic } p, \tag{1.5}
\]

see [25, Theorem 4]. We write

\[
V_K = V_p, \quad \text{and} \quad \theta_K = \theta_p.
\]

**Definition 1.2.1.** We say that elements \( a \) and \( b \) are \( p \)-linearly equivalent if \( \rho_X(a) = \rho_X(b) \) for each indecomposable \( K \)-linear representation \( \rho_X : L \to \mathcal{L}(X) \) over a field \( K \) of characteristic \( p \). In this case, we write

\[
a \equiv b \mod \theta_p, \quad \text{or} \quad a \equiv^p b,
\]

and the corresponding quotient lattice is denoted by \( L/\theta_p \).

**Definition 1.2.2.** We say that the elements \( a, b \in L \) are linearly equivalent, if \( \rho_X(a) = \rho_X(b) \), for each indecomposable \( K \)-linear representation \( \rho_X : L \to \mathcal{L}(X) \) for any field \( K \). In this case we write

\[
a \equiv b \mod \theta, \quad \text{or} \quad a \equiv \theta b,
\]

and the corresponding quotient lattice is denoted by \( L/\theta \).

Clearly, elements \( a \) and \( b \) which are from the same linear equivalence class, are also from the same \( p \)-linear equivalence class.

Let \( P' \) (resp. \( P'' \)) be the family of all distinct indecomposable preprojective representations of the lattice \( D' \) to vector spaces over the field \( K \) (resp. over the field \( \mathbb{Q} \) of rational numbers). Gelfand, Lidsky and Ponomarev in [22] consider the preprojective equivalence \( \tau_\rho \), such that

\[
a \equiv b \mod \tau_\rho \quad \text{if and only if} \quad \rho(a) = \rho(b)
\]

for each \( \rho \in P_k' \) (resp. \( P'' \)). The quotient lattice \( D'/\tau_\rho \) is of interest to authors of [22].

### 1.2.3. Perfect elements and \( p \)-perfect elements

We will distinguish notions of perfect elements in \( L \) and \( p \)-perfect elements in \( L \).

**Definition 1.2.3** ([18]). The element \( a \in L \) in a modular lattice \( L \) is said to be perfect, if for each indecomposable \( K \)-linear representation \( \rho_X : L \to \mathcal{L}(X) \) over any field \( K \), the image \( \rho_X(a) \subseteq X \) of \( a \) is either zero, or \( \rho_X(a) = X \).

**Definition 1.2.4** ([13]). The element \( a \in L \) in a modular lattice \( L \) is said to be \( p \)-perfect, if for each indecomposable \( K \)-linear representation \( \rho_X : L \to \mathcal{L}(X) \) over a field \( K \) of characteristic \( p \), the image \( \rho_X(a) \subseteq X \) of \( a \) is either zero, or \( \rho_X(a) = X \).

Obviously, the sum and intersection of perfect elements is also a perfect element. Thus, perfect (resp. \( p \)-perfect) elements form a sublattice in the lattice \( L \). In [13, Theorem 1] it was shown that for a given characteristic there are at most 16 perfect elements in \( D^4/\theta_\rho \) which do not belong to the cubicles.
In this work we consider modular lattices $D^4$, $D^{2.2.2}$. Some results for these lattices are proved only up to the linear equivalence $\theta$, i.e., these results are true only in $D^4/\theta$ (resp. $D^{2.2.2}/\theta$). The perfect elements are studied in $D^4/\theta$ (resp. $D^{2.2.2}/\theta$).

Ponomarev considered in detail perfect elements for lattices related to the Dynkin diagram $A_n$ [41]. For every Dynkin diagram, he described perfect elements by means of so-called hereditary subsets in the corresponding Auslander–Reiten diagram, [41, Theorem 1.6].

Cylke [10] has constructed a sublattice $B^+(S) \cup B^-(S)$ in the modular lattice $L(S)$ freely generated by a poset $S$ of finite type, or tame type of finite growth. The sublattice $B^+(S) \cup B^-(S)$ is a natural generalization of the sublattices $B^+ \cup B^-$ of perfect elements constructed by Gelfand–Ponomarev [18]. The sublattice $B^+(S) \cup B^-(S)$ is built by means of poset differentiation. Nazarova and Roiter used this technique to construct indecomposable representations of posets and quivers [38,37,43,24].

1.3. Outline of admissible elements

First, in this section, we outline the idea of admissible elements due to Gelfand and Ponomarev [18]. In [18,19], these elements were used to obtain the distributive sublattice of perfect polynomials.\(^5\) We think today, that apart from being helpful in construction of perfect elements, the admissible elements are interesting in themselves.

Further, we outline properties of admissible elements obtained in this work: finite classification, $\varphi$-homomorphism, reduction to atomic elements and periodicity, see Section 1.3.2. In Section 1.3.4 we outline the connection between admissible elements and Herrmann’s endomorphisms used in [27] for the construction of perfect elements.

1.3.1. The idea of admissible elements of Gelfand and Ponomarev

Here, we describe, omitting some details, the idea of constructing the elementary maps and admissible elements of Gelfand–Ponomarev. As we will see below, the admissible elements grow, in a sense, from generators of the lattice or from the lattice’s unity.

Following Bernstein, Gelfand and Ponomarev [2], given a modular lattice $L$ and a field $K$, we have defined a $K$-linear endofunctor\(^6\)

$$\Phi^+ : \text{rep}_K L \longrightarrow \text{rep}_K L.$$  

see Appendix A.2.6 for details. Given a representation $\rho_X : L \longrightarrow \mathcal{L}(X)$ in $\text{rep}_K L$, we denote by

$$\rho_{X^+} : L \longrightarrow \mathcal{L}(X^+)$$

the image representation $\Phi^+$ in $\text{rep}_K L$ under the functor $\Phi^+$.

Construction of the elementary map $\varphi$. Let there exist a map $\varphi$ mapping every subspace $A \in \mathcal{L}(X^+)$ to some subspace $B \in \mathcal{L}(X)$:

$$\varphi : \mathcal{L}(X^+) \longrightarrow \mathcal{L}(X),$$

$$A \overset{\varphi}{\longmapsto} \varphi(A) = B, \quad \text{or} \quad A \overset{\varphi}{\longmapsto} B. \quad \text{(1.6)}$$

It turns out that for many elements $a \in L$ there exists an element $b \in L$ (at least in $L = D^4$ and $D^{2.2.2}$) such that

$$\varphi(\rho_{X^+}(a)) = \rho_X(b) \quad \text{(1.7)}$$

for every pair of representations $(\rho_X, \rho_{X^+})$, where $\rho_{X^+} = \Phi^+ \rho_X$. The map $\varphi$ in (1.7) is constructed in such a way, that $\varphi$ does not depend on $\rho_X$. Thus, we can write

$$a \overset{\varphi}{\longmapsto} b. \quad \text{(1.8)}$$

In particular, Eqs. (1.7) and (1.8) are true for admissible elements, whose definition will be given later, see Section 1.7. Eq. (1.7) is the most important property characterizing the admissible elements.

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\(^5\) Terms ‘lattice element’ and ‘lattice polynomial’ are used synonymously.

\(^6\) A functor from a category $C$ to itself is called an endofunctor on $C$. 
For $D^4$, Gelfand and Ponomarev constructed 4 maps of form (1.6): $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and called them \textit{elementary maps}. In this work, for $D^{2,2,2}$, we construct 3 maps $\varphi_i$, one per chain $x_i \subseteq y_i$, (see Sections 1.6 and 2). Given a sequence of relations

\[ a_1 \overset{\varphi_i}{\longrightarrow} a_2 \overset{\varphi_i}{\longrightarrow} a_3 \overset{\varphi_i}{\longrightarrow} \cdots \overset{\varphi_i}{\longrightarrow} a_{n-1} \overset{\varphi_i}{\longrightarrow} a_n, \]

we say that elements $a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$ grow from the element $a_1$. By abuse of language, we say that elements growing from generators $e_i$ or unity $I$ are \textit{admissible elements}, and the corresponding sequence of indices $\{i, i-1, i-2, \ldots, 2\}$ is said to be the \textit{admissible sequence}. For details of the cases $D^{2,2,2}$ (resp. $D^4$) see Section 1.7.1, (resp. Section 1.7.2).

\subsection*{1.3.2. Reduction of the admissible elements to atomic elements}

Here, we give a brief description of the steps which lead to admissible elements in this work.

\textbf{(A) Finite classification.} The admissible elements (and admissible sequences) can be reduced to a finite number of classes, see Section 1.7. Admissible sequences for $D^{2,2,2}$ (resp. $D^4$) are depicted in Fig. 1.3 (resp. Fig. 1.4). The key property of admissible sequences allowing to do this classification is the following relation between maps $\varphi_i$.

\textbf{For $D^{2,2,2}$ (see Proposition 2.10.2):}

\[ ij i = ik i, \]

\[ \varphi_i \varphi_j \varphi_i = \varphi_i \varphi_k \varphi_i, \quad \text{where } \{i, j, k\} = \{1, 2, 3\}. \]

\textbf{For $D^4$ (see Proposition 4.6.1):}

\[ ik j = il j, \]

\[ \varphi_i \varphi_k \varphi_j = \varphi_i \varphi_l \varphi_j, \quad \text{where } \{i, j, k, l\} = \{1, 2, 3, 4\}. \]

\textbf{(B) $\varphi_i$-homomorphism.} In a sense, the elementary maps $\varphi_i$ are homomorphic with respect to admissible elements. More exactly, we introduce a notion of $\varphi_i$-homomorphic elements. Let

\[ a \overset{\varphi_i}{\rightarrow} \tilde{a} \quad \text{and} \quad p \overset{\varphi_i}{\rightarrow} \tilde{p}. \]

The element $a \in L$ is said to be $\varphi_i$-homomorphic, if

\[ ap \overset{\varphi_i}{\rightarrow} \tilde{a} \tilde{p} \]

for all $p \in L$, see Section 1.6.

\textbf{(C) Reduction to atomic elements.} We permanently apply the following mechanism of creation of admissible elements from more simple elements. Let $\varphi_i$ be any elementary map, and

\[ a \overset{\varphi_i}{\rightarrow} \tilde{a}, \quad b \overset{\varphi_i}{\rightarrow} \tilde{b}, \quad c \overset{\varphi_i}{\rightarrow} \tilde{c}, \quad p \overset{\varphi_i}{\rightarrow} \tilde{p}. \]

Suppose $abc p$ is any admissible element and $a, b, c$ are $\varphi_i$-homomorphic. By means of relation (1.11) we construct a new admissible element

\[ \tilde{a} \tilde{b} \tilde{c} \tilde{p}. \]

The elements $a, b, c$ are called \textit{atomic}, see Section 1.5.

\textbf{(D) Periodicity.} Admissible sequences obtained by methods as in (A) possess some \textit{periodicity}. The corresponding admissible elements are also periodic.

\textbf{Case $D^{2,2,2}$.} In Fig. 1.1 we see an example of admissible elements for $D^{2,2,2}$:

\[ e_{(213)}(21)^{\gamma}, \quad e_{3(213)}(21)^{\gamma}, \quad e_{13(213)}(21)^{\gamma}. \]
Vertical lines\(^7\) in Fig. 1.1 correspond to two types of inclusions of admissible elements:

\[
\begin{align*}
\cdots \subseteq e_{(13)^{y+1}(21)} & \subseteq e_{(13)^{y}(21)} \subseteq \cdots \subseteq e_{(13)(21)} \subseteq e_{(21)}, \\
\cdots \subseteq e_{3(13)^{y+1}(21)} & \subseteq e_{3(13)^{y}(21)} \subseteq \cdots \subseteq e_{3(13)(21)} \subseteq e_{3(21)}, \\
\cdots \subseteq e_{13(13)^{y+1}(21)} & \subseteq e_{13(13)^{y}(21)} \subseteq \cdots \subseteq e_{13(13)(21)} \subseteq e_{13(21)} \subseteq e_{1}, \\
\text{and} & \\
\cdots \subseteq e_{1(21)} & \subseteq e_{1(21)^{y-1}} \subseteq \cdots \subseteq e_{1(21)^{2}} \subseteq e_{1(21)} \subseteq e_{1}, \\
\cdots \subseteq e_{(21)^{y}} & \subseteq e_{(21)^{y-1}} \subseteq \cdots \subseteq e_{(21)^{3}} \subseteq e_{(21)^{2}} \subseteq e_{21}.
\end{align*}
\]

(1.13)

Inclusions (1.13) are easily obtained from Table B.2. From the same table we can see periodicity of indices of atomic elements entering in the decomposition of admissible elements. Two cylindrical helices in Fig. 1.1 correspond to relations

\[
\begin{align*}
e_{(21)^{y}}(21)^{y} \xrightarrow{\varphi_{3}} e_{3(13)^{y}(21)}^{y}, \\
e_{3(13)^{y}(21)}^{y} \xrightarrow{\varphi_{3}} e_{13(13)^{y}(21)}^{y}, \\
e_{13(13)^{y}(21)}^{y} \xrightarrow{\varphi_{3}} e_{13(21)^{y}(21)}^{y}, \\
e_{1(21)^{y}} \xrightarrow{\varphi_{2}} e_{(21)^{y+1}}, \\
e_{(21)^{y}} \xrightarrow{\varphi_{1}} e_{1(21)^{y}}.
\end{align*}
\]

(1.14)

Relations (1.14) are particular cases of the main theorem on admissible elements (Theorem 2.12.1).

Case \(D^{4}\). In Fig. 1.2 we see some examples of admissible elements for \(D^{4}\). Vertical lines on three cylinders in Fig. 1.2 correspond to six series of inclusions of admissible elements:

\[
\begin{align*}
\cdots \subseteq e_{1(41)^{y}} & \subseteq e_{1(41)^{y-1}} \subseteq \cdots \subseteq e_{1(41)} \subseteq e_{1}, \\
\cdots \subseteq e_{(41)^{y}} & \subseteq e_{(41)^{y-1}} \subseteq \cdots \subseteq e_{(41)}, \\
\cdots \subseteq e_{1(31)^{y}(41)} & \subseteq e_{1(31)^{y-1}(41)} \subseteq \cdots \subseteq e_{1(31)(41)} \subseteq e_{1(41)}, \\
\cdots \subseteq e_{3(31)^{y}(41)} & \subseteq e_{3(31)^{y-1}(41)} \subseteq \cdots \subseteq e_{3(31)(41)} \subseteq e_{3(41)}, \\
\cdots \subseteq e_{1(21)^{y}(31)(41)} & \subseteq e_{1(21)^{y-1}(31)(41)} \subseteq \cdots \subseteq e_{1(21)(31)(41)} \subseteq e_{1(31)(41)} \subseteq e_{1(31)^{y}(41)}, \\
\cdots \subseteq e_{(21)^{y}(31)(41)} & \subseteq e_{(21)^{y-1}(31)(41)} \subseteq \cdots \subseteq e_{(21)(31)(41)} \subseteq e_{(31)(41)} \subseteq e_{(31)^{y}(41)}.
\end{align*}
\]

(1.15)

\[\footnote{Note that the order relation in the diagrams in Figs. 1.1 and 1.2 is inverse to the standard use in Hasse diagrams.}\]
Fig. 1.2. Periodicity of admissible elements in $D_4$.

Inclusions (1.15) are obtained from Table 4.3. Three cylindrical helices in Fig. 1.2 correspond to relations

\[ e_{1(41)}^{r-1} \xrightarrow{\phi_1} e_{1(41)}^r, \quad e_{1(41)}^r \xrightarrow{\phi_2} e_{1(41)}^{r-1}, \]

\[ e_{1(31)}^{r-1} \xrightarrow{\phi_1} e_{1(31)}^r, \quad e_{1(31)}^r \xrightarrow{\phi_3} e_{1(31)}^{r-1}, \]

\[ e_{1(21)}^{r-1} \xrightarrow{\phi_1} e_{1(21)}^r, \quad e_{1(21)}^r \xrightarrow{\phi_2} e_{1(21)}^{r-1}. \]

Relation (1.16) is a particular case of Theorem 4.8.1.

1.3.3. Direct construction of admissible elements $D_4^4$

Admissible elements for $D_4^4$ in [18,19] are built recurrently in the length of multi-indices named admissible sequences. In this work we suggest a direct method for creating admissible elements. For $D_2^2, D_2^2$ (resp. $D_4^4$), the admissible sequences and admissible elements form 14 classes (resp. 8 classes) and possess some periodicity properties, see Section 1.7.1, Tables 2.2 and 2.3 (resp. Section 1.7.2, Tables 4.1 and 4.3).

For the definition of admissible polynomials due to Gelfand and Ponomarev and examples obtained from this definition, see Section 4.8.2.

1.3.4. Admissible elements in $D_4^4$ and Herrmann’s endomorphisms

C. Herrmann introduced in [27] the commuting endomorphisms $\gamma_{ik}$ ($i, k = 1, 2, 3$) in $D_4^4$ and used them in the construction of perfect elements, see Section 5.

It turned out, that endomorphisms $\gamma_{ik}$ are also closely connected with admissible elements. First of all, the endomorphism $\gamma_{ik}$ acts on the admissible element $e_{ak}$ such that
There is a similarity between the action of Herrmann’s endomorphism $\gamma_{ik}$ and the action of the elementary map of Gelfand–Ponomarev $\varphi_i$. The endomorphism $\gamma_{ik}$ and the elementary map $\varphi_i$ act, in a sense, in opposite directions, namely the endomorphism $\gamma_{ik}$ adds the index to the beginning of the admissible sequence (see Theorem 5.3.4), and the elementary map $\varphi_i$ adds the index to the end (see Theorem 4.8.1).

Further, every admissible element $e_\alpha$ (resp. $f_\alpha_0$) in $D^4$ has a unified form $e_1a_1^{34}a_2^{24}a_r^{32}$, see Table 5.1, and every admissible element $e_\alpha$ (resp. $f_\alpha_0$) is obtained by means of the sequence of Herrmann’s endomorphisms as follows:

\[
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_1) = e_1a_3^{34}a_t^{24}a_2^{32},
\]

and, respectively,

\[
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{10}) = e_1a_1^{34}a_3^{24}a_r^{32}(e_1a_1^{34} + a_2^{24}a_r^{32}),
\]

see Theorem 5.3.6.

1.4. An outline of the main results of the paper

Throughout the paper, we denote by $\tilde{E}_6$ the following extended Dynkin quiver

\[
\tilde{E}_6 :
\]

\[
\begin{array}{c}
3 \\
\downarrow \\
3' \\
\downarrow \\
1 \rightarrow 1' \rightarrow 0 \leftarrow 2' \leftarrow 2,
\end{array}
\]

that is, the extended Dynkin diagram $\tilde{E}_6$ with centrally oriented arrows. We view it as a partially ordered set in an obvious way. The Dedekind modular lattice $D^{2,2,2}$ is generated by the partially ordered set (1.3) and this corresponds to set $\tilde{E}_6$. It is denoted by $D^{2,2,2}$.

One of the main aims of the paper is to construct admissible and perfect elements for the modular Dedekind lattice $D^{2,2,2}$ associated with the extended Dynkin quiver $\tilde{E}_6$ (see Sections 2 and 3), and admissible elements for the modular lattice $D^4$ associated with the extended Dynkin quiver (see Sections 4 and 5)

\[
Q = \tilde{D}_4 :
\]

\[
\begin{array}{c}
2 \\
\downarrow \\
1 \rightarrow 0 \leftarrow 3 \\
\uparrow \\
4
\end{array}
\]

We show in Section 4.8.2 that the admissible elements for the modular lattice $D^4$ we construct in the paper and the admissible elements for $D^4$ constructed by Gelfand and Ponomarev [18] coincide, up to a linear equivalence, and we obtain the connection between the admissible elements in $D^4$ and Herrmann’s endomorphisms, used in his construction of perfect elements, see Section 5.

In our construction of perfect elements for the lattice $D^{2,2,2}$ an important role is played by various lattice polynomials in $D^{2,2,2}$. In particular, we use atomic, $\varphi$-homomorphic, admissible, cumulative and perfect polynomials. Atomic polynomials form a basis for constructing admissible elements. The cumulative polynomials are constructed
by means of admissible elements, and perfect elements are constructed from the cumulative elements:

\[
\text{Atomic} \quad \Rightarrow \quad \text{Admissible} \quad \Rightarrow \quad \text{Cumulative} \quad \Rightarrow \quad \text{Perfect}
\]

The atomic, admissible and cumulative lattice polynomials are defined in Section 2, and their basic properties are listed there. In particular, we list in Table 2.3 a complete list of admissible polynomials for \( D^{2.2.2} \), see also Section 1.7.

In Section 3, for each \( n \geq 0 \), we describe the sublattices \( H^+(n) \) of perfect elements in \( D^{2.2.2} \) generated by the three chains \( s_1(n) \), \( s_2(n) \), \( s_3(n) \) defined in (3.4) and (3.5). In particular, the cardinality of each \( H^+(n) \) is determined. We also show that the union

\[
H^+ = \bigcup_{n=0}^{\infty} H^+(n)
\]

is a distributive lattice of perfect elements in \( D^{2.2.2} / \theta \). We suppose (Conjecture 1.11.4) that the lattice \( H^+ \cup H^- \) contains all perfect elements of \( D^{2.2.2} / \theta_p \).

The union \( H^+ \) has an interesting architecture and contains distributive lattice \( B^+ \) of perfect elements of \( D^4 / \theta \), see Fig. 1.6, Section 1.11.1.

In Section 4, we construct admissible elements in \( D^4 \), by applying the technique developed in Section 2.

In Section 5 we study Herrmann’s endomorphisms used in his construction of perfect elements in \( D^4 \). We show how the admissible elements introduced in Section 4 are constructed by means of Herrmann’s endomorphisms. Herrmann’s endomorphism \( \gamma_{ij} \) and the elementary map of Gelfand–Ponomarev \( \varphi_i \) act, in a sense, in opposite directions, namely the elementary map \( \varphi_i \) adds the index to the end of the admissible sequence:

\[
\varphi_i \varphi_j (z_{\alpha}) = \varphi_j (z_{\alpha i}),
\]

see Theorem 4.8.1, and the endomorphism \( \gamma_{ik} \) adds the index to the beginning of the admissible sequence:

\[
\gamma_{ik} (z_{\alpha k}) = z_{\alpha k i},
\]

see Theorem 5.3.4.

Not only perfect elements in \( D^4 \), but also the admissible elements \( e_a \) and \( f_{a0} \) in \( D^4 \) can be elegantly obtained by Herrmann’s endomorphisms, see Theorem 5.3.6.

For the convenience of the reader, we list in Appendix A some properties of modular lattices and their representations. In Appendix B, we present proofs of several facts we use in the paper.

Part of the results presented in the present paper was announced in [49] and published in papers [51,52], to which the reader is also referred for some proofs that are not presented here.

### 1.5. Atomic polynomials

Atomic lattice polynomials have a simple lattice definition. They are called atomic because admissible, cumulative and perfect lattice polynomials (as well as invariant in [50]) are defined by means of these polynomials.

For \( D^{2.2.2} \), the definition of \( a_{ij}^n \), \( A_{ij}^n \), \( n \in \{0, 1, 2, \ldots\} \), is cyclic through the indices \( i, j, k \), where the triple \( \{i, j, k\} \) is a permutation of \( \{1, 2, 3\} \). We set

\[
a_{ij}^n = \begin{cases} a_{ij}^n = I & \text{for } n = 0, \\ a_{ij}^n = x_i + y_j a_{ij}^{n-1} & \text{for } n \geq 1. \end{cases}
\]

\[
A_{ij}^n = \begin{cases} A_{ij}^n = I & \text{for } n = 0, \\ A_{ij}^n = y_i + x_j A_{ij}^{n-1} & \text{for } n \geq 1. \end{cases}
\]

The elements \( a_{ij}^n \), \( A_{ij}^n \) are said to be atomic elements in \( D^{2.2.2} \);

Examples of atomic elements in \( D^{2.2.2} \).
\[ a_{12}^1 = x_1 + y_2, \quad A_{12}^1 = y_1 + x_2, \]
\[ a_{22}^1 = x_1 + y_2(x_2 + y_3), \quad A_{22}^1 = y_1 + x_2(y_3 + x_1), \]
\[ a_{22}^2 = x_1 + y_2(x_2 + y_3(x_3 + y_1)), \quad A_{22}^2 = y_1 + x_2(y_3 + x_1(y_2 + x_3)), \]
\[ a_{42}^1 = x_1 + y_2(x_2 + y_3(y_3 + y_1(x_1 + y_2))), \quad A_{42}^1 = y_1 + x_2(y_3 + x_1(y_2 + x_3(y_1 + x_2))), \]
\[ \ldots \]

For \( D^4 \), we define atomic lattice polynomials \( a_{n}^{ij} \), where \( i, j \in \{1, 2, 3, 4\} \), \( n \in \mathbb{Z}_+ \), as follows

\[
a_{n}^{ij} = \begin{cases} a_{n}^{ij} = I & \text{for } n = 0, \\ a_{n}^{ij} = e_i + e_j a_{n-1}^{k} & \text{for } n \geq 1, \end{cases}
\]

(1.22)

where \( \{i, j, k, l\} \) is a permutation of the quadruple \( \{1, 2, 3, 4\} \), and \( \{e_1, e_2, e_3, e_4\} \) are generators of \( D^4 \), see Section 4.2. Examples of the atomic elements in \( D^4 \):

\[
a_{12}^1 = e_1 + e_2, \quad a_{22}^1 = e_1 + e_2(e_3 + e_4), \quad a_{22}^2 = e_1 + e_2(e_3 + e_4(e_1 + e_2)), \quad a_{42}^1 = e_1 + e_2(e_2 + e_3(e_1 + e_2(e_3 + e_4))), \quad \ldots
\]

1.6. Coxeter functors and the elementary maps of Gelfand–Ponomarev for \( D^{2,2,2} \)

Following Bernstein–Gelfand–Ponomarev [2] and Gelfand–Ponomarev [18], we apply here the Coxeter endofunctor

\[
\Phi^+ : \text{rep}_K(D^{2,2,2}) \longrightarrow \text{rep}_K(D^{2,2,2})
\]

of the category \( \text{rep}_K(D^{2,2,2}) \), see Appendices A.2.4 and A.2.6. The functor \( \Phi^+ \) associates to the representation \( \rho_X : D^{2,2,2} \rightarrow \mathcal{L}(X) \) represented by the diagram

\[
\begin{array}{c}
X_3 \\
\downarrow \\
X_1 \leftrightarrow Y_1 \leftrightarrow X_0 \leftrightarrow Y_2 \leftrightarrow X_2,
\end{array}
\]

\[
\rho_X : \quad Y_3 \\
\downarrow \\
X_1 \\
\downarrow \\
\rho_X = \Phi^+ \rho_X : \quad Y_3^+ \\
\downarrow \\
X_1^+ \leftrightarrow Y_1^+ \leftrightarrow X_0^+ \leftrightarrow Y_2^+ \leftrightarrow X_2^+,
\]

with the representation vector space \( X := X_0 \), the representation \( \rho_X^+ = \Phi^+ \rho_X : D^{2,2,2} \rightarrow \mathcal{L}(X^+) \) represented by the diagram

\[
\begin{array}{c}
X_3^+ \\
\downarrow \\
X_1^+ \leftrightarrow Y_1^+ \leftrightarrow X_0^+ \leftrightarrow Y_2^+ \leftrightarrow X_2^+,
\end{array}
\]

Given \( n \geq 1 \), we define \( \rho_X^n = (\Phi^+)^n \rho_X : D^{2,2,2} \rightarrow \mathcal{L}(X^n) \) by the diagram

\[
\begin{array}{c}
X_3^n \\
\downarrow \\
X_1^n \leftrightarrow Y_1^n \leftrightarrow X_0^n \leftrightarrow Y_2^n \leftrightarrow X_2^n,
\end{array}
\]
where we set $X_0 = X$, $X_j^1 = X_j^+$, $X_j^n = (X_j^{n-1})^+$, for $j = 0, 1, 2, 3$, and $Y_j^1 = Y_j^+$, $Y_j^n = (Y_j^{n-1})^+$ for $j = 1, 2, 3$.

Throughout the work the maps $\varphi_i$ play a central role. For exact definition of the maps $\varphi_i$, see Section 2.3. Gelfand and Ponomarev introduced the maps $\varphi_i$ in [18, p. 27] for modular lattices $D'$ and called them elementarly maps.

The images of the generators $y_i \in x_i$, where $i = 1, 2, 3$ and the unit $I$ of the lattice $D_{2,2,2}^2$ under the representation $\rho_{X^n}$ are as follows:

\[ \rho_{X^n}(I) = X_0^n \] for $n \geq 1$, and $\rho_X(I) = X_0$.

\[ \rho_{X^n}(y_i) = Y_i^n \] for $n \geq 1$, and $\rho_X(y_i) = Y_i$,  where $i = 1, 2, 3$.

\[ \rho_{X^n}(x_i) = X_i^n \] for $n \geq 1$, and $\rho_X(x_i) = X_i$, where $i = 1, 2, 3$.

The elementary maps $\varphi_i$ operate from $X_0^1$ to the subspace $Y_1 \subseteq X_0$, see Section 2.3.

We introduce now $\varphi_i$-homomorphic polynomials playing an important role in our further considerations.

An element $a \in D_{2,2,2}^2$ is said to be $\varphi_i$-homomorphic, if

\[ \varphi_i(\rho_X(a)) = \varphi_i(\rho_X(a)) \] for all $p \in D_{2,2,2}^2$.

An element $a \in D_{2,2,2}^2$ is said to be $(\varphi_i, \gamma_k)$-homomorphic, if

\[ \varphi_i(\rho_X(a)) = \varphi_i(\rho_X(\gamma_k a)) \] for all $p \leq \gamma_k$.

Here $\rho_X(a)$, $\rho_X(p)$ are the images of $a$, $p \in D_{2,2,2}^2$ under representation $\rho_{X^n}$, see (1.23). For the detailed definition of $\varphi_i$ and related notions, see Sections 2.2 and 2.3. From now on, we will use notation $\varphi_i(\rho_X(a))$ instead of $\varphi_i(\rho_X(a))$.

The notion of $\varphi_i$-homomorphic elements in $D^3$ is similarly introduced in Section 4.7.

All atomic polynomials are $\varphi_i$-homomorphic. More exactly, for $D_{2,2,2}^2$, we have (see Theorem 2.8.1, Section 2.8):

1. The polynomials $a_{11}^j$ are $\varphi_i$ and $\varphi_j$-homomorphic.
2. The polynomials $A_{11}^j$ are $\varphi_i$-homomorphic.
3. The polynomials $A_{11}^j$ are $(\varphi_i, \gamma_k)$-homomorphic.

For $D^3$, we have (Theorem 4.7.1, Section 4.7):

1. The polynomials $a_{11}^j$ are $\varphi_i$-homomorphic.
2. The polynomials $a_{11}^j$ are $(\varphi_i, e_k)$-homomorphic.

1.7. Admissible sequences and admissible elements

1.7.1. Admissible sequences for $D_{2,2,2}^2$

The admissible lattice polynomials are introduced in Section 2.12. They appear when we use different maps $\varphi_i$, where $i = 1, 2, 3$. The admissible polynomials can be indexed by means of a finite number of types of index sequences, called admissible sequences. Consider a finite sequence of indices $s = i_n \ldots i_1$, where $i_p \in \{1, 2, 3\}$. The sequence $s$ is said to be admissible if

(a) Adjacent indices are distinct ($i_p \neq i_{p+1}$).
(b) In each subsequence $i j i$, we can replace index $j$ by $k$. In other words: $\ldots i j i \ldots = \ldots i k i \ldots$, where all indices $i, j, k$ are distinct.

The admissible sequence with $i_1 = 1$ for $D_{2,2,2}^2$ may be transformed to one of the seven types (Proposition 2.11.1, see Section 2.11),

\[
\begin{align*}
(1) & \ 2(312)^m(21)^n, & (2) & \ 3(213)^m(21)^n, & (3) & \ 13(213)^m(21)^n, \\
(4) & \ 3(12)^m(31)^n, & (5) & \ 2(312)^m(31)^n, & (6) & \ 12(312)^m(31)^n, \\
(7) & \ 1(21)^n = 1(31)^n. & & & & \ \\
\end{align*}
\]
Similarly for \( i_1 = 2, 3 \). For example, for \( i_1 = 2 \), the admissible sequences are:

(1) \((123)^m(12)^n\),  
(2) \(3(123)^m(12)^n\),  
(3) \(23(123)^m(12)^n\),  
(4) \((321)^m(32)^n\),  
(5) \(1(321)^m(32)^n\),  
(6) \(21(321)^m(32)^n\),  
(7) \(2(12)^n = 2(32)^n\). \hspace{1cm} (1.27)

A description of all admissible sequences for \( D^{2.2} \) is given in Table 2.2, and a description of all admissible elements is given in Table 2.3. For every

\[ z_\alpha = e_\alpha, f_\alpha, g_\alpha \]

from Table 2.3, where \( \alpha \) is an admissible sequence, the following statement takes place:

Let \( \alpha \) be an admissible sequence \( i_n \ldots i_1 \) and \( i \neq i_n \). Then \( i\alpha \) is also an admissible sequence and

\[ \varphi_i \rho_X(z_\alpha) = \rho_X(z_{i\alpha}) \]

for each representation \( \rho_X \) (Theorem 2.12.1, Section 2.12; see Fig. 1.3).

### 1.7.2. Admissible sequences for \( D^4 \)

The admissible sequence for \( D^4 \) is defined as follows. Consider a finite sequence of indices \( s = i_n \ldots i_1 \), where \( i_p \in \{1, 2, 3\} \). The sequence \( s \) is said to be admissible if

(a) Adjacent indices are distinct \( i_p \neq i_{p+1} \).
(b) In each subsequence \( ijl \), we can replace index \( j \) by \( k \). In other words: \( \ldots ijl \ldots = \ldots ikl \ldots \), where all indices \( i, j, k, l \) are distinct.

Any admissible sequence with \( i = 1 \) for \( D^4 \) may be transformed to one of the next 8 types (Proposition 4.1.3, Table 4.1):

(1) \((21)^i(41)^j(31)^k = (21)^i(31)^k(41)^j\),
(2) \((31)^i(41)^j(21)^k = (31)^i(21)^k(41)^j\),
(3) \((41)^i(31)^j(21)^k = (41)^i(21)^k(31)^j\),
(4) \((21)^i(41)^j(21)^k = 1(21)^i(41)^j(21)^k = 1(21)^i(21)^j(41)^k\),
(5) \((21)^i(31)^k(41)^j = 2(31)^{i+1}(41)^j^{i-1}(21)^i\),
(6) \((31)^i(21)^j(31)^k = 3(21)^{i+1}(41)^j^{i-1}(31)^i\),
(7) \((41)^i(31)^j(41)^k = 3(31)^{i+1}(21)^j^{i-1}(41)^i\),
(8) \((21)^i(41)^j(21)^k = (21)^i(21)^{i+1}(31)^{i-1} = (21)^i(21)^j(41)^k\) = \((13)^s(21)^{i+1}(41)^j^{i-1} = (12)^{i+1}(41)^j(31)^{i-1} = (12)^{i+1}(31)^{i-1}(41)^j\). \hspace{1cm} (1.28)

Let \( \alpha \) be an admissible sequence \( i_n \ldots i_1 \) and \( i \neq i_n \), where \( i_s \in \{1, 2, 3, 4\} \). Let \( z_\alpha = e_\alpha, f_\alpha, g_\alpha \) be admissible elements for \( D^4 \) from Table 4.3. Then \( i\alpha \) is also an admissible sequence and

\[ \varphi_i \rho_X(z_\alpha) = \rho_X(z_{i\alpha}) \]

for each representation \( \rho_X \). \hspace{1cm} (1.29)

see Theorem 4.8.1, Fig. 1.4. The admissible elements for \( D^4 \) form a finite family and are directly constructed, see Table 4.3.

Admissible polynomials given by Table 4.3 coincide in \( D^4/\theta \) (see Section 1.2.2) with polynomials constructed by Gelfand and Ponomarev. For small lengths of the admissible sequences, we prove that these polynomials coincide in \( D^4 \), see Propositions 4.8.3–4.8.5 from Section 4.8.2.

### 1.8. Inclusion of admissible elements in the modular lattice

One can easily check, with definitions from Table 2.3, that

\[ f_\alpha \subseteq e_\alpha, \quad g_\alpha \subseteq e_\alpha \]  \hspace{1cm} (1.30)
It is more difficult to prove the following theorem (see Theorem 2.13.1, Section 2.13, Appendix B):

**Theorem.** For every admissible sequence $\alpha$ and $\alpha_i$, where $i = 1, 2, 3$ from Table 2.2, we have

$$e_{\alpha_i} \subseteq g_{\alpha_0}.$$  

This proof uses case-by-case consideration of admissible sequences $\alpha$ from Table 2.3.
1.9. Examples of the admissible elements

1.9.1. The lattice $D^{2,2,2}$

Let $n$ be the length of an admissible sequence $\alpha$, let $k$ and $p$ be indices defined by Table 2.3. For $n = 1$: $\alpha = 1$, (Table 2.3, Line 7, $k = 0$). We have

$$e_1 = y_1, \quad f_1 = x_1 \subseteq e_1, \quad g_{10} = y_1(y_2 + y_3).$$

For $n = 2$: $\alpha = 21$, (Table 2.3, Line 9, $k = 0, p = 0$). Here,

$$e_1 = y_2a_1^{13} = y_2(x_1 + y_3),$$
$$e_2 = y_1(x_2 + y_3) \subseteq g_{10}, \quad \text{(permutation (21) \rightarrow (12)).}$$
$$f_{21} = y_2y_3 \subseteq e_21,$$
$$g_{210} = e_21(x_2 + A_1^{13}) = y_2(x_1 + y_3)(x_2 + x_3 + y_1).$$

For $n = 3$: $\alpha = 321$, (Table 2.3, Line 10, $k = 0, p = 0$). In this case:

$$e_{321} = y_3A_1^{12}A_1^{23} = y_3(y_1 + x_2)(y_2 + x_3),$$
$$e_{213} = y_2(y_3 + x_1)(x_1 + x_2) \subseteq g_{210}, \quad \text{(permutation (321) \rightarrow (213)).}$$
$$f_{321} = x_3A_1^{12} = x_3(y_1 + x_2) \subseteq e_{321},$$
$$g_{3210} = e_{321}(y_1y_3 + a_1^{12}) = y_3(y_1 + x_2)(y_2 + x_3)(x_1 + y_2 + y_1y_3).$$

By the modular law (A.1) of $D^{2,2,2}$ and since $x_1 \subseteq y_1$, we have

$$y_1(x_1 + y_3) = x_1 + y_1y_3 \quad \text{and}$$
$$e_{3213} = y_3(x_3 + y_2)(x_2 + y_1(x_1 + y_3)) \subseteq g_{3210}.$$

Further,

$$f_{1321} = y_1y_2a_1^{13} = y_1y_2(x_1 + y_3) \subseteq e_{1321}$$

and

$$g_{13210} = e_{1321}(x_1 + a_1^{32}A_1^{32})$$
$$= e_{1321}(x_1 + (x_3 + y_2)(y_3 + x_2))$$
$$= y_1(x_1 + y_3)(x_3 + y_2(x_2 + y_1))(x_1 + (x_3 + y_2)(y_3 + x_2)).$$

1.9.2. The lattice $D^4$

Admissible sequences and admissible elements are taken from Table 4.3. For $n = 1$: $\alpha = 1$, (Table 4.3, Line G11, $r = 0, s = 0, t = 0$). We have

$$e_1 = e_1 \quad \text{(admissible element $e_1$ coincides with generator $e_1$),}$$
$$f_{10} = e_1(e_2 + e_3 + e_4) \subseteq e_1.$$

For $n = 2$: $\alpha = 21$, (Table 4.3, Line G21, $r = 0, s = 0, t = 1$). We have

$$e_{21} = e_2(e_3 + e_4),$$
$$f_{210} = e_2(e_3 + e_4)(e_4 + e_3(e_1 + e_2) + e_1)$$
$$= e_2(e_3 + e_4)(e_4 + e_2(e_1 + e_3) + e_1)$$
$$= e_2(e_3 + e_4)(e_2(e_1 + e_4) + e_2(e_1 + e_3)) \subseteq e_{21}.$$

For $n = 3$: Consider two admissible sequences: $\alpha = 121$ and $\alpha = 321 = 341.$
(1) \( \alpha = 121 \), (Table 4.3, Line G11, \( r = 0, s = 0, t = 1 \)). We have
\[
e_{121} = e_1a_2^{34} = e_1(e_3 + e_4(e_1 + e_2)) \\
= e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)),
\]
\[
f_{1210} = e_{121}(e_1a_1^{32} + a_1^{24}a_1^{34}) \\
= e_{121}(e_1(e_2 + e_3) + (e_2 + e_4)(e_3 + e_4)) \\
= e_1(e_3 + e_4(e_1 + e_2))(e_1(e_2 + e_3) + (e_2 + e_4)(e_3 + e_4)) \subseteq e_{121}.
\]

(2) \( \alpha = 321 = 341 \), (Table 4.3, Line G31, \( t = 0, s = 0, r = 1 \)). We have
\[
e_{321} = e_{341} = e_3a_1^{21}a_1^{14} \\
= e_3(e_2 + e_1)(e_4 + e_1),
\]
\[
f_{3210} = f_{3410} = e_{321}(a_1^{12} + e_3a_1^{24}) \\
= e_{321}(e_1 + e_4(e_3 + e_2) + e_3(e_2 + e_4)) \\
= e_{321}(e_1 + (e_3 + e_2)(e_4 + e_2)(e_3 + e_4)) \\
= e_3(e_2 + e_1)(e_4 + e_1)(e_1 + (e_3 + e_2)(e_4 + e_2)(e_3 + e_4)) \subseteq e_{321}.
\]

For \( n = 4 \): \( \alpha = 2341 = 2321 = 2141 \), (Table 4.3, Line F21, \( s = 0, t = 1, r = 1 \)). We have
\[
e_{2141} = e_{241}a_1^{34} \\
= e_2(e_4 + e_1(e_3 + e_2))(e_3 + e_4),
\]
\[
f_{21410} = e_{2141}(a_2^{14} + e_1a_2^{24}) \\
= e_{2141}(e_2(e_4 + e_1(e_3 + e_2)) + e_1(e_2 + e_4(e_1 + e_3))) \subseteq e_{2141}.
\]

1.10. Cumulative polynomials

1.10.1. The lattice \( D^{2,2,2} \)

The cumulative polynomials \( x_t(n), y_t(n) \), where \( t = 1, 2, 3 \), and \( x_0(n) \) (all of length \( n \)) are sums of all admissible elements of the same length \( n \), where \( n \) is the length of the multi-index. In other words, cumulative polynomials are as follows

\[
x_t(n) = \sum f_{i_n...i_{2t}}, \quad t = 1, 2, 3,
\]
\[
y_t(n) = \sum e_{i_n...i_{2t}}, \quad t = 1, 2, 3,
\]
\[
x_0(n) = \sum g_{i_n...i_{20}}.
\]

From (1.30) and Theorem 2.13.1 we deduce that:
\[
x_t(n) \subseteq y_t(n) \subseteq x_0(n), \quad t = 1, 2, 3.
\]

The cumulative polynomials satisfy the same inclusions as the corresponding generators in \( D^{2,2,2} \).

Examples of the cumulative polynomials in \( D^{2,2,2} \).

For \( n = 1 \):
\[
x_1(1) = f_1 = x_1,
\]
\[
y_1(1) = e_1 = y_1,
\]
\[
x_0(1) = g_0 = I,
\]
i.e., the cumulative polynomials for \( n = 1 \) coincide with the generators of \( D^{2,2,2} \).

For \( n = 2 \): By the modular law (A.1) and since \( y_3(x_1 + y_2) \subseteq x_1 + y_3 \), we have
\[
x_1(2) = f_21 + f_31 = y_2y_3, \quad (f_{21} = f_{31} = y_2y_3),
\]
\[
y_1(2) = e_12 + e_31 = y_2(x_1 + y_3) + y_3(x_1 + y_2) \\
= (x_1 + y_3)(y_2 + y_3(x_1 + y_2)) = (x_1 + y_3)(y_2 + y_3)(x_1 + y_2),
\]
\[
x_0(2) = g_{10} + g_{20} + g_{30} = y_1(y_2 + y_3) + y_2(y_1 + y_3) + y_3(y_2 + y_3) \\
= (y_2 + y_3)(y_1 + y_3)(y_1 + y_2).
\]
For \( n = 3 \): (see Fig. 1.3)
\[
x_1(3) = f_{321} + f_{121} + f_{231},
\]
\[
y_1(3) = e_{321} + e_{121} + e_{231}.
\]

From Table 2.3 (Line 8, \( k = 0 \)) \( f_{121} = y_1(x_2 + x_3) \) and by Section 1.9.1 it follows that
\[
x_1(3) = x_3(y_1 + x_2) + y_1(x_2 + x_3) + x_2(y_1 + x_3)
= (x_2 + x_3)(y_1 + x_3).
\]

Again, from Table 2.3 (Line 8, \( k = 0 \)) \( e_{121} = y_1(x_2 + y_3)(y_2 + x_3) \) and by Section 1.9.1, we have
\[
y_1(3) = y_3(y_1 + x_2)(y_2 + x_3) + y_1(x_2 + y_3)(y_2 + x_3) + y_2(y_1 + x_3)(y_3 + x_2).
\]

Finally,
\[
x_0(3) = g_{210} + g_{310} + g_{320} + g_{120} + g_{230} + g_{130}
= \sum y_i(x_j + y_k)(y_j + x_i + x_k),
\]
where the sum runs over all permutations \( \{i, j, k\} \) of \( \{1, 2, 3\} \).

1.10.2. The lattice \( D^4 \)

According to Gelfand and Ponomarev we consider elements \( e_t(n) \) and \( f_0(n) \) used for construction of the perfect elements, [18, p. 7, p. 53].

The cumulative elements \( e_t(n) \), where \( t = 1, 2, 3, 4 \) and \( f_0(n) \) are sums of all admissible elements of the same length \( n \), where \( n \) is the length of the multi-index. So, for the case \( D^4 \), the cumulative polynomials are defined as follows:
\[
e_t(n) = \sum e_{i_1\ldots i_2t}, \quad t = 1, 2, 3, 4,
\]
\[
f_0(n) = \sum f_{i_1\ldots i_20},
\]
where admissible elements \( e_\alpha \) and \( f_{00} \) are defined in Table 4.3, see also Sections 1.7.2 and 1.9.2.

Examples of the cumulative polynomials in \( D^4 \).

For \( n = 1 \):
\[
e_1(1) = e_1,
\]
\[
f_0(1) = 1,
\]
i.e., the cumulative polynomials for \( n = 1 \) coincide with the generators of \( D^4 \).

For \( n = 2 \): We use here the following property:
\[
e_\alpha(e_1a_1^{34} + a_3^{24}a_t^{32}) = e_\alpha(a_t^{43} + e_2a_{s-1}^{41}a_{r+1}^{31}),
\]
see Lemma 4.8.2, heading (2). For the type G11 in Table 4.3, we have
\[
e_\alpha(e_1a_{2r+1}^{34} + a_{2r}^{24}a_2^{32}) = e_\alpha(a_{2r+1}^{43} + e_2a_2^{41}a_{2r}^{31}).
\]
Taking in (1.32), \( r = s = t = 0 \) we have
\[
f_{10} = e_1(a_1^{43} + e_2) = e_1(e_2 + e_3 + e_4).
\]

Thus,
\[
e_1(2) = e_2 + e_3 + e_4
\]
\[
= e_2(e_3 + e_4) + e_3(e_2 + e_4) + e_4(e_2 + e_3),
\]
\[
f_0(2) = f_{10} + f_{20} + f_{30} + f_{40}
\]
\[
= e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4) + e_4(e_1 + e_2 + e_3).
\]
For $n = 3$:
\[
\begin{align*}
    e_1(3) &= e_{321} + e_{231} + e_{341} + e_{431} + e_{241} + e_{421}, \\
    f_0(3) &= (f_{120} + f_{130} + f_{140}) + (f_{210} + f_{230} + f_{240}) + (f_{310} + f_{320} + f_{340}) + (f_{310} + f_{320} + f_{340}),
\end{align*}
\]
and so forth, see also Section 1.9.2.

1.11. Perfect elements

1.11.1. The sublattice $H^+(n)$ of the perfect elements in $D^{2,2,2}$

The perfect elements (see Appendix A.2) are constructed in the following way:

For $n = 0$:
\[
\begin{align*}
    a_0(0) &= y_j + y_k, \\
    b_0(0) &= x_i + y_j + y_k, \\
    c_1(0) &= c_2(0) = c_3(0) = \sum y_i.
\end{align*}
\]

For $n \geq 1$:
\[
\begin{align*}
    a_i(n) &= x_j(n) + x_k(n) + y_j(n+1) + y_k(n+1), \\
    b_i(n) &= a_i(n) + x_i(n+1) \\
    &= x_j(n) + x_k(n) + x_i(n+1) + y_j(n+1) + y_k(n+1), \\
    c_i(n) &= a_i(n) + y_i(n+1) \\
    &= x_j(n) + x_k(n) + y_i(n+1) + y_j(n+1) + y_k(n+1).
\end{align*}
\]

Proposition 1.11.1 (See Proposition 2.13.3, Section 2.12). If $z_t(n)$ is one of the cumulative polynomials $x_t(n)$, $y_t(n)$, where $t = 1, 2, 3$, or $x_0$, then
\[
\sum_{i=1,2,3} q_i \rho_X(z_t(n)) = \rho_X(z_t(n+1)).
\]

We will also show (Proposition 2.10.1) that the property of element $z$ to be perfect follows from the same property of element $u$ if
\[
\sum_{i=1,2,3} q_i \rho_X(z) = \rho_X(u).
\]

The elements $z_t(n)$ are not perfect, but perfect elements are expressed as sums of cumulative polynomials.

Proposition 1.11.2 (See Proposition 3.1.1, Section 3.1). The elements $a_i(n)$, $b_i(n)$, $c_i(n)$ are perfect for all $n$.

The elements $a_i(n) \subseteq b_i(n) \subseteq c_i(n)$, where $i = 1, 2, 3$, generate the 64-element distributive sublattice $H^+(n)$ of perfect elements for $n \geq 1$ (see Fig. 1.5) and the 27-element distributive sublattice $H^+(n)$ of perfect elements for $n = 0$. The sublattice $H^+(0)$ contains only 27 elements because $c_1(0) = c_2(0) = c_3(0)$.

1.11.2. Perfect cubes in the free modular lattice $D^r$

In [18], Gelfand and Ponomarev constructed the sublattice $B$ of perfect elements for the free modular lattice $D^r$ with $r$ generators:
\[
B = B^+ \bigcup B^-, \quad \text{where } B^+ = \bigcup_{n=1}^\infty B^+(n), \quad B^- = \bigcup_{n=1}^\infty B^-(n).
\]

They proved that every sublattice $B^+(n)$ (resp. $B^-(n)$) is $2^r$-element Boolean algebra, so-called Boolean cube (which can be also named perfect Boolean cube) and these cubes are ordered in the following way. Every element of the cube
Fig. 1.5. The 64-element distributive lattice $H^+(n)$ of perfect elements. Two generator systems: \{a_i \subseteq b_i \subseteq c_i\}, Section 1.11.1 and \{p_i \subseteq q_i \subseteq s_i\}, Section 3.2.2.

Fig. 1.6. Comparison between perfect sublattices in $D^4$ and $D^{2,2,2}$.

$B^+(n)$ is included in every element of the cube $B^+(n+1)$, see Fig. 1.6, i.e.,

\[
\begin{align*}
  v^+(n) & \in B^+(n) \\
  v^+(n+1) & \in B^+(n+1)
\end{align*}
\]
\implies v^+(n+1) \subseteq v^+(n).

By analogy, the dual relation holds:

\[
\begin{align*}
  v^-(n) & \in B^-(n) \\
  v^-(n+1) & \in B^-(n+1)
\end{align*}
\]
\implies v^-(n) \subseteq v^-(n+1).
1.11.3. Perfect elements in $D^4$

Perfect elements in $D^4$ (similarly, for $D^r$, see [18, p. 6]) are constructed by means of cumulative elements (1.31) as follows:

$$h_i(n) = \sum_{j \neq i} e_j(n).$$  \hspace{1cm} (1.35)

Elements (1.35) generate perfect Boolean cube $B^+(n)$ from Section 1.11.2, see [18].

Examples of perfect elements.

For $n = 1$:

$$h_1(1) = e_2 + e_3 + e_4, \quad h_2(1) = e_1 + e_3 + e_4,$$
$$h_3(1) = e_1 + e_2 + e_4, \quad h_4(1) = e_1 + e_2 + e_3.$$

For $n = 2$, by (1.33) we have

$$h_1(2) = e_2(2) + e_3(2) + e_4(2)$$
$$= (e_{12} + e_{32} + e_{42}) + (e_{13} + e_{23} + e_{43}) + (e_{14} + e_{24} + e_{34})$$
$$= e_1(e_3 + e_4) + e_3(e_1 + e_4) + e_4(e_1 + e_3)$$
$$+ e_1(e_2 + e_4) + e_2(e_1 + e_4) + e_4(e_1 + e_2)$$
$$+ e_1(e_2 + e_3) + e_2(e_1 + e_3) + e_3(e_1 + e_2)$$
$$= (e_1 + e_3)(e_1 + e_4)(e_3 + e_4) + (e_1 + e_2)(e_1 + e_4)(e_2 + e_4)$$
$$+ (e_1 + e_2)(e_1 + e_3)(e_2 + e_3).$$  \hspace{1cm} (1.37)

Let $h^{\text{max}}(n)$ (resp. $h^{\text{min}}(n)$) be the maximal (resp. minimal) element in the cube $B^+(n)$.

$$h^{\text{max}}(n) = \sum_{i=1,2,3,4} h_i(n),$$
$$h^{\text{min}}(n) = \bigcap_{i=1,2,3,4} h_i(n).$$  \hspace{1cm} (1.38)

1.11.4. Perfect elements in $D^4$ by C. Herrmann

C. Herrmann introduced in [27,29] polynomials $q_{ij}$ and associated endomorphisms $\gamma_{ij}$ of $D^4$ playing the central role in his construction of perfect polynomials. For $\{i, j, k, l\} = \{1, 2, 3, 4\}$, define

$$q_{ij} = q_{ji} = q_{kl} = q_{lk} = (e_i + e_j)(e_k + e_l),$$

and

$$\gamma_{ij} f(e_1, e_2, e_3, e_4) = f(e_1 q_{ij}, e_2 q_{ij}, e_3 q_{ij}, e_4 q_{ij}),$$

see Section 5.3.

Herrmann’s construction of perfect elements $s_n, t_n$ and $p_{i,n}$, where $i = 1, 2, 3, 4$, is as follows:

$$s_0 = I, \quad s_1 = e_1 + e_2 + e_3 + e_4,$$
$$s_{n+1} = \gamma_{12}(s_n) + \gamma_{13}(s_n) + \gamma_{14}(s_n),$$
$$t_0 = I, \quad t_1 = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4),$$
$$t_{n+1} = \gamma_{12}(t_n) + \gamma_{13}(t_n) + \gamma_{14}(t_n),$$
$$p_{i,0} = I, \quad p_{i,1} = e_i + t_1, \quad \text{where } i = 1, 2, 3, 4,$$
$$p_{i,n+1} = \gamma_{ij}(p_{j,n}) + \gamma_{ik}(p_{k,n}) + \gamma_{il}(p_{l,n}), \quad \text{where } i = 1, 2, 3, 4, \text{ and } \{i, j, k, l\} = \{1, 2, 3, 4\},$$
Table 1.1
Perfect elements in $B^+(n)$

<table>
<thead>
<tr>
<th>N</th>
<th>Gelfand–Ponomarev definition</th>
<th>Herrmann definition</th>
<th>Sum of cumulative polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sum_{i=1,2,3,4} h_i(n)$</td>
<td>$s_n$</td>
<td>$\sum_{i=1,2,3,4} e_i(n)$</td>
</tr>
<tr>
<td>2</td>
<td>$h_1(n)$</td>
<td>$p_{2,n} + p_{3,n} + p_{4,n}$</td>
<td>$e_2(n) + e_3(n) + e_4(n)$</td>
</tr>
<tr>
<td>3</td>
<td>$h_2(n)$</td>
<td>$p_{1,n} + p_{3,n} + p_{4,n}$</td>
<td>$e_1(n) + e_3(n) + e_4(n)$</td>
</tr>
<tr>
<td>4</td>
<td>$h_3(n)$</td>
<td>$p_{1,n} + p_{2,n} + p_{4,n}$</td>
<td>$e_1(n) + e_2(n) + e_4(n)$</td>
</tr>
<tr>
<td>5</td>
<td>$h_4(n)$</td>
<td>$p_{1,n} + p_{2,n} + p_{3,n}$</td>
<td>$e_1(n) + e_2(n) + e_3(n)$</td>
</tr>
<tr>
<td>6</td>
<td>$h_1(n)h_2(n)$</td>
<td>$p_{3,n} + p_{4,n}$</td>
<td>$e_2(n) + e_4(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>7</td>
<td>$h_1(n)h_3(n)$</td>
<td>$p_{2,n} + p_{4,n}$</td>
<td>$e_2(n) + e_4(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>8</td>
<td>$h_1(n)h_4(n)$</td>
<td>$p_{2,n} + p_{3,n}$</td>
<td>$e_2(n) + e_3(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>9</td>
<td>$h_2(n)h_3(n)$</td>
<td>$p_{1,n} + p_{4,n}$</td>
<td>$e_1(n) + e_4(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>10</td>
<td>$h_2(n)h_4(n)$</td>
<td>$p_{1,n} + p_{2,n}$</td>
<td>$e_1(n) + e_2(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>11</td>
<td>$h_3(n)h_4(n)$</td>
<td>$p_{1,n} + p_{3,n}$</td>
<td>$e_1(n) + e_2(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>12</td>
<td>$h_1(n)h_2(n)h_3(n)$</td>
<td>$p_{4,n}$</td>
<td>$e_4(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>13</td>
<td>$h_1(n)h_2(n)h_4(n)$</td>
<td>$p_{3,n}$</td>
<td>$e_3(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>14</td>
<td>$h_1(n)h_3(n)h_4(n)$</td>
<td>$p_{2,n}$</td>
<td>$e_2(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>15</td>
<td>$h_2(n)h_3(n)h_4(n)$</td>
<td>$p_{1,n}$</td>
<td>$e_1(n) + f_0(n + 1)$</td>
</tr>
<tr>
<td>16</td>
<td>$\cup_{i=1,2,3,4} h_i(n)$</td>
<td>$t_n$</td>
<td>$f_0(n + 1) = f_0(n + 1)$</td>
</tr>
</tbody>
</table>

see Section 5.4.2. Then,

$$s_n = \sum_{r+s+t=n-1} \gamma^1_2 \gamma^1_3 \gamma^1_4 \left( \sum_{i=1,2,3,4} e_i \right) = \sum_{i=1,2,3,4} e_i(n) \simeq h^\text{max}(n),$$

$$t_n = \sum_{r+s+t=n} \gamma^1_2 \gamma^1_3 \gamma^1_4 \left( \sum_{i=1,2,3,4} e_i(n) \right) = f_0(n + 1) \simeq h^\text{min}(n),$$

where $a \simeq b$ means modulo linear equivalence, see Theorems 5.4.3 and 5.4.4. Thus, the lattice of perfect elements generated by elements $s_n, t_n, p_{i,n}$ introduced by Herrmann [27] coincides modulo linear equivalence with the Gelfand–Ponomarev perfect elements, see Table 1.1.

Table 1.1 gives two systems of generators in the sublattice of the perfect elements $B^+(n)$ in $D^4$: coatoms $h_i(n)$ (by Gelfand–Ponomarev) vs. atoms $p_{i,n}$ (by C. Herrmann). Compare Table 1.1 with Table 3.1 and Proposition 3.2.5 describing the similar situation for the case $D^{2,2,2}$.

1.11.5. The union of sublattices $H^+(n)$ in the lattice $D^{2,2,2}/\theta$

Let us return to the lattice $D^{2,2,2}$. Consider the modular lattice $H^+$ generated by the union $\bigcup_{n=0}^{\infty} H^+(n) \mod \theta$ and the modular lattice $H^-$ generated by the union $\bigcup_{n=0}^{\infty} H^-(n) \mod \theta$, where $H^-(n)$ is the dual modular lattice for $H^+(n)$. The union $\bigcup_{n=0}^{\infty} H^+(n)$ is not organized as simple as $\bigcup_{n=0}^{\infty} B^+(n)$ in the case of $D^4$. Denote by $U_n$ the lower 8-element cube of $H^+(n)$ and by $V_{n+1}$ the upper 8-element cube of $H^+(n + 1)$, see Figs. 1.6 and 1.7.

Proposition (See Proposition 3.6.1, Section 3.6). The union $U_n \cup V_{n+1}$ is the 16-element Boolean algebra in $D^{2,2,2}/\theta$.

It is necessary to draw 8 additional edges in the places where $H^+(n)$ joins $H^+(n + 1)$.

Theorem (See Theorem 3.7.2, Section 3.7). The union $\bigcup_{n=0}^{\infty} H^+(n)$ is a distributive lattice mod $\theta$. The diagram $H^+$ is obtained by uniting the diagrams $H^+(n)$ and joining the cubes $U_n$ and $V_{n+1}$ for all $n \geq 0$, i.e., it is necessary to draw 8 additional edges for all $n \geq 0$, see Fig. 1.7.

Denote by $H^-$ the lattice dual to the lattice $H^+$. 
Remark 1.11.3. For a comparison between sublattices of perfect elements

\[ H^+ \subseteq D^{2,2}/\theta \text{ and } B^+ \subseteq D^4/\theta, \]

compare the Hasse diagrams of \( H^+ \) and \( B^+ \) in Fig. 1.6. From Fig. 1.6 we see that the perfect sublattice \( B^+ \) can be injectively mapped (imbedded) into the perfect sublattice \( H^+ \) mod \( \theta \). Indeed,

\[ p_1(n) + p_2(n) + p_3(n) = b_1(n)b_2(n)b_3(n) \subseteq c_1(n)c_2(n)c_3(n), \]

see Section 3.2.2, Proposition 3.2.5, Table 3.1.

Conjecture 1.11.4. The lattice \( H^+ \cup H^- \) contains all perfect elements of \( D^{2,2}/\theta_p \).

1.12. Note on the base field

Observe that all constructions of this paper are carried out uniformly (independently on the base field). Consider an example of the typical arguments dealing with the base field.

Let \( y_1, y_2 \) be generators of the lattice \( D^{2,2} \). Assume, \( \rho_X(y_1 + y_2) = 0 \) for all indecomposable \( \rho_X \). Then \( Y_1 = Y_2 = X_1 = X_2 = 0 \) and \( X_3 = K^m, Y_3 = K^n, X_0 = K^l \), where \( m \leq n \leq l \), and we have

\[ \begin{array}{c}
K^m \\
\downarrow \\
\rho_X : \\
\downarrow \\
K^n \\
\downarrow \\
0 \leftrightarrow 0 \leftrightarrow K^l \leftrightarrow 0 \leftrightarrow 0.
\end{array} \]

If representation \( \rho_X \) is indecomposable then one of three cases holds

\[ m = n = l = 1, \quad \rho_X = \rho_X^3, \]

\[ m = 0, \quad l = n = 1, \quad \rho_X = \rho_X^3, \]

\[ m = n = 0, \quad l = 1, \quad \rho_X = \rho_X^0. \]

see Proposition 2.9.3, Table A.1. This argumentation does not depend on the base field \( K \).
1.13. Bibliographic notes

1.13.1. Solvable and unsolvable word problems

Herrmann in [27,30] and Cylke in [9] proved that modulo $\theta_p$ perfect elements constructed by Gelfand and Ponomarev in $D^4$ are the only ones.

Herrmann proved in [28] that the word problem in $D^4$ is unsolvable, and Freese in [15] proved that the word problem in $D^4$ is unsolvable for $r \geq 5$. Furthermore, Herrmann proved that the modular lattice freely generated by a partially ordered set $P$ containing one of the lattices

\[ D^4, \quad D^{2.2.2}, \quad D^{1.2.3}, \]

has an unsolvable word problem [28, p. 526].

Thus, the modular lattice $D^{2.2.2}$, which is of interest to us, has also an unsolvable word problem.

Cylke in [10, p. 145] showed that the word problem in $D^4/\theta$ is solvable and gave examples of pairs of linearly equivalent but not equal elements of $D^4$. Herrmann and Huhn in [26, Corollary 9] and also Hutchinson and Czédli in [25] have shown that for certain varieties the word problem for the free lattice in these varieties is solvable. In particular, the word problem is solvable for free lattices $FV(P)$, $FV_\rho(P)$ freely generated by poset $P$, see (A.9). Thus, the word problem is solvable for $D^4/\theta$, $D^{2.2.2}/\theta$, $D^4/\theta_p$ and $D^{2.2.2}/\theta_p$.

1.13.2. Decidability and representation type

If a quiver has only finitely many indecomposable representations, it is called a quiver of finite representation type. If in each finite-dimension $d \geq 1$, the indecomposable $K$-linear representations of the quiver of dimension $d$ form at most finitely many one-parameter families, up to a finite number of representations, then the quiver is called of tame representation type. If the representation theory of the quiver is at least as complicated as the representation theory of the double loop quiver, then the quiver is called of wild representation type. The reader is referred to [14] and [48] for a precise definition of tame and wild representation type. We recall from [17] and [36] (see also [37] and [12]) that each of quivers $\mathbb{B}_6$ and $\mathbb{B}_4$ is tame.

M. Prest found the important relation between decidability and the representation type. Let $R$ be a ring (or $K$-algebra). The theory of $R$-modules is said to be decidable if there is an algorithm checking whether any sentence is true in every $R$-module or not. For example, the word problem for the ring $R$ is “interpretable” in the theory of $R$-modules: two algebraic combinations $t_1$ and $t_2$, of elements of $R$ define the same element of $R$ if and only if the sentence $v t_1 = v t_2$ for each $v \in M$ for every $R$-module $M$, see [42, Chapter 17.1]. Recall, that the category of $K$-representations of the quiver $Q$ can be considered as the category of the left $K Q$-modules for the path algebra $K Q$, see Appendix A.2.3, Proposition A.2.4. The following theorem makes clear the relation between decidability of the path algebra $K Q$ and the representation type of the quiver $Q$:

**Theorem 1.13.1** ([42, Section 17.4, Theorem 17.22]). Let $R$ be the path algebra of a quiver without relations. Then the theory of $R$-modules is decidable if and only if $R$ is of tame or finite representation type.

1.13.3. Howe–Huang projective invariants of quadruples

We mention in [51, Sect. A.10.4] a connection between some Howe–Huang projective invariants of quadruples [31], the Grassmann–Cayley algebra [3], and admissible elements in $D^4$.

2. Elementary maps $\varphi_i$ and atomic, admissible, cumulative polynomials

2.1. Construction of the Coxeter functor $\Phi^+$ by means of the lattice operations

Following Gelfand and Ponomarev [18,19] we construct the representation $\Phi^+ \rho_X$ by means of the lattice operations. Our construction is different from the classic definition of the Coxeter functor of Bernstein–Gelfand–Ponomarev [2].

The representation $\rho_X^+ := \Phi^+ \rho_X$ (see Section 1.6) is constructed in the space

\[ X_0^+ = \{ (\eta_1, \eta_2, \eta_3) \mid \eta_i \in Y_i, \sum \eta_i = 0 \}, \]
where $i$ runs over the set $\{1, 2, 3\}$. Set

$$R = \bigoplus_{i=1,2,3} Y_i,$$

i.e.,

$$R = \{ (\eta_1, \eta_2, \eta_3) \mid \eta_i \in Y_i, i = 1, 2, 3 \}.$$

Then we see that $X_0^1 \subseteq R$. Hereafter $\eta_i$ denotes a vector from the space $Y_i$ and $\xi_i$ denotes a vector from the space $X_i$.

Introduce the spaces $G_i, H_i, G'_i, H'_i \subseteq R$, where $i = 1, 2, 3$:

$$G_1 = \{ (\eta_1, 0, 0) \mid \eta_1 \in Y_1 \}, \quad H_1 = \{ (\xi_1, 0, 0) \mid \xi_1 \in X_1 \},$$

$$G_2 = \{ (0, \eta_2, 0) \mid \eta_2 \in Y_2 \}, \quad H_2 = \{ (0, \xi_2, 0) \mid \xi_2 \in X_2 \},$$

$$G_3 = \{ (0, 0, \eta_3) \mid \eta_3 \in Y_3 \}, \quad H_3 = \{ (0, 0, \xi_3) \mid \xi_3 \in X_3 \},$$

$$G'_1 = \{ (\xi_1, \eta_2, \eta_3) \mid \xi_1 \in X_1, \eta_i \in Y_i \}, \quad H'_1 = \{ (0, \eta_2, \eta_3) \mid \eta_i \in Y_i \},$$

$$G'_2 = \{ (\eta_1, \xi_2, \eta_3) \mid \xi_2 \in X_2, \eta_i \in Y_i \}, \quad H'_2 = \{ (\eta_1, 0, \eta_3) \mid \eta_i \in Y_i \},$$

$$G'_3 = \{ (\eta_1, \eta_2, \xi_3) \mid \xi_3 \in X_3, \eta_i \in Y_i \}, \quad H'_3 = \{ (\eta_1, \eta_3, 0) \mid \eta_i \in Y_i \}.$$  \hfill (2.1)

Now, let

$$Y_i^1 = G'_i X_0^1,$$  \hfill (2.2)

$$X_i^1 = H'_i X_0^1,$$  \hfill (2.3)

$$\rho_X^+ \Phi^+ \rho_X = [X_i^1 \subseteq Y_i^1 \subseteq X_0^1 ] \mid i = 1, 2, 3].$$  \hfill (2.4)

This construction of the Coxeter functor $\Phi^+$ is well-defined as follows from Appendix A.2.6.

2.2. The associated representations $v^0, v^1$

With representations $\rho_X$ and $\rho_X^+$ we associate the following representations $v^0, v^1$ in $R$:

$$v^0(y_i) = X_0^1 + G_i, \quad v^0(x_i) = X_0^1 + H_i, \quad i = 1, 2, 3,$$  \hfill (2.5)

$$v^1(y_i) = X_0^1 G'_i, \quad v^1(x_i) = X_0^1 H'_i, \quad i = 1, 2, 3.$$  \hfill (2.6)

Let $\mu : X_0^1 \longrightarrow R$ be an inclusion. Then

$$v^1(a) = \mu \rho_X^+(a) \quad \text{for every } a \in D^{2,2,2}. \hfill (2.7)$$

Let $\nabla : R \longrightarrow X_0$ be a projection, $\nabla : (\eta_1, \eta_2, \eta_3) \longrightarrow \eta_i$. Then $\ker \nabla = X_0^1$.

Lemma 2.2.1. For every $a \in D^{2,2,2}$ and for every subspace $B \subseteq R$, we have

$$\nabla (v^0(a)B) = \nabla (v^0(a)) \nabla (B).$$  \hfill (2.8)

Proof. The inclusion

$$\nabla (v^0(a)B) \subseteq \nabla (v^0(a)) \nabla (B)$$

is clear, because $v^0(a)B \subseteq v^0(a)$ and $v^0(a)B \subseteq B$. Conversely, let $z \in \nabla (v^0(a)) \nabla (B)$. Then there exist vectors

$$u \in v^0(a) \text{ and } v \in B \text{ such that } \nabla (u) = \nabla (v).$$  \hfill (2.9)

According to (2.5)

$$v^0(a) = X_0^1 + A.$$
for some subspace $A \subseteq R$. From (2.9), we have
\[ v - u = w \in \ker \nabla = X^1_0, \quad \text{i.e.,} \]
\[ v = u + w \in X^1_0 + A = v^0(a), \quad \text{and hence} \ v \in v^0(a)B. \]
Thus, $z = \nabla(v) \in \nabla(v^0(a)B)$ and (2.8) is proved. \( \square \)

From (2.8) it follows that the map $\nabla : R \to R \setminus X^1_0$ gives a representation by the formula
\[ a \mapsto \nabla(v^0(a)). \]
We have
\[ \nabla(v^0(y_i)) = \nabla G_i = Y_i = \rho_X(y_i), \quad i = 1, 2, 3, \quad (2.10) \]
\[ \nabla(v^0(x_j)) = \nabla H_i = X_i = \rho_X(x_j), \quad i = 1, 2, 3, \quad (2.11) \]
\[ \nabla(v^0(a)) = \rho_X(a) \quad \text{for every} \ a \in D^{2,2,2}. \quad (2.12) \]

2.3. The elementary maps $\varphi_i$

Following [18], we introduce the linear maps
\[ \varphi_i : X^1_0 \to X_0, (\eta_1, \eta_2, \eta_3) \mapsto \eta_i. \]
Let $\pi_i$ be a projection $R$ on $G_i$. Then
\[ \varphi_i = \nabla \pi_i \mu. \quad (2.13) \]
It follows from the definition that
\[ \varphi_1 + \varphi_2 + \varphi_3 = 0. \quad (2.14) \]
From (2.14) we immediately deduce that
\[ \varphi_1 B + \varphi_2 B = \varphi_1 B + \varphi_3 B = \varphi_1 B + \varphi_2 B \quad (2.15) \]
for every subspace $B \subseteq X^1_0$.

2.4. Basic relations for the maps $\psi_i$ and $\varphi_i$

We introduce the maps $\psi_i : D^{2,2,2} \to \mathcal{L}(R)$ useful in the study of maps $\varphi_i$. Set
\[ \psi_i(a) = X^1_0 + G_i(H'_i + v^1(a)). \quad (2.16) \]
The map $\psi_i$ is said to be a joint map.

**Proposition 2.4.1.** The joint maps $\psi_i$ satisfy the following basic relations:

1. $\psi_i(x_i) = X^1_0$.
2. $\psi_i(x_j) = \psi_k(x_j) = v^0(y_iy_k)$.
3. $\psi_i(y_i) = v^0(x_jy_j + y_k)$.
4. $\psi_i(y_j) = v^0(y_i(x_j + y_k))$.
5. $\psi_i(I) = v^0(y_iy_j + y_k)$.
6. $\psi_i(y_jy_k) = v^0(y_i(x_j + y_k))$.

**Proof.** (1) From (2.6) and (2.16), we have $\psi_i(x_i) = X^1_0 + G_iH'_i$. Since $G_iH'_i = 0$, we have $\psi_i(x_i) = X^1_0$.

(2) From (2.6) and (2.16), we have $\psi_i(x_j) = X^1_0 + G_i(H'_i + X^1_0H'_j)$. Since $H'_j = G_i + G_k$ (where $i, j, k$ are distinct), it follows that
\[ \psi_i(x_j) = X^1_0 + G_i(G_j + G_k + X^1_0(G_i + G_k)). \]
By the permutation property (A.2)
\[ \psi_i(x_j) = X_0^1 + G_i(X_0^1 + (G_j + G_k)(G_i + G_k)) = X_0^1 + G_i(X_0^1 + G_k + G_j(G_i + G_k)). \]
Since \( G_j(G_i + G_k) = 0 \), it follows that
\[ \psi_i(x_j) = X_0^1 + G_i(X_0^1 + G_k) = (X_0^1 + G_i)(X_0^1 + G_k) = v^0(y_i y_k). \]

(3) As above, \( \psi_i(y_i) = X_0^1 + G_i(H_i' + X_0^1 G_i'). \) Since \( H_i' \subseteq G_i' \), and by the modular law (A.1), we have
\[ \psi_i(y_i) = X_0^1 + G_i G_i'(H_i' + X_0^1) \]

as above, \( \psi_i(y_i) = X_0^1 + G_i G_i'(H_i' + X_0^1) \). Since \( G_i' H_i = H_i \) and \( H_i' = G_i + G_k \) (where \( i, j, k \) are distinct), we have
\[ \psi_i(y_i) = X_0^1 + H_i'(G_j + G_k + X_0^1) = (X_0^1 + H_i')(G_j + G_k + X_0^1) = v^0(x_j y_k). \]

(4) Again, \( \psi_i(y_j) = X_0^1 + G_i(H_i' + X_0^1 G_j') \). From \( G_i \subseteq G_j' \) for \( i \neq j \) and by the permutation property (A.2), we have
\[ \psi_i(y_j) = X_0^1 + G_i(H_i' + X_0^1). \]
Since \( H_i' G_j' = H_j + G_k \), we have
\[ \psi_i(y_j) = X_0^1 + G_i(H_i' + X_0^1) = (X_0^1 + G_i)(X_0^1 + H_j + G_k) = v^0(x_j + y_k). \]

(5) Similarly,
\[ \psi_i(I) = X_0^1 + G_i(H_i' + X_0^1) = X_0^1 + G_i(G_j + G_k + X_0^1) = (X_0^1 + G_i)(G_j + G_k + X_0^1) = v^0(y_j y_k). \]

(6) As above,
\[ \psi_i(y_j y_k) = X_0^1 + G_i(H_i' + X_0^1 G_j' G_k'). \]
From \( G_j' G_k' = H_j + H_k + G_i \) and by the permutation (A.2), we have
\[ \psi_i(y_j y_k) = X_0^1 + G_i(((G_j + G_k)(H_j + H_k + G_i) + X_0^1) = X_0^1 + G_i(H_j + H_k + G_i(G_j + G_k) + X_0^1). \]
The intersection vanishes: \( G_i(G_j + G_k) = 0 \), and hence
\[ \psi_i(y_j y_k) = X_0^1 + G_i(H_j + H_k + X_0^1) = v^0(y_i x_j x_k). \]

Remark 2.4.2 (The Lattice Description of the Projection, [18]). If \( \pi: R \to R \) is a projection, and \( A \) is a subspace of \( R \), then
\[ \pi A = \operatorname{Im} \pi (\ker \pi + A). \] (2.17)
Indeed,
\[ v \in \operatorname{Im} \pi (\ker \pi + A) \implies v = \pi v \in \pi (\ker \pi + A), \quad \text{i.e., } v \in \pi A. \]

Conversely,
\[ v \in \pi A \implies v \in \operatorname{Im} \pi, \quad \text{besides } v = \pi v \implies \pi v \in \pi A \implies v \in \ker \pi + A. \]

Proposition 2.4.3 (A Relation Between the \( \varphi_i \) and \( \psi_i \)). The following properties establish a relation between the \( \varphi_i \) and \( \psi_i \):

1. Let \( a, b \subseteq D^{2,2}.2 \). If \( \psi_i(a) = v^0(b) \), then \( \varphi_i \rho_X^+(a) = \rho_X(b) \).
2. Let \( c \subseteq D^{2,2}.2 \). Then \( \nabla \psi_i(c) = \varphi_i \rho_X^+(c) \).
3. Let \( a, b, c \subseteq D^{2,2}.2 \). If \( \psi_i(a) = v^0(b) \) and \( \psi_i(ac) = \psi_i(a) \psi_i(c) \), then
\[ \varphi_i \rho_X^+(ac) = \varphi_i \rho_X^+(a) \varphi_i \rho_X^+(c). \] (2.18)
Proof. (1) From (2.13) we deduce that \( \varphi_i \rho_X^+(a) = \nabla \pi_i \mu \rho_X^+(a) \). By (2.7), we have \( \varphi_i \rho_X^+(a) = \nabla \pi_i v^1(a) \). By (2.17) for \( \text{Im} \pi_i = G_i \) and \( \ker \pi_i = H_i' \), we have
\[
\nabla \pi_i v^1(a) = G_i (H_i' + v^1(a)),
\]
and
\[
\nabla \pi_i v^1(a) = \nabla (G_i(H_i' + v^1(a))) = \nabla (X_0^1 + G_i(H_i' + v^1(a))) = \nabla (\psi_i(a)) = \nabla (v^0(b)).
\]
From (2.12) \( \nabla (\psi_i(a)) = \rho_X(b) \).
(2) By definition (2.16) of \( \psi_i \), we have
\[
\nabla \psi_i(c) = \nabla (X_0^1 + G_i(H_i' + v^1(c))).
\]
Since \( X_0^1 \subseteq \ker \nabla \), we have \( \nabla \psi_i(c) = \nabla (G_i(H_i' + v^1(c))) \). By lattice description of the projection (2.17), we have \( \nabla \psi_i(c) = \nabla (\pi_i v^1(c)) \). By (2.7) \( \nabla \psi_i(c) = \nabla (\pi_i(\mu \rho_X^+(c))) \). Finally, by definition (2.13) of \( \varphi_i \), we have \( \nabla \psi_i(c) = \varphi_i \rho_X^+(c) \).
(3) By the hypothesis \( \psi_i(ac) = v^0(a) \psi_i(c) \). By Lemma 2.2.1 and Eq. (2.8), we have
\[
\nabla \psi_i(ac) = \nabla \psi_i(a) \nabla \psi_i(c).
\]
The proof follows from (2). \( \square \)

From Propositions 2.4.1 and 2.4.3 we have

Corollary 2.4.4. For the elementary map \( \varphi_i \), the following basic relations hold:

1. \( \varphi_i \rho_X^+(x_i) = 0 \),
2. \( \varphi_i \rho_X^+(x_j) = \varphi_k \rho_X^+(x_j) = \rho_X(y_j y_k) \),
3. \( \varphi_i \rho_X^+(y_i) = \rho_X(x_i y_j + y_k) \),
4. \( \varphi_i \rho_X^+(y_j) = \rho_X(y_j x_i + y_k) \),
5. \( \varphi_i \rho_X^+(I) = \rho_X(y_j x_i + y_k) \),
6. \( \varphi_i \rho_X^+(y_j y_k) = \rho_X(y_i x_j + x_k) \).

Proof. It is necessary to explain only (1). Since the \( i \)th coordinate in the space \( H_i' \) is 0, it follows that
\[
\varphi_i \rho_X^+(x_i) = \nabla \pi_i \mu \rho_X^+(x_i) = \nabla \pi_i v^1(x_i) = \nabla \pi_i (X_0^1 H_i' ) = 0. \quad \square
\]

2.5. Additivity and multiplicativity of the joint maps \( \psi_i \)

Proposition 2.5.1. The map \( \psi_i \) is additive and quasimultiplicative\(^8\) with respect to the lattice operations \( + \) and \( \cap \). The following relations hold for every \( a, b \in D^{2,2,2} \):

1. \( \psi_i(a) + \psi_i(b) = \psi_i(a + b) \),
2. \( \psi_i(a) \psi_i(b) = \psi_i((a + x_i)(b + x_j x_k)) \),
3. \( \psi_i(a) \psi_i(b) = \psi_i(a(b + x_i + x_j x_k)) \).

Proof. (1) By the modular law (A.1)
\[
\psi_i(a) + \psi_i(b) = X_0^1 + G_i (H_i' + v^1(a)) + G_i (H_i' + v^1(b))
\]
\[
= X_0^1 + G_i \left( H_i' + v^1(a) + G_i (H_i' + v^1(b)) \right) = X_0^1 + G_i \left( (H_i' + v^1(b))(H_i' + G_i) + v^1(a) \right).
\]
Since \( H_i' + G_i = R \), it follows that
\[
\psi_i(a) + \psi_i(b) = X_0^1 + G_i \left( H_i' + v^1(b) + v^1(a) \right) = X_0^1 + G_i (H_i' + v^1(b + a)).
\]

\(^8\) The map \( \psi_i \) is multiplicative if \( \psi_i(a) \psi_i(b) = \psi_i(ab) \). We will say that \( \psi_i \) is quasimultiplicative if element \( ab \) in this relation is a little deformed, or more exactly, if \( ab \) is deformed to \( (a + x_i)(b + x_j x_k) \) or to \( a(b + x_i + x_j x_k) \) as in (2) and (3) of Proposition 2.5.1.
(2) By definition (2.6) \( v^1(b) \subseteq X_0^1 \), and by the permutation property (A.3) we have
\[
X_0^1 + G_i(H_i' + v^1(a)) = X_0^1 + H_i'(G_i + v^1(a)).
\]
By the modular law (A.1) and by (A.3) we have
\[
\psi_i(a)\psi_i(b) = (X_0^1 + G_i(H_i' + v^1(a)))(X_0^1 + G_i(H_i' + v^1(b)))
= X_0^1 + G_i(H_i' + v^1(a))(X_0^1 + H_i'(G_i + v^1(b))).
\]
Further,
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i(X_0^1(H_i' + v^1(a)) + H_i'(G_i + v^1(b))).
\]
Since
\[
X_0^1(H_i' + v^1(a)) = X_0^1H_i' + v^1(a) \quad \text{and} \quad X_0^1H_i' = v^1(x_i),
\]
we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i(v^1(x_i) + v^1(a) + H_i'(G_i + v^1(b))).
\]
By the permutation property (A.2) and by (2.19) we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i\left(H_i' + (v^1(x_i) + v^1(a))(G_i + v^1(b))\right).
\]
Since
\[
G_i = H_j'H_k' \quad \text{and} \quad v^1(x_i) + v^1(a) = v^1(x_i + a) = X_0^1(v^1(x_i + a)),
\]
it follows that
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i\left(H_i' + v^1(x_i + a)(X_0^1H_j'H_k' + v^1(b))\right)
= X_0^1 + G_i\left(H_i' + v^1(x_i + a)(v^1(x_jx_k) + v^1(b))\right)
= X_0^1 + G_i\left(H_i' + v^1(x_i + a)v^1(x_jx_k + b)\right) = \psi_i((a + x_i)(b + x_jx_k)).
\]
(3) From (2.19) and since \( v^1(x_i) = X_0^1H_i' \subseteq H_i' \), we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i\left(v^1(a) + H_i'(G_i + v^1(b) + v^1(x_i))\right).
\]
Again, by (A.2) we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i\left(H_i' + v^1(a)(G_i + v^1(b) + v^1(x_i))\right)
= X_0^1 + G_i\left(H_i' + v^1(a)(X_0^1H_j'H_k' + v^1(b) + v^1(x_i))\right).
\]
Thus, \( \psi_i(a)\psi_i(b) = \psi_i(a(b + x_i + x_jx_k)) \). \( \square \)

**Corollary 2.5.2 (Atomic Multiplicativity).**

(a) Let one of the following inclusions hold:

(i) \( x_i + x_jx_k \subseteq a \),

(ii) \( x_i + x_jx_k \subseteq b \),

(iii) \( x_j \subseteq a \), \( x_jx_k \subseteq b \),

(iv) \( x_i \subseteq b \), \( x_jx_k \subseteq a \).

Then the joint map \( \psi_i \) operates as a homomorphism on the elements \( a \) and \( b \) with respect to the lattice operations \( + \) and \( \cap \), i.e.,
\[
\psi_i(a) + \psi_i(b) = \psi_i(a + b), \quad \psi_i(a)\psi_i(b) = \psi_i(a)\psi_i(b).
\]
The joint map follows from Table 2.1.

**Proposition 2.5.1**

Table 2.1

<table>
<thead>
<tr>
<th>1.1</th>
<th>( a_{ij}^{n} = x_{i} + y_{j}a_{n-1}^{jk} ) (definition)</th>
<th>1.2</th>
<th>( A_{n}^{ij} = y_{i} + x_{j}A_{n-1}^{ki} ) (definition)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>( a_{ij}^{0} \supseteq a_{ij}^{1} \supseteq \cdots \supseteq a_{ij}^{n} \supseteq \cdots \supseteq x_{i} + x_{j} )</td>
<td>2.2</td>
<td>( A_{0}^{ij} \supseteq A_{1}^{ij} \supseteq \cdots \supseteq A_{n}^{ij} \supseteq \cdots )</td>
</tr>
<tr>
<td>3.1</td>
<td>( x_{k}A_{n}^{ij} = x_{k}a_{n}^{ij} )</td>
<td>3.2</td>
<td>( A_{n}^{ij} = y_{i} + x_{j}a_{n-1}^{jk} )</td>
</tr>
<tr>
<td>4.1</td>
<td>( y_{i}y_{j}A_{n}^{k} = y_{i}y_{j}a_{n}^{jk} )</td>
<td>4.2</td>
<td>( y_{i}a_{n+1}^{jk} = y_{i}(x_{i} + y_{j}A_{n}^{k}) )</td>
</tr>
<tr>
<td>5.1</td>
<td>( a_{ij}^{n}a_{m}^{k} = a_{ij}^{n}a_{m}^{k} ) for ( n \leq m + 1 )</td>
<td>5.2</td>
<td>( y_{i}(x_{j} + x_{k})a_{n}^{ij} = y_{i}(x_{j} + x_{k})A_{n}^{ij} )</td>
</tr>
<tr>
<td>6.1</td>
<td>( A_{n}^{ij}a_{m}^{k} = y_{i}a_{n+1}^{ij} + x_{k}A_{n}^{ij} )</td>
<td>6.2</td>
<td>( A_{n}^{ij}a_{m}^{k} = y_{i}a_{n+1}^{ij} + x_{j}A_{n-1}^{ki} )</td>
</tr>
<tr>
<td>7.1</td>
<td>( y_{i}(x_{j} + x_{k}) + y_{i}y_{j}a_{n+2}^{jk} = y_{i}a_{n}^{ik} )</td>
<td>7.2</td>
<td>( y_{i}(y_{j} + x_{k})A_{n+2}^{kj} = y_{i}A_{n}^{kj} )</td>
</tr>
<tr>
<td>8.1</td>
<td>( x_{i}(y_{j} + y_{k}) + y_{i}y_{j}A_{n+1}^{ij} = y_{i}(y_{j} + y_{k})a_{n}^{ij} )</td>
<td>8.2</td>
<td>( y_{i}(y_{j} + x_{k})A_{n+2}^{kj} = y_{i}(y_{j} + y_{k})A_{n}^{ij} )</td>
</tr>
</tbody>
</table>

(b) The joint map \( \psi_{i} \) applied to the following atomic elements is the intersection preserving map, i.e., multiplicative with respect to the operation \( \cap \):

\[
\psi_{i}(ba_{n}^{ij}) = \psi_{i}(b)\psi_{i}(a_{n}^{ij}) \quad \text{for every } b \subseteq D^{2,2,2},
\]

\[
\psi_{j}(ba_{n}^{ij}) = \psi_{j}(b)\psi_{j}(a_{n}^{ij}) \quad \text{for every } b \subseteq D^{2,2,2},
\]

\[
\psi_{i}(bA_{n}^{ij}) = \psi_{i}(b)\psi_{i}(A_{n}^{ij}) \quad \text{for every } b \subseteq D^{2,2,2}.
\]

(c) The element \( x_{i} \) can be inserted in any additive expression under the action of the joint map \( \psi_{i} \):

\[
\psi_{i}(b + x_{i}) = \psi_{i}(b) \quad \text{for every } b \subseteq D^{2,2,2}.
\]

**Proof.** (a) Follows from Proposition 2.5.1.

(b) Let us consider \( a_{n}^{ij} \). For \( n \geq 2 \), we have

\[
a_{n}^{ij} = x_{i} + y_{j}a_{n-1}^{jk} = x_{i} + y_{j}(x_{j} + y_{k}a_{n-2}^{ki}) = x_{i} + x_{j} + y_{j}y_{k}a_{n-2}^{ki},
\]

thus \( a_{n}^{ij} \supseteq x_{i} + x_{j} \). Now, we consider atomic element \( A_{n}^{ij} \). For \( n \geq 2 \), we have

\[
A_{n}^{ij} = y_{i} + x_{j}A_{n-1}^{ki} = y_{i} + x_{j}(y_{k} + x_{k}A_{n-2}^{ij}) \supseteq x_{i} + x_{j}x_{k}.
\]

For \( n = 1 \), the relation is proved directly.

(c) Follows from Corollary 2.4.4, (1) since we have

\[
\psi_{i}(b + x_{i}) = \psi_{i}(b) + \psi_{i}(x_{i}) \quad \text{and} \quad \psi_{i}(x_{i}) = X_{0}^{1} \subseteq \psi_{i}(b)
\]

for every \( b \subseteq D^{2,2,2} \). □

2.6. Basic properties of atomic elements \( a_{n}^{ij} \) and \( A_{n}^{ij} \)

The atomic elements are defined in Section 1.5. These polynomials are building bricks in the further constructions.

The basic properties of atomic elements are given in Table 2.1. These properties will be used in further considerations. For convenience, we also give the definitions of \( a_{n}^{ij}, A_{n}^{ij} \) in relations (1.1) and (1.2) of Table 2.1. Indices \( i, j, k \) in Table 2.1 are distinct and \( \{i, j, k\} = \{1, 2, 3\} \). Properties 3.1, 3.2, 4.1, 4.2, 5.1, 5.2 from Table 2.1 mean that atomic elements \( a_{n}^{ij} \) and \( A_{n}^{ij} \) coincide for many lattice polynomials and can substitute for one another in many cases.

**Proof of properties 2.1–8.2 (Table 2.1).**

(2.1) First, \( a_{n-1}^{ij} = x_{i} + y_{j}a_{n-2}^{jk}, a_{n}^{ij} = x_{i} + y_{j}a_{n-1}^{jk} \). By induction, if \( a_{n-2}^{jk} \supseteq a_{n-1}^{jk}, \) then \( a_{n-1}^{ij} \supseteq a_{n}^{ij} \). For \( n = 1 \), we have \( a_{1}^{ij} = I \supseteq a_{1}^{ij} \). Further,

\[
a_{n}^{ij} = x_{i} + y_{j}(x_{j} + y_{k}a_{n-2}^{ki}) = x_{i} + x_{j} + y_{j}y_{k}a_{n-2}^{ki} \supseteq x_{i} + x_{j}.
\]
(2.2) Similarly,
\[ A_{n-1}^{ij} = y_i + x_j A_{n-2}^{kj} \geq y_i + x_j A_{n-1}^{ij} = A_n^{ij}. \]

(3.1) By the induction hypothesis \( x_j A_{n-1}^{ki} = x_j a_{n-1}^{jk} \), then
\[ x_k A_n^{ij} = x_k (y_i + x_j A_{n-1}^{ki}) = x_k (y_i + x_j a_{n-1}^{jk}). \]

By property 1.1 from Table 2.1 we have \( a_{n-1}^{jk} \geq x_k \), thus by the permutation property (A.2) we deduce
\[ x_k A_n^{ij} = x_k (x_j + y_i a_{n-1}^{jk}) = x_k a_n^{ij}. \]

(3.2) By property 3.1 from Table 2.1 we have \( A_n^{ij} = y_i + x_j A_{n-1}^{ki} = y_i + x_j a_{n-1}^{jk} \). \( \square \)

(4.1) We have
\[ y_i y_j a_{n-2}^{jk} = y_i y_j (x_i + y_k a_{n-1}^{jk}) = y_i y_j (x_i + y_k (y_j + y_j a_{n-2}^{jk})). \]
i.e.,
\[ y_i y_j a_{n-2}^{jk} = y_i y_j (x_i + x_k + y_k y_j a_{n-2}^{jk}). \] (2.26)

On the other hand,
\[ y_i y_j A_{n-2}^{ij} = y_i (y_k + x_i A_{n-1}^{jk}) = y_i y_j (y_k + x_i (y_j + x_j A_{n-2}^{ij})). \]

Since \( A_{n-2}^{ij} \geq x_i \), we have by the permutation property (A.2):
\[ y_i y_j A_{n-2}^{ij} = y_i y_j (y_k + x_i (y_j + x_j A_{n-2}^{ij})). \]

Further, since \( y_j A_{n-2}^{ij} \geq y_i y_j \), again, by (A.2), we have
\[ y_i y_j A_{n-2}^{ij} = y_i y_j (x_i + y_k (y_j + y_j A_{n-2}^{ij})) = y_i y_j (x_i + x_k + y_k y_j A_{n-2}^{ij}). \]

By the induction hypothesis
\[ y_i y_j A_{n-2}^{ij} = y_i y_j (x_i + x_k + y_k y_j a_{n-2}^{jk}) \]
and by (2.26) we have \( y_i y_j a_{n-2}^{jk} = y_i y_j a_{n-2}^{ij} \). \( \square \)

(4.2) By property 4.1 we have
\[ y_i a_{n+1}^{ij} = y_i (x_i + y_j a_{n}^{jk}) = x_i + y_i y_j a_{n}^{jk} = x_i + y_i y_j A_{n}^{kj} = y_i (x_i + y_j A_{n}^{kj}). \]

(5.1) By property 2.1 and (A.2) we have
\[ a_n^{ij} a_n^{jk} = (x_i + y_j a_{n-1}^{jk}) a_n^{jk} = (y_j + x_i a_{n-1}^{jk}) a_n^{jk}. \]

By property 3.1, \( a_n^{ij} a_m^{jk} = (y_j + x_i a_{n-1}^{jk}) a_m^{jk} = A_n^{ij} a_m^{jk}. \) \( \square \)

(5.2) We have
\[ y_i (x_j + x_k) a_n^{ij} = y_i (x_j + x_k) (x_i + y_j a_{n-1}^{jk}). \]

Since \( a_{n-1}^{jk} \geq x_j + x_k \), it follows that
\[ y_i (x_j + x_k) a_n^{ij} = y_i (x_j + x_k) (y_j + x_i a_{n-1}^{jk}) \]
\[ = y_i (x_j + x_k) (y_j + x_i A_{n-1}^{ji}) = y_i (x_j + x_k) A_n^{ij}. \]

(6.1) Since \( y_i a_{m-1}^{ij} \subseteq A_n^{ij} \), it follows that
\[ A_n^{ij} a_m^{ki} = A_n^{ij} (x_k + y_i a_{m-1}^{ij}) = x_k A_n^{ij} + y_i a_{m-1}^{ij}. \]
(6.2) Since \( x_j A_{n-1}^{ki} \subseteq a_{m}^{jk} \), it follows that
\[
A_{m}^{ij} a_{m}^{jk} = a_{m}^{jk} \quad (y_i + x_j A_{n-1}^{ki}) = y_i a_{m}^{jk} + x_j A_{n-1}^{ki}. \]
\( \square \)

(7.1) Since
\[
y_i(x_j + x_k) + y_i y_j a_{n-2}^{ik} = y_i(x_j + x_k + y_i y_j a_{n-2}^{ik}) = y_i(x_k + y_j(x_j + y_i a_{n-2}^{ik})),
\]
by definition (1.1, Table 2.1) we have
\[
y_i(x_k + y_j(x_j + y_i a_{n-2}^{ik})) = y_i a_{n}^{kj}. \]
\( \square \)

(7.2) As above, using the modular law (A.1) and the permutation property (A.2) we have
\[
y_i y_j + y_i (x_j + x_k) A_{n-2}^{ki} = y_i ((x_j + x_k) A_{n-2}^{ki} + y_i y_j) = y_i (x_k + x_j A_{n-2}^{ki} + y_i y_j) = y_i (x_k + y_j(y_i + x_j A_{n-2}^{ki})) = y_i (x_k + y_j A_{n-1}^{ij}) \]
\[
= y_i (y_j + x_k A_{n-1}^{ij}) = y_i A_{n}^{jk}. \]
\( \square \)

(8.1) Since
\[
x_i (y_j + y_k) + y_i y_j A_{n-1}^{kj} = (y_j + y_k) (x_i + y_i y_j A_{n-1}^{kj}),
\]
by property 4.1 we have
\[
x_i (y_j + y_k) + y_i y_j A_{n-1}^{kj} = (y_j + y_k) (x_i + y_i y_j A_{n-1}^{kj}) = y_i (y_j + y_k) (x_i + y_j A_{n-1}^{kj}) = y_i (y_j + y_k) A_{n}^{ij}. \]
\( \square \)

(8.2) As above,
\[
y_i y_j + x_j (y_i + y_k) A_{n-1}^{ki} = y_j (y_i + y_k) (y_i + x_j A_{n-1}^{ki}) = y_i (y_j + y_k) A_{n}^{ij}. \]
\( \square \)

2.7. Action of the maps \( \psi_i \) and \( \varphi_i \) on atomic elements

**Proposition 2.7.1** (Action of Joint Maps on Atomic Elements). The joint maps \( \psi_i \) applied to the atomic elements \( a_n^{ij} \) and \( A_n^{ij} \) \((n \geq 1)\) satisfy the following relations:

1. \( \psi_i(a_n^{ij}) = v^0(y_i A_n^{ki}), \)
2. \( \psi_j(a_n^{ij}) = \begin{cases} v^0(y_j A_n^{ki}) & \text{for } n > 1, \\ v^0(y_j(y_i + y_k) A_n^{ij}) & \text{for } n = 1. \end{cases} \)
3. \( \psi_k(y_i a_n^{ij}) = v^0(y_k A_n^{ij}) \)
4. \( \psi_i(A_n^{ij}) = v^0(y_i(y_j + y_k) a_n^{ik}), \)
5. \( \psi_j(y_k A_n^{ij}) = v^0(y_j(x_k + y_i) a_n^{ik}), \)
6. \( \psi_k(y_j A_n^{ij}) = v^0(y_k a_n^{ij}). \)

**Remark 2.7.2.** The joint map \( \psi_k \) increases the lower index \( n \) of the atomic element and swaps indices \( i \) and \( j \) in headings (3) and (6). The new atomic elements appear in this way.

\( ij \implies ji, \quad n \implies n + 1. \)

In headings (1) and (2), the lower index \( n \) does not grow and the upper pair \( ij \) becomes \( kj \). The first index of the pair is changed and the atomic elements “\( a \)” are transformed to atomic elements “\( A \)”:

\( ij \implies kj, \quad a \implies A. \)

In headings (4) and (5), the lower index \( n \) does not grow and the upper pair \( ij \) becomes \( ik \). The second index of the pair is changed and the atomic elements “\( A \)” are transformed to atomic elements “\( a \)”:

\( ij \implies ik, \quad A \implies a. \)

Map \( \psi_k \) transforms \( a \implies A \) in headings (1), (2), (3) and \( A \implies a \) in headings (4), (5), (6).
Remark 2.7.3. It is not so interesting to consider $\psi_j(A_n^{ij})$ because

$$\psi_j(A_n^{ij}) = \psi_j(y_i + x_jA_n^{kj}) = \psi_j(y_i).$$

By basic relations (Proposition 2.4.1) $\psi_j(y_i) = v^0(y_j(x_i + y_k))$.

For further moving of indices, see Sections 2.11 and 2.12 devoted to the admissible elements.

Proof of Proposition 2.7.1. Uses a simultaneous induction on (1)–(6). For convenience, without loss of generality, we let $i = 1, j = 2$.

(1) For $n = 1$, by Proposition 2.4.1 we have

$$\psi_1(a_1^{12}) = \psi_1(x_1 + y_2) = \psi_1(y_2) = v^0(y_1(x_2 + y_3)) = v^0(y_1 A_1^{32}).$$

For $n \geq 2$, by Corollary 2.5.2 we have

$$\psi_1(a_n^{12}) = \psi_1(x_1 + y_2a_{n-1}^{23}) = \psi_1(y_2a_{n-1}^{23}).$$

Since

$$y_2a_{n-1}^{23} = y_2(x_2 + y_3 a_{n-2}^{31}) = x_2 + y_2 y_3 a_{n-2}^{31}$$

and $a_{n-2}^{31} \subseteq y_1$, we see by multiplicativity (Corollary 2.5.2) that

$$\psi_1(a_n^{12}) = \psi_1(x_2) + \psi_1(y_2 y_3) \psi_1(a_{n-2}^{31}).$$

By basic relations (Proposition 2.4.1)

$$\psi_1(a_n^{12}) = v^0(y_1 y_3) + v^0(y_1(x_2 + y_3)) \psi_1(a_{n-2}^{31}).$$

By the induction hypothesis from (2) we have $\psi_1(a_{n-2}^{31}) = v^0(y_1 A_{n-2}^{21})$ and

$$\psi_1(a_n^{12}) = v^0(y_1 y_3 + y_1(x_2 + y_3) A_{n-2}^{21}).$$

By property 7.2 from Table 2.1 we have

$$\psi_1(a_n^{12}) = v^0(y_1 A_n^{32}). \quad \Box$$

(2) For $n = 1$, we have

$$\psi_2(a_1^{12}) = \psi_2(x_1) + \psi_2(y_2) = v^0(y_2 y_3 + x_2(y_1 + y_3))$$

$$= v^0(y_2(y_1 + y_3)(y_3 + x_2)) = v^0(y_2(y_1 + y_3) A_1^{32}).$$

For $n \geq 2$, we have $\psi_2(a_n^{12}) = \psi_2(x_1) + \psi_2(y_2 a_{n-1}^{23})$. By the multiplicativity property (Corollary 2.5.2) we have

$$\psi_2(a_n^{12}) = v^0(y_2 y_3) + \psi_2(y_2) \psi_2(a_{n-1}^{23}).$$

By the induction hypothesis, from (1) we have $\psi_2(a_{n-1}^{23}) = v^0(y_2 A_{n-1}^{13})$ and

$$\psi_2(a_n^{12}) = v_0(y_2 y_3 + x_2(y_1 + y_3 A_{n-1}^{13})).$$

Since $A_{n-1}^{13} \subseteq y_1 + y_3$, we see that

$$\psi_2(a_n^{12}) = v^0(y_2 y_3 + x_2 A_{n-1}^{13}) = v^0(y_2 y_3 + x_2 A_{n-1}^{13}) = v^0(y_2 A_n^{32}). \quad \Box$$

(3) We have

$$\psi_3(y_1 a_n^{12}) = \psi_3(y_1(x_1 + y_2 a_{n-1}^{23})) = \psi_3(x_1 + y_1 y_2 a_{n-1}^{23}).$$

Further, by multiplicativity (Corollary 2.5.2) we have

$$\psi_3(y_1 a_n^{12}) = \psi_3(x_1) + \psi_3(y_1 y_2) \psi_3(a_{n-1}^{23}).$$

From basic relations (Proposition 2.4.1) it follows that

$$\psi_3(y_1 a_n^{12}) = v^0(y_3 y_2) + v^0(y_3(x_1 + x_2)) \psi_3(a_{n-1}^{23}).$$
By induction, from (2) \( \psi_3(y_1a_{n}^{12}) = y_0(y_3y_2 + y_3(x_1 + x_2)A_{n-1}^{13}) \). Again, from Table 2.1, property 7.2 we have \( \psi_3(y_1a_{n}^{12}) = y_0(y_3A_{n+1}^{13}) \). In this case \( \psi_k \) increases the lower index of the atomic element \( a_{n}^{ij} \). \( \square \)

(4) As above,
\[
\psi_1(A_{n}^{12}) = \psi_1(y_1 + x_2A_{n}^{31}) = \psi_1(y_1) + \psi_1(x_2A_{n}^{31}).
\]

From Table 2.1, property 3.1 we have
\[
\psi_1(A_{n}^{12}) = \psi_1(y_1) + \psi_1(x_2a_{n-1}^{13}) = \psi_1(y_1) + \psi_1(x_2)\psi_1(a_{n-1}^{13}).
\]

By basic relations (Proposition 2.4.1) and by induction hypothesis, from (1) we have
\[
\psi_1(A_{n}^{12}) = y_0(x_1(y_2 + y_3) + y_1y_3A_{n-1}^{23} A_{n}^{21}).
\]

From Table 2.1, property 8.1 we have
\[
\psi_1(A_{n}^{12}) = y_0(y_1(y_2 + y_3)a_{n}^{13}).
\]

(5) Again, \( \psi_2(y_3A_{n}^{12}) = \psi_2(y_3(y_1 + x_2A_{n-1}^{31})) \). Since \( y_3 \subseteq A_{n-1}^{31} \), we have by the permutation property (A.2):
\[
\psi_2(y_3A_{n}^{12}) = \psi_2(y_3(x_2 + y_1A_{n-1}^{31})).
\]

By (2.22)–(2.25) and Corollary 2.5.2 we have
\[
\psi_2(y_3A_{n}^{12}) = \psi_2(y_3)(\psi_2(x_2) + \psi_2(y_1A_{n-1}^{31})) = \psi_2(y_3)\psi_2(y_1A_{n-1}^{31}).
\]

By the induction hypothesis, from (6) it follows that
\[
\psi_2(y_3A_{n}^{12}) = y_0(y_2(x_3 + y_1)a_{n}^{13})
\]

because \( \psi_2(y_1A_{n}^{31} - 1) = y_0(y_2a_{n}^{13}) \). \( \square \)

(6) We have
\[
\psi_2(y_1A_{n}^{31}) = \psi_2(y_1(y_3 + x_1A_{n}^{23} + 1)) = \psi_2(y_1y_3 + x_1A_{n}^{23} - 1) = \psi_2(y_1y_3) + \psi_2(x_1)\psi_2(a_{n-1}^{23} - 1).
\]

By the induction hypothesis, from (2) it follows that
\[
\psi_2(y_1A_{n}^{31}) = y_0(y_2(x_1 + x_3) + y_2y_3A_{n-1}^{12}).
\]

From Table 2.1, property 4.1 we have
\[
\psi_2(y_1A_{n}^{31}) = y_0(y_2(x_1 + x_3) + y_2y_3a_{n-1}^{21}).
\]

Finally, from Table 2.1, property 7.1 \( \psi_2(y_1A_{n}^{31}) = y_0(y_2a_{n+1}^{13}) \). In this case \( \psi_k \) increases the lower index of the atomic element \( A_{n}^{ij} \). \( \square \)

From Propositions 2.4.3 and 2.7.1 we have

**Corollary 2.7.4 (Action of the Elementary Maps).** The elementary maps \( \varphi_i \) applied to the atomic elements \( a_{n}^{ij} \) and \( A_{n}^{ij} \) (\( n \geq 1 \)) satisfy the following relations:

1. \( \varphi_i \rho \varphi^+(a_{n}^{ij}) = \rho X(y_1A_{n}^{k}) \),
2. \( \varphi_j \rho \varphi^+(a_{n}^{ij}) = \left\{ \begin{array}{ll} \rho X(y_jA_{n}^{k}) & \text{for } n > 1, \\
\rho X(y_jA_{n+1}^{k}) & \text{for } n = 1. \end{array} \right. \)
3. \( \varphi_k \rho \varphi^+(y_ia_{n}^{ij}) = \rho X(y_kA_{n+1}^{ij}) \),
4. \( \varphi_i \rho \varphi^+(A_{n}^{ij}) = \rho X(y_1(y_j + y_k)a_{n}^{ij}) \),
5. \( \varphi_j \rho \varphi^+(y_ka_{n}^{ij}) = \rho X(y_j(x_k + y_i)a_{n}^{ij}) \),
6. \( \varphi_k \rho \varphi^+(y_1A_{n}^{ij}) = \rho X(y_kA_{n+1}^{ij}) \).
2.8. \( \varphi_i \)-homomorphic theorem

**Theorem 2.8.1.** (1) The polynomials \( a_{ij}^{ij} \) are \( \varphi_i \) - and \( \varphi_j \)-homomorphic.

(2) The polynomials \( A_n^{ij} \) are \( \varphi_i \)-homomorphic.

(3) The polynomials \( A_n^{ij} \) are \( (\varphi_j, y_k) \)-homomorphic.

**Proof.** (1), (2) follow from multiplicity (2.22)–(2.24), Propositions 2.7.1 and 2.4.3. □

(3) For convenience, without loss of generality, we let \( i = 1, j = 2, k = 3 \). We need to prove that

\[
\varphi_2 \rho_X + (A_n^{ij} p) = \varphi_2 \rho_X + (y_3 A_n^{ij} \varphi_2 \rho_X + (p)) \quad \text{for every} \quad p \subseteq y_3.
\]

(2.27)

Since \( \psi_2(A_n^{ij} p) = \psi_2((y_1 + x_2 A_n^{31}) p) \) and \( p \subseteq y_3 \subseteq A_n^{31} \), we have by the permutation property (A.2):

\[
\psi_2(A_n^{ij} p) = \psi_2((x_2 + y_1 A_n^{31}) p) = \psi_2(x_2 + y_1 A_n^{31}) \psi_2(p).
\]

The element \( x_2 \) can be dropped according to (2.25), and hence,

\[
\psi_2(A_n^{ij} p) = \psi_2(y_1 A_n^{31}) \psi_2(p).
\]

By property (6) from Proposition 2.7.1 we have

\[
\psi_2(A_n^{ij} p) = v^0(y_2 a_n^{13}) \psi_2(p).
\]

(2.28)

Since \( y_2 a_n^{13} = y_2(y_1 + y_3) a_n^{13} \), we have \( v^0(y_2 a_n^{13}) = v^0(y_2(y_1 + y_3)) v^0(a_n^{13}) \) and by (5) from Proposition 2.4.1 we have \( v^0(y_2 a_n^{13}) = \psi_2(I) v^0(a_n^{13}) \). Further, \( \psi_2(I) \varsupseteq \psi_2(p) \) and (2.28) are equivalent to

\[
\psi_2(A_n^{ij} p) = v^0(a_n^{13}) \psi_2(p).
\]

(2.29)

We have \( \psi_2(p) \subseteq \psi_2(y_3) \) together with \( p \subseteq y_3 \), and therefore \( \psi_2(p) = \psi_2(p) \psi_2(y_3) \). From (2.29) and Proposition 2.4.1 we have

\[
\psi_2(A_n^{ij} p) = v^0(a_n^{13}) \psi_2(p) \psi_2(y_3) = v^0(a_n^{13}) \psi_2(p) v^0(y_2(x_1 + y_3)) = v^0(y_2(x_1 + y_3) a_n^{13}) \psi_2(p).
\]

By (5) from Proposition 2.7.1, \( \psi_2(A_n^{ij} p) = \psi_2(y_3 A_n^{ij}) \psi_2(p) \). Applying projection \( \nabla \) from Section 2.2 we get

\[
\nabla \psi_2(A_n^{ij} p) = \nabla \psi_2(y_3 A_n^{ij}) \nabla \psi_2(p)
\]

(2.30)

because \( \psi_2(y_3 A_n^{ij}) = v^0(y_2(x_1 + y_3) a_n^{13}) \), see Lemma 2.2.1, relation (2.8).

By Proposition 2.4.3 \( \nabla \psi_i(c) = \varphi_i \rho_X + (c) \) for every \( c \subseteq D^{2,2,2} \), thus we get (2.27), and therefore heading (3) of the \( \varphi \)-homomorphic Theorem 2.8.1 is proven. □

The final proposition of this section shows when the lower indices of atomic elements are increased by the action of some elementary mappings \( \varphi_i \):

**Corollary 2.8.2.** The index growth takes place in the following cases:

(1) \( \varphi_k \rho_x + (y_i a_n^{ij}) = \rho_x (y_k A_n^{ij+1}) \);

(2) \( \varphi_k \rho_x + (y_j a_n^{ij}) = \rho_x (y_k a_n^{ij+1}) \).

2.9. The perfectness of \( y_1 + y_j \)

To prove the perfectness of the element \( y_1 + y_2 \), two simple lemmas are necessary.

**Lemma 2.9.1.** Let \( A, B \subseteq X, A, B \) be subspaces in the finite-dimensional vector space \( X \). There exists a subspace \( C \subseteq B \), such that

\[
C + AB = B \quad \text{and} \quad CA = 0.
\]

(2.31)
Indeed, take any direct complement $C$ of $AB$, i.e., $C \oplus AB = B$. If $v \in CA$ is non-zero, then $v \in CA \subseteq BA$ and the sum $C + AB$ is not direct. □

**Lemma 2.9.2.** If $U, V, W \subseteq X$ and $(U + V)W = 0$, then

$$U(V + W) = UV \quad \text{and} \quad V(U + W) = UV.$$  \hspace{1cm} (2.32)

Indeed,

$$U(V + W) \subseteq (U + V)(V + W) = (U + V)W + V = V,$$

i.e., $U(V + W) \subseteq UV$. The inverse inclusion is obvious. Similarly,

$$V(U + W) \subseteq (U + V)(U + W) = U + W(U + V) = U,$$

i.e., $V(U + W) \subseteq UV$ and the inverse inclusion is obvious. □

**Proposition 2.9.3.** (a) The element $y_i + y_j$ where $i \neq j$, is perfect in $D^{2,2,2}$.  

(b) If $\rho_X(y_i + y_j) = 0$ for some indecomposable representation $\rho_X$, where $i \neq j$, then $\rho_X$ is one from the seven projective representations (Table A.1).

**Proof.** (a) Consider $y_1 + y_2$. Let $B = X_3, A = Y_1 + Y_2$ in (2.31), then there exists a subspace $C \subseteq X_3$ such that

$$C + X_3(Y_1 + Y_2) = X_3, \quad C(Y_1 + Y_2) = 0.$$ \hspace{1cm} (2.33)

Therefore $Y_3 \supseteq C + Y_3(Y_1 + Y_2)$. Again use (2.31) with

$$B = Y_3, \quad A = C + Y_3(Y_1 + Y_2).$$

Then, there exists a subspace $D \subseteq Y_3$ such that

$$D + C + Y_3(Y_1 + Y_2) = Y_3, \quad D(C + Y_3(Y_1 + Y_2)) = 0.$$ \hspace{1cm} (2.34)

Using (2.32) with $U = Y_3(Y_1 + Y_2), V = C$ and $W = D$, we obtain from (2.32) and (2.34):

$$U(V + W) = Y_3(Y_1 + Y_2)(C + D) = CY_3(Y_1 + Y_2).$$

By (2.33) we have $Y_3(Y_1 + Y_2)(C + D) = 0$. Since $D + C \subseteq Y_3$, we see that $(Y_1 + Y_2)(C + D) = 0$. Therefore, $D + C$ and $Y_1 + Y_2$ form a direct sum. We complement this sum to $X_0$:

$$(D + C) \oplus (Y_1 + Y_2) \oplus F = X_0.$$ \hspace{1cm} (2.35)

We will show that the decomposition (2.35) forms a decomposition of the representation $\rho_X$ in $X = X_0$. First,

$$Z = Z(Y_1 + Y_2) + Z(D + C + F) \quad \text{for} \quad Z = Y_1, Y_2, X_1, X_2$$ \hspace{1cm} (2.36)

because $Z \subseteq Y_1 + Y_2$ and the sum (2.35) is direct. By (2.33) we have

$$X_3 \subseteq X_3(Y_1 + Y_2) + X_3(D + C + F) \subseteq X_3,$$

i.e.,

$$X_3 = X_3(Y_1 + Y_2) + X_3(D + C + F).$$ \hspace{1cm} (2.37)

Similarly, by (2.34) we have

$$Y_3 \subseteq Y_3(Y_1 + Y_2) + Y_3(D + C + F) \subseteq Y_3,$$

i.e.,

$$Y_3 = Y_3(Y_1 + Y_2) + Y_3(D + C + F).$$ \hspace{1cm} (2.38)

So, by (2.36)--(2.38) $Y_1 + Y_2$ and $D + C + F$ form a decomposition of the representation $\rho_X$. Since $\rho_X(y_1 + y_2) = Y_1 + Y_2$, we have either $\rho_X(y_1 + y_2) = X_0$ or $\rho_X(y_1 + y_2) = 0$, i.e., $y_1 + y_2$ is a perfect element. □

(b) If $\rho_X(y_1 + y_2) = 0$, then $Y_1 = X_1 = Y_2 = X_2 = 0$ and $\rho_X$ is one of $\rho^{x_0}, \rho^{y_3}, \rho^{z_3}$, see Table A.1. □
Corollary 2.9.4. The elements $y_1 + y_2 + x_3$ and $y_1 + y_2 + y_3$ are perfect.

Proof. Let $Y_1 + Y_2 = X_0 = X$. Then $Y_1 + Y_2 + X_3 = X_0$. If $Y_1 + Y_2 = 0$, then, in the indecomposable representation $\rho_X : D^{2,2,2} \rightarrow \mathcal{L}(X_0)$, we have $X_0 = X_3 = X_3$. The same for the element $y_1 + y_2 + y_3$.

If $\rho_X(y_1 + y_2) = 0$, then $Y_1 = Y_2 = Y_3 = 0$ and $\rho_X$ is one of $\rho_y, \rho_y^3, \rho_y^3$, see Table A.1. \hfill $\Box$

2.10. The elementary map $\varphi_i$: A way to construct new perfect elements

In this section we list three fundamental properties of the elementary map $\varphi_i$. Proposition 2.10.1 gives a way to construct new perfect elements from already existing ones. Proposition 2.10.2 motivates the construction of admissible sequences and admissible elements.

Proposition 2.10.1. Let $z$ be a perfect element in $D^{2,2,2}$ and, for every indecomposable representation $\rho_X$, let

$$\varphi_i \rho_X^+(z) + \varphi_j \rho_X^+(z) = \rho_X(u), \quad \text{where} \ i \neq j. \tag{2.39}$$

Then $u$ is also the perfect element.

Proof. If $\rho_X^+(z) = 0$, then $\rho_X(u) = 0$. If $\rho_X^+(z) = X_0^1$, then by Corollary 2.4.4 we have

$$\rho_X(u) = \varphi_i X_0^1 + \varphi_j X_0^1 = \varphi_i \rho_X^+(I) + \varphi_j \rho_X^+(I) = Y_1(Y_j + Y_k) + Y_j(Y_i + Y_k) = (Y_i + Y_j)(Y_i + Y_k)(Y_j + Y_k)$$

and by Proposition 2.9.3 the element $u$ is also perfect. \hfill $\Box$

Proposition 2.10.2. For $\{i, j, k\} = \{1, 2, 3\}$, the following relations hold

$$\varphi_i \varphi_j \varphi_i + \varphi_i \varphi_k \varphi_i = 0, \tag{2.40}$$

$$\varphi_i^3 = 0. \tag{2.41}$$

Proof. For every vector $v \in X_0^1$, by definition of $\varphi_i$, we have $(\varphi_i + \varphi_j + \varphi_k)(v) = 0$, see Eq. (2.14). In other words, $\varphi_i + \varphi_j + \varphi_k = 0$. Therefore,

$$\varphi_i \varphi_j \varphi_i + \varphi_i \varphi_k \varphi_i = \varphi_i(\varphi_j + \varphi_k)\varphi_i = -\varphi_i^3.$$ 

So, it suffices to prove that $\varphi_i^3 = 0$. For every $z \subseteq D^{2,2,2}$, by headings (1), (3) and (5) of Corollary 2.4.4 we have

$$\varphi_i^3 \rho_X^3(z) \subseteq \varphi_i^2 \rho_X^2(\varphi_i \rho_X^+ (I)) = \varphi_i^2 (\rho_X^2(y_i(y_j + y_k))) \subseteq \varphi_i^2 (\rho_X^2(y_i)) = \varphi_i (\varphi_i \rho_X^2(y_i))$$

$$\varphi_i (\varphi_i \rho_X (x_i(y_j + y_k))) \subseteq \varphi_i \rho_X (x_i) = 0. \hfill \Box$$

Corollary 2.10.3. The relation

$$\varphi_i \varphi_j \varphi_i (B) = \varphi_i \varphi_k \varphi_i (B) \tag{2.42}$$

takes place for every subspace $B \subseteq X^3$, where $X^3$ is the representation space of $\rho_X^3$. \hfill $\Box$

2.11. List of the admissible sequences

Recall that the admissible sequences are introduced in Section 1.7. Let us construct new admissible sequences acting by the elementary map $\varphi_i$.

Let the action of $\varphi_i$ on an admissible sequence $\alpha$ be defined so that

1. The index $i$ is added in front of the sequence $\alpha = i_1i_2 \ldots i_n$, i.e.,

$$\varphi_i(i_1i_2 \ldots i_n) = ii_1i_2 \ldots i_n. \tag{2.43}$$

2. New sequence $\varphi_i(\alpha)$ should also be an admissible sequence, in other words, $i \neq i_1$. 

Table 2.2
For $D^{2.2.7}$ the admissible sequences starting with $i_1 = 1$

| $N$ | Admissible sequence | Action $\varphi_1$ | Action $\varphi_2$ | Action $\varphi_3$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(213)^m(21)^n$</td>
<td>$13(213)^{m-1}(21)^{n+1}$</td>
<td>–</td>
<td>$3(213)^m(21)^n$</td>
</tr>
<tr>
<td>2</td>
<td>$3(213)^m(21)^n$</td>
<td>$13(213)^m(21)^n$</td>
<td>$(213)^m(21)^n$</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>$(213)^m(21)^n$</td>
<td>–</td>
<td>$(213)^m(21)^n$</td>
<td>$3(213)^m(21)^n$</td>
</tr>
<tr>
<td>4</td>
<td>$(312)^m(31)^n$</td>
<td>$12(312)^{m-1}(31)^{n+1}$</td>
<td>$2(312)^m(31)^n$</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>$(231)^m(31)^n$</td>
<td>$12(312)^m(31)^n$</td>
<td>–</td>
<td>$(312)^m(31)^n$</td>
</tr>
<tr>
<td>6</td>
<td>$(123)^m(31)^n$</td>
<td>–</td>
<td>$2(312)^m(31)^n$</td>
<td>$(312)^m(31)^n$</td>
</tr>
<tr>
<td>7</td>
<td>$(1(21)^n = (1(31)^n) $</td>
<td>–</td>
<td>$(21)^n+1$</td>
<td>$(31)^n+1$</td>
</tr>
</tbody>
</table>

The rule $i \neq i_1$ in the definition of admissible sequence is motivated by the property $\varphi_i = -(\varphi_j + \varphi_k)$, so that $\varphi_i$ can be always replaced by $\varphi_j + \varphi_k$.

**Proposition 2.11.1.** The admissible sequence starting with $i_1 = 1$ may be transformed to one of the seven types given in Table 2.2, column 1 (similarly for $i_1 = 2$ and $i_1 = 3$, see (1.27)). Here, $n \in \{1, 2, 3, \ldots\}$, $m \in \{0, 1, 2, 3, \ldots\}$. In case 7, $n \geq 0$. In cases 1 and 4, $m > 0$.

For $m = 0$ cases 1 and 4 are not given in the table: in case 1 (resp. case 4), action $\varphi_1$ transforms $(21)^n$ to $(1(21)^n$ (resp. $(31)^n$ to $(1(31)^n$).

**Proof.** It suffices to prove that every action of $\varphi_i$ on any admissible sequence from the list (Table 2.2, column 1) gives us again an admissible sequence from the same list.

**Line 1**, action $\varphi_1$. Applying $\varphi_1$ to the sequence $(213)^m(21)^n$ we get

$$1(213)^m(21)^n = 1(213)(213)^{m-1}(21)^n.$$  

Since $1213(2\ldots) = 1321(2\ldots)$, we see that $1(213)^m(21)^n = 132(1(213)^{m-1}(21)^n)$. By the induction hypothesis

$$1(213)^{m-1}(21)^n = 13(213)^{m-2}(21)^{n+1},$$  

so

$$1(213)^m(21)^n = 132(13(213)^{m-2}(21)^{n+1}) = 13(213)^{m-1}(21)^{n+1},$$  

i.e., we get a sequence from Line 3.

**Line 1**, action $\varphi_3$. Applying $\varphi_3$ to the sequence $(213)^m(21)^n$ we just get a sequence from Line 2.

**Line 2**, action $\varphi_1$. Action $\varphi_1$ on the $(213)^m(21)^n$ leads to Line 3.

**Line 2**, action $\varphi_2$. Applying $\varphi_2$ to the sequence $3(213)^m(21)^n$, we get

$$23(213)^m(21)^n = 23(213)(213)^{m-1}(21)^n.$$  

Since $23213\ldots = 21323\ldots$, we have $23(213)^m(21)^n = 213(23(213)^{m-1}(21)^n)$. Again, by the induction hypothesis

$$23(213)^{m-1}(21)^n = (213)^{m-1}(21)^{n+1},$$  

so

$$23(213)^m(21)^n = 213((213)^{m-1}(21)^{n+1}) = (213)^m(21)^{n+1},$$  

i.e., a sequence from Line 1.

**Line 3**, action $\varphi_2$. Applying $\varphi_2$ we immediately get Line 1.

**Line 3**, action $\varphi_3$. Apply $\varphi_3$ to the sequence $13(213)^m(21)^n$. Again,

$$313(213)^m(21)^n = 313(213)(213)^{m-1}(21)^n = 312(313(213)^{m-1}(21)^n).$$  

By the induction hypothesis

$$313(213)^{m-1}(21)^n = 312(313(213)^{m-1}(21)^{n+1}) = 3(213)^m(21)^{n+1}. $$  

**Lines 4, 5, 6.** Similarly as for Lines 1, 2, 3.

**Line 7.** Applying $\varphi_2$ and $\varphi_3$ we get Line 1 and Line 4 with $m = 0$, respectively: $(21)^{n+1}$ and $(31)^{n+1}. $
Table 2.3:
For $D^{2,2,2}$: the admissible elements starting with $i_1 = 1$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha$</th>
<th>$f_\alpha$</th>
<th>$e_\alpha$</th>
<th>$g_{a0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\gamma$</td>
<td>$y_1y_2a^{13}<em>q a^{12}</em>{k-1}$</td>
<td>$y_2A^{12}_{k-1}a^{21}_q A^{13}_q a^{13}_q$</td>
<td>$e_\alpha(x_1 + a^{32}_q A^{12}_q)$</td>
</tr>
<tr>
<td>2</td>
<td>$3\gamma$</td>
<td>$y_3(x_1 + x_2)A^{23}<em>q a^{31}</em>{k-1}$</td>
<td>$y_3a^{31}_{k-1}A^{12}_q + a^{13}_q A^{23}_q$</td>
<td>$e_\alpha(y_2) + A^{12}_q a^{31}_q$</td>
</tr>
<tr>
<td>3</td>
<td>$13\gamma$</td>
<td>$y_3y_1a^{32}_{q+1} A^{13}_k$</td>
<td>$y_1A^{21}_{k-1}a^{13}_q + a^{32}_q A^{13}_q a^{13}_q$</td>
<td>$e_\alpha(x_3 + a^{21}_q A^{13}_k)$</td>
</tr>
<tr>
<td>4</td>
<td>$(213)\gamma$</td>
<td>$y_2(x_3 + x_1)A^{12}_q a^{13}_k$</td>
<td>$y_2a^{12}_k A^{31}_q + a^{32}_q A^{13}_q a^{12}_q + a^{13}_q A^{23}_q$</td>
<td>$e_\alpha(y_1) + A^{31}_q a^{12}_q$</td>
</tr>
<tr>
<td>5</td>
<td>$(3213)\gamma$</td>
<td>$y_2y_3a^{21}_{q+2} A^{13}_k$</td>
<td>$y_2A^{13}_k a^{32}_k + a^{21}_q A^{13}_q a^{32}_k$</td>
<td>$e_\alpha(x_2 + a^{13}_q A^{23}_q)$</td>
</tr>
<tr>
<td>6</td>
<td>$13(213)\gamma$</td>
<td>$y_1(x_2 + x_3)A^{31}<em>q a^{12}</em>{k-1}$</td>
<td>$y_1a^{12}_k A^{23}_q + a^{31}_q A^{13}_q a^{12}_q + a^{13}_q A^{23}_q$</td>
<td>$e_\alpha(y_1) + A^{23}_q a^{12}_q$</td>
</tr>
<tr>
<td>7</td>
<td>$(121)^{2k}$</td>
<td>$x_1A^{23}_q a^{12}_k$</td>
<td>$y_1A^{12}_k A^{31}_q + a^{32}_q A^{13}_q a^{12}_q + a^{13}_q A^{23}_q$</td>
<td>$e_\alpha(y_2 a^{12}_q + y_3 a^{31}_q)$</td>
</tr>
<tr>
<td>8</td>
<td>$(121)^{2k+1}$</td>
<td>$y_1(x_2 + x_3)A^{12}_k a^{31}_q$</td>
<td>$y_1A^{12}_k a^{31}_q a^{31}_q + a^{23}_q A^{13}_q a^{12}_q$</td>
<td>$e_\alpha(y_1 a^{31}_q a^{31}_q + y_1 A^{13}_q a^{13}_q)$</td>
</tr>
<tr>
<td>9</td>
<td>$\beta$</td>
<td>$y_2y_3a^{21}_q A^{13}_k$</td>
<td>$y_2A^{21}_k a^{13}_q a^{32}_q a^{13}_q + a^{13}_q A^{13}_q$</td>
<td>$e_\alpha(x_2 + a^{13}_q A^{13}_q)$</td>
</tr>
<tr>
<td>10</td>
<td>$3\beta$</td>
<td>$x_3 A^{12}_k a^{31}_q$</td>
<td>$y_3a^{31}_q A^{12}_k a^{12}_q A^{23}_q$</td>
<td>$e_\alpha(y_1 + A^{31}_q A^{12}_q)$</td>
</tr>
<tr>
<td>11</td>
<td>$13\beta$</td>
<td>$y_1y_2a^{13}_q A^{12}_k$</td>
<td>$y_1A^{12}_k A^{13}_q a^{12}_q a^{13}_q + a^{31}_q A^{13}_q a^{12}_q$</td>
<td>$e_\alpha(x_1 + a^{32}_q A^{13}_q)$</td>
</tr>
<tr>
<td>12</td>
<td>$(213)\beta$</td>
<td>$x_2A^{12}_q a^{13}_q a^{31}_q$</td>
<td>$y_2a^{31}_q A^{13}_q a^{12}_q + a^{13}_q A^{23}_q a^{12}_q$</td>
<td>$e_\alpha(y_2 + A^{12}_q a^{31}_q A^{13}_q)$</td>
</tr>
<tr>
<td>13</td>
<td>$3(213)\beta$</td>
<td>$y_3y_1a^{32}_q A^{21}_k$</td>
<td>$y_3a^{32}_q A^{21}_k a^{21}_q A^{21}_k a^{13}_q$</td>
<td>$e_\alpha(x_3 + a^{21}_q A^{21}_k)$</td>
</tr>
<tr>
<td>14</td>
<td>$13(213)\beta$</td>
<td>$x_1A^{23}_q a^{12}_k a^{31}_q$</td>
<td>$y_1A^{12}_k A^{23}_q a^{31}_q a^{12}_q A^{21}_k a^{13}_q$</td>
<td>$e_\alpha(y_1 + A^{31}_q a^{21}_q A^{21}_k)$</td>
</tr>
</tbody>
</table>

In Lines 1–6: $\gamma = (213)^2 p (21)^{2k}$, $k > 0$, $p \geq 0$. In Lines 7–8: $k \geq 0$.
In Lines 9–14: $\beta = (213)^2 p (21)^{2k+1}$, $k \geq 0$, $p \geq 0$. In all lines, $q = k + 3p$.

2.12. The theorem on the admissible element classes

The lattice polynomials indexed by admissible sequences are also said to be admissible. We define admissible elements $f_\alpha$, $e_\alpha$, $g_{a0}$ in Table 2.3. By definition $g_0 = I$.

Theorem 2.12.1 (On Classes of Admissible Elements). Let $\alpha = i_n i_{n-1} \ldots 1$ be an admissible sequence and $i \neq i_n$. Then $i\alpha$ is admissible and, for $z_\alpha = f_\alpha, e_\alpha, g_{a0}$ from Table 2.3, the following relation holds:

$$
\varphi_i \rho X(z_\alpha) = \rho X(z_{i\alpha}).
$$

(2.44)

For the proof of Theorem 2.12.1, see [51, Appendix B]. □

2.13. The inclusion theorem and the cumulative polynomials

Theorem 2.13.1. For every admissible sequences $\alpha, \alpha_i$, $(i = 1, 2, 3)$ from Table 2.2, the following inclusion holds

$$
e_{\alpha_i} \subseteq g_{a0}, \quad i = 1, 2, 3.
$$

(2.45)

For the proof of Theorem 2.13.1, see Appendix B.3. □

It is easy to check by the definitions from Table 2.3 that

$$
f_\alpha \subseteq e_\alpha, \quad g_{a0} \subseteq e_\alpha.
$$

(2.45)

So, from Theorem 2.13.1 (Inclusion Theorem) and inclusions (2.45) we get

$$
f_{\alpha_i} \subseteq e_{\alpha_i} \subseteq g_{a0}.
$$

(2.46)

\footnote{Please do not confuse polynomials $e_{\alpha_i}$ considered here with polynomials $e_{i\alpha}$ considered, for example, in Theorem 2.12.1.}
Recall the definition of cumulative polynomials from Section 1.10. Cumulative polynomials of length \( n \) are constructed as sums of all admissible elements of the same length \( n \), where \( n \) is the length of the multi-index:

\[
\begin{align*}
x_t(n) &= \sum f_{i_n \ldots i_2 i_1}, \quad t = 1, 2, 3, \\
y_t(n) &= \sum e_{i_n \ldots i_2 i_1}, \quad t = 1, 2, 3, \\
x_0(n) &= \sum g_{i_n \ldots i_2 i_1}.
\end{align*}
\]

From (2.46) and Theorem 2.12.1 the next inclusions take place:

**Corollary 2.13.2.**

\[
x_t(n) \subseteq y_t(n) \subseteq x_0(n), \quad t = 1, 2, 3. \tag{2.47}
\]

Thus, cumulative polynomials satisfy the same inclusions as the corresponding generators in \( D^{2,2,2} \).

**Proposition 2.13.3 (On the Cumulative Polynomials).** If \( z_t(n) \) is one of the cumulative polynomials \( x_t(n), y_t(n)(t = 1, 2, 3) \) or \( x_0(n) \), then

\[
\sum_{i=1,2,3} \phi_i \rho_{X^+}(z_t(n)) = \rho_X(z_t(n + 1)). \tag{2.48}
\]

**Proof.** By Theorem 2.12.1

\[
\phi_i \rho_{X^+}(z_{\alpha}) = \rho_X(z_{i\alpha}) \quad \text{for every } z_{\alpha} = f_{\alpha}, e_{\alpha}, g_{\alpha 0}.
\]

The latter relation is true only under the condition that \( i \alpha \) is admissible, i.e., \( i \neq i_n \) for \( \alpha = i_n i_{n-1} \ldots i_1 \). By (2.15) in the sum of (2.48) we can always exclude one of the elementary maps \( \phi_1, \phi_2, \phi_3 \) such that \( i \neq i_n \). \( \square \)

3. Perfect polynomials in \( D^{2,2,2} \)

**3.1. Perfectness, distributivity and cardinality of \( H^+(n) \)**

Recall the definition of the elements \( a_i(n), b_i(n), c_i(n) \). The perfect elements (see Appendix A.2.1) are constructed in the following way, see Definition (1.34) from Section 1.11.1:

For \( n = 0 \):

\[
\begin{align*}
a_0(0) &= y_j + y_k, \\
b_0(0) &= x_i + y_j + y_k, \\
c_0(0) &= c_2(0) = c_3(0) = \sum y_i.
\end{align*}
\]

For \( n \geq 1 \):

\[
\begin{align*}
a_i(n) &= x_j(n) + x_k(n) + y_j(n + 1) + y_k(n + 1), \\
b_i(n) &= a_i(n) + x_i(n + 1) = x_j(n) + x_k(n) + x_i(n + 1) + y_j(n + 1) + y_k(n + 1), \\
c_i(n) &= a_i(n) + y_i(n + 1) = x_j(n) + x_k(n) + y_i(n + 1) + y_j(n + 1) + y_k(n + 1).
\end{align*}
\]

By definition we have

\[
a_i(n) \subseteq b_i(n) \subseteq c_i(n). \tag{3.2}
\]

The perfectness of the elements \( a_0(0), b_0(0), c_0(0) \) is proved in Proposition 2.9.3 and Corollary 2.9.4. The perfectness of the elements \( a_1(1), b_1(1), c_1(1) \) is proved in Sections C.2.1–C.2.3 of [51].

**Proposition 3.1.1.** (a) For every element \( v_i(n) = a_i(n), b_i(n), c_i(n) \), the following relation holds:

\[
\rho_X(v_i(n)) = \sum_{k=1,2,3} \phi_k \rho_{X^+}(v_i(n - 1)). \tag{3.3}
\]
(b) The elements \( a_i(n), b_i(n), c_i(n) \) are perfect for every \( n \geq 1 \) and \( i = 1, 2, 3 \).

The proof of Proposition 3.1.1 is given in Section C.2.4 of [51]. □

Let us consider sublattice \( H^+(0) \) generated by 3 chains

\[
s_i(0) = \{ a_i(0) \subseteq b_i(0) \}, \quad i = 1, 2, 3
\]

and sublattice \( H^+(n) \) (\( n \geq 1 \)) generated by 3 chains

\[
s_i(n) = \{ a_i(n) \subseteq b_i(n) \subseteq c_i(n) \}, \quad i = 1, 2, 3.
\]

**Proposition 3.1.2.** Every element of the lattice \( H^+(n) \), where \( n \geq 0 \), is of the form \( v_1v_2v_3 \), where \( v_i \in \{ a_i(n), b_i(n), c_i(n), I_n \} \).

**Proof.** Let \( v_1v_2v_3 \) and \( v'_1v'_2v'_3 \) be two elements of \( H^+(n) \). Then

\[
v_1v_2v_3 + v'_1v'_2v'_3 = \bigcap_{i=1}^{3} \sup(v_i, v'_i).
\]

Indeed, if \( v_i \supseteq v'_i \) for every \( i = 1, 2, 3 \), then \( v_1v_2v_3 + v'_1v'_2v'_3 = v_1v_2v_3 \). If \( v_1 \supseteq v'_1 \) and \( v_2 \subseteq v'_2, v_3 \subseteq v'_3 \), then by (3.77) we have

\[
v_1v_2v_3 + v'_1v'_2v'_3 = v_1v'_2v'_3(v_2v_3 + v'_1) = v_1v'_2v'_3.
\] □

**Corollary 3.1.3.** (a) The lattice \( H^+(0) \) contains \( \leq 27 \) elements.

(b) The lattice \( H^+(n) \) for \( n > 0 \) contains \( \leq 64 \) elements.

According to definition of perfect elements (see Appendix A.2.1), \( H^+(n) \) forms a sublattice of perfect elements in \( D_{2.2}^{+} \).

**Proposition 3.1.4.** (a) \( H^+(n) \) is a distributive sublattice for every \( n \geq 0 \).

(b) \( |H^+(0)| = 27 \) and \( |H^+(n)| = 64 \), for each \( n \geq 1 \).

**Proof.** For the proof of (a) see Section 3.8 below.

(b) The equalities \( |H^+(0)| = 27 \) and \( |H^+(n)| = 64 \) are proved in Section C.2.6 of [51].

Assume that \( n \geq 1 \), and that \( |H^+(n)| = 64 \). We show that \( |H^+(n+1)| = 64 \). Let \( v_1(n)v_2(n)v_3(n) \) and \( v'_1(n)v'_2(n)v'_3(n) \) be two distinct elements in the \( H^+(n) \) and \( \rho_X \) is the preprojective representation separating them. For \( n = 1 \), the representation \( \rho_X \) is given by Tables C.6, C.7 from [51]. For example,

\[
\begin{align*}
\rho_X(v_1(n)v_2(n)v_3(n)) &= 0, \\
\rho_X(v'_1(n)v'_2(n)v'_3(n)) &= X^1_0.
\end{align*}
\]

Therefore,

\[
\text{there exists an } i \text{ such that } \rho_X(v_i(n)) = 0, \quad (3.8)
\]

and

\[
\rho_X(v'_i(n)) = X^1_0 \quad \text{for every } i. \quad (3.9)
\]

For every preprojective representation \( \rho = \rho_X \), there exists preprojective representation \( \tilde{\rho} \) such that \( \Phi^+ \tilde{\rho} = \rho \).

(Naturally, \( \tilde{\rho} = \Phi^- \rho \), see Appendix A.2.2, [2]). We will write \( \tilde{\rho} = \rho_X^- \). Thus,

\[
\Phi^+ \rho_X^- = \rho_X.
\]

Then, by (3.3) for \( i \) from (3.8) we have

\[
\rho_X^-(v_i(n+1)) = \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^-(v_i(n)) = \sum_{p=1,2,3} \varphi_p \rho_X(v_i(n)) = 0, \quad (3.10)
\]
i.e.,
\[ \rho_X^{-}(v_1(n+1)v_2(n+1)v_3(n+1)) = 0. \] (3.11)

On the other hand,
\[ \rho_X^{-}(v'_i(n+1)) = \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^{-}(v'_i(n)) = \sum_{p=1,2,3} \varphi_p \rho_X(v'_i(n)) = \sum_{p=1,2,3} \varphi_p X_0^n. \] (3.12)

By Corollary 2.4.4 we have
\[ \sum_{p=1,2,3} \varphi_p X_0^n = \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^{-}(I) \]
\[ = \rho_X^{-}(y_1(y_2+y_3) + y_2(y_1+y_3) + y_3(y_1+y_2)) = \rho_X^{-}(\bigcap_{i \neq j}(y_i + y_j)). \] (3.13)

Since \( y_i + y_j \) is perfect and \( \rho_X^{-} = \Phi^- \rho \) is non-projective, by Proposition 2.9.3 we have
\[ \rho_X^{-}(y_i+y_j) = X_0 \quad \text{for every } \{i, j\}, i \neq j. \] (3.14)

Thus, by (3.12)–(3.14)
\[ \rho_X^{-}(v'_i(n+1)) = X_0 \quad \text{for every } i, \]
and therefore
\[ \rho_X^{-}(v'_1(n+1)v'_2(n+1)v'_3(n+1)) = X_0. \] (3.16)

Thus, by (3.11) and (3.16) the representation \( \rho_X^{-} \) separates \( v_1(n+1)v_2(n+1)v_3(n+1) \) and \( v'_1(n+1)v'_2(n+1)v'_3(n+1) \).

\[ \square \]

**Corollary 3.1.5.** \( H^+(n) \) is a distributive 64-element lattice for \( n \geq 1 \). It is the direct product (Fig. 1.5) of three chains \( \{a_i(n) \subseteq b_j(n) \subseteq c_i(n)\}. \) \( \square \)

### 3.2. The additive form for elements of \( H^+(n) \)

#### 3.2.1. Sums and intersections

**Proposition 3.2.1.** (a) For every perfect elements \( v_i, v_j, v_k \), the following relation holds
\[ \sum_{p=1,2,3} \varphi_p \rho_X^+(\bigcap_{t=i,j,k} v_t) = \bigcap_{t=i,j,k} \left( \sum_{p=1,2,3} \varphi_p \rho_X^+(v_t) \right). \] (3.17)

(b) For every perfect element \( v \) and every \( u \), the following relation holds
\[ \sum_{p=1,2,3} \varphi_p \rho_X^+(vu) = \sum_{p=1,2,3} \varphi_p \rho_X^+(v) \sum_{p=1,2,3} \varphi_p \rho_X^+(u). \] (3.18)

**Proof.** (a) If \( \rho_X^+(v_t) = 0 \) for some \( t \), then
\[ \rho_X^+\left(\bigcap_{t=i,j,k} v_t\right) = 0 \quad \text{and} \quad \sum_{p=1,2,3} \varphi_p \rho_X^+(v_t) = 0 \]
and (3.17) is true. If \( \rho_X^+(v_t) = X_0^n \) for every \( t \), then by Corollary 2.4.4 and (3.13) we have
\[ \rho_X^+\left(\bigcap_{t=i,j,k} v_t\right) = X_0^n = \rho_X^+(I), \]
\[ \sum_{p=1,2,3} \varphi_p \rho_X^+(v_t) = \rho_X^+\left(\bigcap_{i \neq j}(y_i + y_j)\right). \] (3.19)
Hereafter
\[ \bigcap_{i \neq j} (y_i + y_j) \text{ means } (y_1 + y_2)(y_1 + y_3)(y_2 + y_3). \]

From (3.19) we have
\[ \sum_{p=1,2,3} \varphi_p \rho X^+ \left( \bigcap_{t=i,j,k} v_t \right) = \sum_{p=1,2,3} \varphi_p \rho X^+(I) = \rho X \left( \bigcap_{i \neq j} (y_i + y_j) \right), \]  
(3.20)
and again (3.17) is true. Note that by Proposition 2.9.3 in (3.20) \( \rho X(\bigcap(y_i + y_j)) = X_0 \) except for the case where \( \rho X \) is a projective representation (Table A.1), for which \( \rho X(\bigcap(y_i + y_j)) = 0 \).

(b) As in (a), if \( \rho X^+(vu) = 0 \), then
\[ \rho X^+(vu) = 0 \quad \text{and} \quad \sum_{p=1,2,3} \varphi_p \rho X^+(v) = 0 \]
and (3.18) is true.

If \( \rho X^+ \) is projective, then \( \rho X^+ = \Phi^+ \rho X = 0 \) and (3.18) is true as well.

Consider the case \( \rho X^+ = \Phi^+ \rho X \neq 0 \), i.e., \( \rho X \) is not a projective representation. If \( \rho X^+(v) = X_0^1 \), then \( \rho X^+(vu) = \rho X^+(u)X_0^1 = \rho X^+(u) \) and
\[ \sum_{p=1,2,3} \varphi_p \rho X^+(vu) = \sum_{p=1,2,3} \varphi_p \rho X^+(u). \]  
(3.21)

On the other hand,
\[ \sum_{p=1,2,3} \varphi_p \rho X^+(v) = \sum_{p=1,2,3} \varphi_p \rho X^+(I) = \rho \left( \bigcap_{i \neq j} (y_i + y_j) \right) \]  
(3.22)
and by Proposition 2.9.3 we have \( \sum \varphi_p \rho X^+(v) = X_0 \) and
\[ \sum_{p=1,2,3} \varphi_p \rho X^+(v) \sum_{p=1,2,3} \varphi_p \rho X^+(u) = \left( \sum_{p=1,2,3} \varphi_p \rho X^+(u) \right) X_0 = \sum_{p=1,2,3} \varphi_p \rho X^+(u). \]  
(3.23)
So, (3.18) is true. \( \square \)

All considerations given below up to the end of Section 3.2 are represented modulo linear equivalence, see Appendix A.2.1.

Proposition 3.2.2. The next relation takes place modulo linear equivalence
\[ x_0(n + 2) \simeq \bigcap_{i=1,2,3} a_i(n). \]  
(3.24)

Proof. We will prove (3.24) for \( n = 0, 1 \) without restriction mod \( \theta \).

Case \( n = 0 \). By Table 2.3, Line 7, \( k = 0 \), we have
\[ x_0(2) = g_{10} + g_{20} + g_{30} = \sum_{r,s,t} y_r(y_s + y_t) = \bigcap_{r \neq s} (y_r + y_s). \]  
(3.25)
The sum is taken over all permutations \( \{r, s, t\} \) of \( \{1, 2, 3\} \). On the other hand, by Table C.5 from [51],
\[ \bigcap_{i=1,2,3} a_i(0) = \bigcap_{r \neq s} (y_r + y_s). \]  
(3.26)
Case $n = 1$. We have
\[ x_0(3) = g_{210} + g_{120} + g_{310} + g_{130} + g_{230} + g_{320}. \]  
(3.27)

By Table 2.3, Line 9, $k = 0$, $p = 0$, $q = 0$, we have, for example,
\[ g_{210} = y_2 q_1^{13} (x_2 + A_1^{13}) = y_2 (x_1 + y_3) (x_2 + x_3 + y_1). \]

From here,
\[ g_{210} + g_{310} = y_2 (x_1 + y_3) (x_2 + x_3 + y_1) + y_3 (x_1 + y_2) (x_2 + x_3 + y_1) = \]
\[ (x_1 + y_3) (x_1 + y_2) [y_2 (x_2 + x_3 + y_1) + y_3 (x_2 + x_3 + y_1)] = \]
\[ (x_1 + y_3) (x_1 + y_2) [x_2 + y_2 (x_3 + y_1) + x_3 + y_3 (x_2 + y_1)] = \]
\[ (x_1 + y_3) (x_1 + y_2) [(x_3 + y_2) (x_3 + y_1) + (x_2 + y_3) (x_2 + y_1)]. \]

Thus,
\[ g_{210} + g_{310} = (x_1 + y_3) (x_1 + y_2) [(x_3 + y_2) (x_3 + y_1) + (x_2 + y_3) (x_2 + y_1)], \]
\( g_{120} + g_{320} = (x_2 + y_3) (x_2 + y_1) [(x_3 + y_1) (x_3 + y_2) + (x_1 + y_3) (x_1 + y_2)], \)
\( g_{130} + g_{230} = (x_3 + y_2) (x_3 + y_1) [(x_2 + y_1) (x_2 + y_3) + (x_1 + y_2) (x_1 + y_3)]. \)

On the other hand, by (3.80) we have
\[ t_1(1) = x_1(1) + y_1(2) = x_1 + e_{21} + e_{31} = \]
\[ x_1 + y_2 (x_1 + y_3) + y_3 (x_1 + y_2) = (x_1 + y_2) (x_1 + y_3), \]
i.e.,
\[ g_{210} + g_{310} = t_1(1) (t_2(1) + t_3(1)), \]
\( g_{120} + g_{320} = t_2(1) (t_1(1) + t_3(1)), \)
\( g_{130} + g_{230} = t_3(1) (t_1(1) + t_2(1)), \)
(3.29)

and therefore
\[ x_0(3) = \bigcap_{i \neq j} (t_i(1) + t_j(1)) = \bigcap_{k=1,2,3} a_k(1). \]  
(3.30)

For $n = 0$, $1$, relation (3.24) is true in $D^{2,2,2}$ not only in $D^{2,2,2}/\theta$.

Induction step. As in (3.10),
\[ \rho_X^{-} (x_0(n + 1)) = \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^{-} (x_0(n)) = \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^{-} \left( \bigcap_{i=1,2,3} a_i(n - 2) \right). \]  
(3.31)

By (3.17), Proposition 3.2.1
\[ \rho_X^{-} (x_0(n + 1)) = \bigcap_{i=1,2,3} \left( \sum_{p=1,2,3} \varphi_p \Phi^+ \rho_X^{-} a_i(n - 2) \right), \]  
or
\[ \rho_X^{-} (x_0(n + 1)) = \bigcap_{i=1,2,3} \left( \sum_{p=1,2,3} \varphi_p \rho_X^+ a_i(n - 2) \right). \]  
(3.32)

From here, by (3.3), Proposition 3.1.1
\[ \rho_X (x_0(n + 1)) = \bigcap_{i=1,2,3} \rho_X (a_i(n - 1)), \]  
(3.33)

which is to be proved. \( \square \)

Corollary 3.2.3. For $n \geq 2$, the elements $x_0(n)$ are perfect.
Table 3.1
The multiplicative and additive forms of $H^+(n)$

<table>
<thead>
<tr>
<th>$N$</th>
<th>The multiplicative form</th>
<th>The additive form</th>
<th>The number of polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1(n)$</td>
<td>$s_j(n) + s_k(n)$</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>$b_1(n)$</td>
<td>$s_j(n) + s_k(n) + p_l(n)$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>$c_1(n)$</td>
<td>$s_j(n) + q_k(n)$</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>$a_1(n)a_2(n)$</td>
<td>$s_j(n)$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>$b_1(n)b_2(n)$</td>
<td>$s_j(n) + p_l(n) + q_j(n)$</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>$c_1(n)c_2(n)$</td>
<td>$s_j(n) + q_k(n)$</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>$a_1(n)b_2(n)$</td>
<td>$s_j(n) + q_j(n)$</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>$a_1(n)c_2(n)$</td>
<td>$s_j(n) + p_l(n)$</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>$b_1(n)c_2(n)$</td>
<td>$s_j(n) + q_j(n)$</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>$a_1(n)a_2(n)b_2(n)$</td>
<td>$p_k(n)$</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
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<td>$q_k(n)$</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
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<td>3</td>
</tr>
<tr>
<td>13</td>
<td>$b_1(n)b_2(n)c_2(n)$</td>
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<td>3</td>
</tr>
<tr>
<td>14</td>
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<td>3</td>
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<td>3</td>
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<tr>
<td>16</td>
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<td>$p_l(n) + q_j(n)$</td>
<td>6</td>
</tr>
<tr>
<td>17</td>
<td>$a_1(n)a_2(n)c_2(n)$</td>
<td>$x_0(n+2)$</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
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<td>$p_l(n) + p_j(n) + p_k(n)$</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>$c_1(n)c_2(n)c_2(n)$</td>
<td>$q_k(n) + q_j(n) + q_k(n)$</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>$l_n = a_1(n) + a_2(n)$</td>
<td>$s_j(n) + s_l(n) + s_k(n)$</td>
<td>1</td>
</tr>
</tbody>
</table>

for all $[i, j], i \neq j$

**Proof.** It follows from the fact that the intersection of perfect elements is also perfect. □

**Conjecture 3.2.4.** Relation (3.24) takes place in $D^{2.2.2}$ for all integer $n \geq 0$, not only in $D^{2.2.2}/\theta$.

For $n = 0$, this conjecture was proven above in (3.25), (3.26) and, for $n = 1$, it was proven in (3.30).

3.2.2. Other generators: $p_i(n) \subseteq q_i(n) \subseteq s_i(n)$

Let us introduce the following polynomials:

\[
\begin{align*}
p_i(n) &= x_0(n+2) + x_i(n+1), \\
q_i(n) &= x_0(n+2) + y_i(n+1), \\
s_i(n) &= x_0(n+2) + y_i(n+1) + x_i(n).
\end{align*}
\]

(3.34)

Compare two generator systems in Fig. 1.5. The generators $a_i(n) \subseteq b_i(n) \subseteq c_i(n)$ are given by the upper edges of $H^+(n)$. The generators $p_i(n) \subseteq q_i(n) \subseteq s_i(n)$ are given by the lower edges of $H^+(n)$.

**Proposition 3.2.5.** Table 3.1 gives relations between the upper and lower generators. These relations are true mod $\theta$.

For a proof, see Section C.2.7 [51]. □

3.3. Coupling together $H^+(n)$ and $H^+(n+1)$ in $D^{2.2.2}/\theta$

We consider ordering of $H^+(n)$ and $H^+(n+1)$ relative to each other. The way $H^+(n)$ is connected with $H^+(n+1)$ is not as simple as free modular lattice $D^r$ in [19,20], see Sections 1.11.2 and 1.11.5. For the way $H^+(n)$ is connected with $H^+(n+1)$, see the Hasse diagram in Fig. 1.6.

3.4. Boolean algebras $U_n$ and $V_{n+1}$

Consider the lower cube $U_n \subseteq H^+(n)$ and the upper cube $V_{n+1} \subseteq H^+(n+1)$:

\[
\begin{align*}
U_n &= \{b_1b_2b_3, a_1b_2b_3, b_1a_2b_3, b_1b_2a_3, a_1b_2a_3, a_1a_2a_3\}, \\
V_{n+1} &= \{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_1\tilde{c}_2, \tilde{c}_2\tilde{c}_3, \tilde{c}_1\tilde{c}_2\tilde{c}_3, \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3\}.
\end{align*}
\]

(3.35)
Consider case $n = 1$. We will prove that $(\Phi^-)^2 \rho^{x_0}$ (see Table A.1) separates $U_1$ and $V_2$. It is sufficient to prove that

$$(\Phi^-)^2 \rho^{x_0}(z) = X_0 \quad \text{for every } z \subseteq U_1,$$

$$(\Phi^-)^2 \rho^{x_0}(z) = 0 \quad \text{for every } z \subseteq V_2.$$ 

For the generators $a_i(1)$ and $b_i(1)$, we have

$$(\Phi^-)^2 \rho^{x_0}(a_i(1)) = X_j + X_k + \cdots \neq 0.$$ 

The generators $a_i(n)$ and $b_i(n)$ are defined in (1.34); for $n = 1$, these generators are explicitly written in [51, Appendix C, Equation C.19–C.20].
The generators $a_i(1)$, $b_i(1)$ are perfect, i.e.,

$$(\Phi^-)^2 \rho^{x_0}(a_i(1)) = X_0, \quad (\Phi^-)^2 \rho^{x_0}(b_i(1)) = X_0,$$

and therefore the first relation of (3.38) is true.

On the other hand, since

$$c_i(2) = x_j(2) + x_k(2) + \sum_{q=1,2,3} y_q(3)$$

(3.39)

and $\Phi^+(\Phi^-)^2 \rho^{x_0} = \Phi^- \rho^{x_0}$, we have

$$(\Phi^-)^2 \rho^{x_0}(c_i(2)) = \sum_{p=1,2,3} \varphi_p \Phi^- \rho^{x_0}\left(x_j + x_k + \sum_{q=1,2,3} y_q(2)\right).$$

(3.40)

According to Table A.1 we have $\Phi^- \rho^{x_0}(x_j) = 0$ for every $j$, i.e.,

$$(\Phi^-)^2 \rho^{x_0}(c_i(2)) = \sum_{p=1,2,3} \varphi_p \Phi^- \rho^{x_0}\left(\sum_{q=1,2,3} y_q(2)\right).$$

(3.41)

By Table A.1 we have $\rho^{x_0}(y_j) = 0$. Further,

$$\Phi^- \rho^{x_0}(y_j(2)) = \sum_{q=1,2,3} \varphi_q \Phi^+ \Phi^- \rho^{x_0}(y_j) = \sum_{q=1,2,3} \varphi_q \rho^{x_0}(y_j) = 0.$$

Thus, we have $(\Phi^-)^2 \rho^{x_0}(c_i(2)) = 0$ and the second relation of (3.38) is true. The induction step is proved as in Proposition 3.1.4 above. \(\square\)

### 3.6. The Boolean algebra $U_n \cup V_{n+1}$

#### Proposition 3.6.1. $U_n \cup V_{n+1}$ is a 16-element Boolean algebra in $D^{2,2,2}/\theta$.

**Proof.** According to Proposition A.1.7 it suffices to prove relations (A.12)–(A.14) for

$$c_i = c_i(n+1), \quad d_i = a_i(n)b_j(n)b_k(n) = p_j(n) + p_k(n)$$

(see Fig. 3.1 and Table 3.2), i.e.,

$$c_i(n+1) \subseteq p_j(n) + p_k(n).$$

(3.42)
\[ p_j(n) + p_k(n) \subseteq c_i(n + 1) + p_i(n) + p_k(n), \]
\[ (p_j(n) + p_k(n))c_j(n + 1) \subseteq c_i(n + 1) \mod \theta, \]  
(3.43)

(3.44)

where \( \{i, j, k\} \) is a permutation of \( \{1, 2, 3\} \). Relations (3.42) and (3.43) are true in \( D^{2,2,2} \), relation (3.44) we prove only in \( D^{2,2,2}/\theta \).

According to the definition of perfect elements from Appendix A.2.1

\[ c_i(n + 1) = x_j(n + 1) + x_k(n + 1) + \sum_{t=1,2,3} y_t(n + 2). \]

By definition (3.34)

\[ p_j(n) = x_0(n + 2) + x_j(n + 1), \]
\[ p_j(n) + p_k(n) = x_0(n + 2) + x_j(n + 1) + x_k(n + 1). \]

(1) Then the inclusion (3.42) follows from

\[ \sum_{t=1,2,3} y_t(n + 2) \subseteq x_0(n + 2), \]

which, in turn, comes from the Corollary 2.13.2 concerning cumulative polynomials.

(2) Inclusion (3.43) follows from the relation

\[ x_j(n + 1) \subseteq c_i(n + 1). \]

(3) The last inclusion (3.44) is a little more difficult. It is equivalent to

\[ (p_j(n) + p_k(n))(x_j(n + 1) + x_k(n + 1) + \sum_{t=1,2,3} y_t(n + 2)) \]

\[ \subseteq x_j(n + 1) + x_k(n + 1) + \sum_{t=1,2,3} y_t(n + 2) \mod \theta. \]  
(3.46)

From (3.45) and since \( x_k(n + 1) \subseteq p_j(n) \), we see that (3.46) is equivalent to

\[ x_k(n + 1) + \sum_{t=1,2,3} y_t(n + 2) + (p_j(n) + p_k(n))x_i(n + 1) \]

\[ \subseteq x_j(n + 1) + x_k(n + 1) + \sum_{t=1,2,3} y_t(n + 2) \mod \theta. \]  
(3.47)

To prove (3.47), it suffices to check that

\[ (p_j(n) + p_k(n))x_i(n + 1) \subseteq y_i(n + 2) \mod \theta, \]  
(3.48)

i.e.,

\[ a_i(n)b_j(n)b_k(n)x_i(n + 1) \subseteq y_i(n + 2) \mod \theta. \]  
(3.49)

Since \( x_i(n + 1) \subseteq b_j(n)b_k(n) \) (by relation (A.12)), we see that (3.49) is equivalent to

\[ a_i(n)x_i(n + 1) \subseteq y_i(n + 2) \mod \theta. \]  
(3.50)

**Conjecture 3.6.2.** Relation (3.50) takes place in \( D^{2,2,2} \), not only in \( D^{2,2,2}/\theta \). In this case Proposition 3.6.1 is also true in \( D^{2,2,2} \), and \( U_n \cup V_{n+1} \) is a 16-element Boolean algebra in \( D^{2,2,2} \).

So, let us prove (3.50). Let \( n = 1, i = 1 \). We have to prove that

\[ a_1(1)x_1(2) \subseteq y_1(3). \]  
(3.51)

According to (1.34) we have

\[ a_1(1) = x_2 + x_3 + y_3(x_2 + y_1) + y_1(x_2 + y_3) + y_1(x_3 + y_2) + y_2(x_3 + y_1) \]
\[ = (x_2 + y_3)(x_2 + y_1) + (x_3 + y_2)(x_3 + y_1). \]
Further, according to Section 1.10.1, we have
\[ x_1(2) = f_{21} + f_{31} = y_2y_3 \]
and
\[
y_1(3) = y_3(y_1 + x_2)(y_2 + x_3) + y_1(x_2 + y_3)(x_3 + y_2) + y_2(y_1 + x_3)(y_3 + x_2)
\]
\[ = (x_2 + y_3)(x_3 + y_2)[y_3(y_1 + x_2)(y_2 + x_3) + y_1 + y_2(y_1 + x_3)(y_3 + x_2)]
\]
\[ = (x_2 + y_3)(x_3 + y_2)[y_3(y_1 + x_2)(x_3 + y_2y_3) + y_1 + y_2(y_1 + x_3)(x_2 + y_2y_3)]
\]
\[ = (x_2 + y_3)(x_3 + y_2)(y_3 + x_2)(x_3 + y_1 + y_2y_3)(x_2 + y_1 + y_2y_3)(y_1 + x_2 + x_3). \]

Now, (3.51) is true since
\[ x_1(2) = y_2y_3 \subseteq (x_2 + y_3)(x_3 + y_2)(x_3 + y_1 + y_2y_3)(x_2 + y_1 + y_2y_3) \]
and
\[ a_1(1) = (x_2 + y_3)(x_2 + y_1) + (x_3 + y_2)(x_3 + y_1) \subseteq y_1 + x_2 + x_3. \]

Therefore, (3.51) is true, and (3.50) is true for \( n = 1 \).

**Induction step.** Let (3.50) be true for \( n \), i.e.,
\[ \rho_X^+(a_i(n)x_i(n + 1)) \subseteq \rho_X^+(y_i(n + 2)) \]
for every indecomposable representation \( \rho_X \). Applying \( \sum \varphi_i \) we have
\[ \sum_{p=1,2,3} \varphi_p \rho_X^+(a_i(n)x_i(n + 1)) \subseteq \sum_{p=1,2,3} \varphi_p \rho_X^+(y_i(n + 2)). \]

Since \( a_i(n) \) is perfect, we have according to (3.18) and Proposition 3.2.1
\[ \sum_{p=1,2,3} \varphi_p \rho_X^+(a_i(n)) \sum_{p=1,2,3} \varphi_p \rho_X^+(x_i(n + 1)) \subseteq \sum_{p=1,2,3} \varphi_p \rho_X^+(y_i(n + 2)). \]

By Propositions 2.13.3 and 3.1.1
\[ \rho_X(a_i(n + 1)x_i(n + 2)) \subseteq \rho_X(y_i(n + 3)). \]

So, the induction step and Proposition 3.6.1 are proven. \( \square \)

**Corollary 3.6.3 (Connection Edges).** The following 8 inclusions hold:
\[ c_i(n + 1) \subseteq a_i(n)b_j(n)b_k(n), \quad i = 1, 2, 3, \quad i \neq j \neq k \]
\[ c_i(n + 1)c_j(n + 1) \subseteq a_i(n)a_j(n)b_k(n), \quad k = 1, 2, 3, \]
\[ \bigcap_{i=1,2,3} c_i(n + 1) \subseteq \bigcap_{i=1,2,3} a_i(n), \quad k \neq 1, 2, 3 \]
\[ \sum_{i=1,2,3} c_i(n + 1) \subseteq \bigcap_{i=1,2,3} b_i(n). \]

The 8 inclusions (3.56)–(3.59) correspond to the 8 edges shown in Fig. 1.7 or Fig. A.2.

**Proof.** Inclusions (3.56) follow from (3.42), inclusions (3.57), (3.58) and (3.59) follow from (3.56), see Table 3.1. \( \square \)

3.7. The theorem on perfect elements in \( D^{2,2,2}/\theta \)

We will prove the theorem describing connection construction of sublattices \( H^+(n) \mod \theta \).
**Proposition 3.7.1.** (a) For every \( u \in U_n \) and every \( v \in H^+(n+1) \), we have \( u + v \in U_n \).

(b) For every \( u \in H^+(n) \) and every \( v \in V_{n+1} \), we have \( uv \in V_{n+1} \) (in \( D^{2,2}/\theta \)).

**Proof.** (a) According to Table 3.2, the elements of \( U_n \) are

\[
 p_i(n), \quad p_i(n) + p_j(n), \quad \sum p_i(n), \quad x_0(n + 2).
\]

(3.60)

It suffices to verify heading (1) for the generators \( p_i(n + 1), q_i(n + 1), s_i(n + 1) \) of \( H^+(n+1) \). According to definition (3.34) it suffices to verify that adding the elements

\[
x_0(n + 3), \quad x_i(n + 2), \quad y_i(n + 2)
\]
does not lead out of the elements of type (3.60). According to (2.45) and Corollary 2.13.2 we have

\[
x_0(n + 3) \subseteq y_i(n + 2) \subseteq x_0(n + 2)
\]

(3.61)

and adding the element \( x_0(n + 3) \) does not lead out of \( U_n \). From (3.61) it follows that

\[
x_i(n + 2) \subseteq y_i(n + 2) \subseteq p_i(n) = x_0(n + 2) + x_i(n + 1).
\]

(3.62)

Therefore adding \( y_i(n + 2) \) and \( x_i(n + 2) \) does not lead out of \( U_n \). Finally,

\[
x_i(n + 1) \subseteq p_i(n), \quad p_j(n) + x_i(n + 1) = p_j(n) + x_i(n + 1),
\]

(3.63)

i.e., addition \( x_i(n + 1) \) does not lead out of \( U_n \) either. \( \square \)

(b) Every \( u \in H^+(n) \) is of the form \( u_1u_2u_3 \), where (Fig. 1.5)

\[
u_i \in \{ a_i(n) \subseteq b_i(n) \subseteq c_i(n) \subseteq I \}.
\]

If \( u_i \neq a_i(n) \) for every \( i \), then

\[
v \subseteq \sum c_i(n + 1) \subseteq \bigcap b_i(n) \subseteq u
\]

and \( uv = v \). So, it suffices to consider the elements \( u \in H^+(n) \) of the type \( a_i(n)u', a_i(n)a_j(n)u' \) and \( a_i(n)a_j(n)a_k(n) \), where \( u' \supseteq v \), i.e., it suffices to prove that (in \( D^{2,2}/\theta \))

\[
va_i(n) \in V_{n+1} \quad \text{for every } v \subseteq V_{n+1}.
\]

(3.64)

By Corollary 3.6.3 we have \( c_i(n + 1) \subseteq a_i(n) \) and

\[
c_i(n + 1)a_i(n) = c_i(n + 1),
\]

(3.65)

and so (3.64) is proven for \( v = c_i(n + 1) \).

Now we consider the case \( i \neq j \), for example, \( a_2(n)c_1(n + 1) \). Further, we have

\[
c_1(n + 1) = x_3(n + 1) + x_2(n + 1) + \sum_{i=1,2,3} y_i(n + 2),
\]

(3.66)

\[
x_3(n + 1) \subseteq a_2(n),
\]

\[
\sum_{i=1,2,3} y_i(n + 2) \subseteq x_0(n + 2) \simeq \bigcap a_i(n) \subseteq a_2(n).
\]

The last relation follows from the (3.24) and it is true mod \( \theta \). Therefore

\[
c_1(n + 1)a_2(n) = x_3(n + 1) + \sum_{i=1,2,3} y_i(n + 2) + x_2(n + 1)a_2(n).
\]

(3.67)

According to (3.50) we have

\[
x_2(n + 1)a_2(n) \subseteq c_2(n + 1) \mod \theta.
\]
and
\[ c_1(n + 1)a_2(n) \subseteq x_3(n + 1) + \sum_{i=1,2,3} y_i(n + 2) + x_2(n + 1)c_2(n + 1) \]
\[ = c_2(n + 1) \left[ x_3(n + 1) + \sum_{i=1,2,3} y_i(n + 2) + x_2(n + 1) \right] \]
\[ = c_2(n + 1)c_1(n + 1) \subseteq a_2(n)c_1(n + 1) \mod \theta. \]

Thus,
\[ c_1(n + 1)a_2(n) \cong c_1(n + 1)c_2(n + 1), \] (3.68)

and (3.64) is proven in \( D^{2,2,2}/\theta \) for \( v = c_j(n + 1), i \neq j \). Finally, by (3.65) and (3.68) we have
\[ a_j(n) \sum_{i=1,2,3} c_i(n + 1) = a_j(n)[c_i(n + 1) + c_j(n + 1)] \]
\[ = c_j(n + 1) + c_i(n + 1)a_j(n) \cong c_j(n + 1) + c_i(n + 1)c_j(n + 1), \]
i.e.,
\[ a_j(n) \sum_{i=1,2,3} c_i(n + 1) \cong c_j(n + 1). \] (3.69)

**Theorem 3.7.2.** The union \( \bigcup_{n=0}^{\infty} H^+(n) \) is a distributive lattice \( \mod \theta \). The diagram \( H^+ \) is obtained by uniting the diagrams of \( H^+(n) \) for \( n \geq 0 \) and joining the cubes \( U_n \) and \( V_{n+1} \) for all \( n \geq 0 \), i.e., it is necessary to draw 8 additional edges for all \( n \geq 0 \) (Fig. 1.7).

**Proof.** We have to show that the sum and the intersection do not lead out of \( H^+ \). The distributivity follows from the absence of diamonds \( M_3 \), see Proposition A.1.3, [5]. If
\[ u \in H^+(n), \quad v \in H^+(m) \]
and these lattices are not adjacent, i.e., \( n \leq m - 2 \), then \( v \subseteq u \). Indeed, by Proposition 3.6.1 (Fig. 1.7) we have (in \( D^{2,2,2}/\theta \)) that
\[ v \subseteq \bigcap_{i=1,2,3} c_i(m) \subseteq \bigcap_{i=1,2,3} b_i(m - 1) \subseteq \bigcap_{i=1,2,3} c_i(m - 1) \subseteq \bigcap_{i=1,2,3} a_i(m - 2) \subseteq u. \]

So, we consider only the case \( u \in H^+(n), v \in H^+(n + 1) \). Every element \( u \in H^+(n) \) is of the form \( u_1 + u_2 \), where \( u_1 \in H^+(n) \) and \( u_2 \in U_n \). By Proposition 3.7.1 we have
\[ u + v = u_1 + u_2 + v = u_1 + (u_2 + v) \in H^+(n). \] (3.70)

Similarly, every element \( v \) from \( H^+(n + 1) \) is of the form \( v_1v_2 \), where \( v_1 \in H^+(n + 1) \) and \( v_2 \in V_{n+1} \). By Proposition 3.7.1
\[ vu = v_1v_2u = v_1(v_2u) \in H^+(n + 1) \quad \text{(in} \ D^{2,2,2}/\theta \text{)}. \] (3.71)

So, Theorem 3.7.2 follows from (3.70) and (3.71). This concludes the proof. \( \square \)

3.8. The distributivity of \( H^+(n) \)

Our proof of the following proposition is founded on the well-known result of Jonsson [34] (see Proposition A.1.6):

A modular lattice generated by the chains \( s_1, s_2, s_3 \) is distributive if and only if every sublattice \( [v_1, v_2, v_3] \), where \( v_1 \) is some element from \( s_1 \), is distributive.

**Proposition 3.1.4.** \( H^+(n) \) is distributive sublattices for every \( n \geq 0 \).
Proof. By Jonsson’s criterion it suffices to prove that
\[ v_i v_k + v_j v_k = (v_i + v_j)v_k \]  \hspace{1cm} (3.72)
for distinct \( i, j, k \). In our case, every two generators from different chains compose the same sum
\[ v_i + v_j = I_n, \quad \text{where} \ i \neq j. \]  \hspace{1cm} (3.73)
For \( n \geq 1 \), we have
\[ I_n = \sum_{i=1,2,3} x_i(n) + \sum_{i=1,2,3} y_i(n + 1). \]  \hspace{1cm} (3.74)
For \( n = 0 \), we have
\[ I_0 = \sum_{i=1,2,3} y_i. \]  \hspace{1cm} (3.75)
By (3.72) and (3.73) it suffices to prove
\[ v_i v_k + v_j v_k = v_k \]  \hspace{1cm} (3.76)
for distinct \( i, j, k \). We will omit index \( n \) in the polynomials \( a_i(n), b_j(n), c_i(n) \).
Suppose that
\[ a_i a_j + a_k = I_n. \]  \hspace{1cm} (3.77)
Then, \( b_i a_j + a_k = I_n \) and \( c_i a_j + a_k = I_n \). Further, by the modular law (A.1) we have
\[ a_i = a_i(a_i a_j + a_k) = a_i a_j + a_i a_k, \]
\[ b_i = b_i(a_i a_j + a_k) = b_i a_j + b_i a_k, \]
\[ c_i = c_i(a_i a_j + a_k) = c_i a_j + c_i a_k. \]  \hspace{1cm} (3.78)
Now, if \( a_i \subseteq v_i \), where \( v_i = b_l \) or \( c_i \), then
\[ a_i = a_i a_j + a_i a_k \subseteq a_i v_j + a_i v_k \subseteq a_i(v_j + a_i v_k) \subseteq a_i, \]
i.e., \( a_i = a_i v_j + a_i v_k \). The same for \( b_i, c_i \):
\[ a_i = a_i v_j + a_i v_k, \]
\[ b_i = b_i v_j + b_i v_k, \]
\[ c_i = c_i v_j + c_i v_k. \]  \hspace{1cm} (3.79)
So, (3.79) and (3.76) follow from (3.77). Therefore, it suffices to prove (3.77). For \( n = 0 \) we have \( a_i = y_j + y_k \) and
\[ I_n \supseteq a_i a_j + a_k = (y_j + y_k)(y_i + y_k) + y_i + y_j \supseteq y_k + y_i + y_j = I_n. \]
Set
\[ t_i(n) = x_i(n) + y_i(n + 1). \]  \hspace{1cm} (3.80)
Obviously, \( a_i(n) = t_j(n) + t_k(n) \). By (3.74), for \( n \geq 1 \), we have (parameter \( n \) is dropped)
\[ I_n \supseteq a_i a_j + a_k = (t_j + t_k)(t_i + t_k) + t_i + t_j \supseteq t_k + t_i + t_j = I_n. \]
The distributivity of \( H^+(n) \) is proven. \( \square \)

4. Atomic and admissible polynomials in \( D^4 \)

4.1. Admissible sequences in \( D^4 \)

For definition of admissible sequences in the case of the modular lattice \( D^4 \), see Section 1.7.2. Essentially, the fundamental property of this definition is
\[ ijk = ilk \quad \text{for all} \ [i, j, k, l] = \{1, 2, 3, 4\}. \]  \hspace{1cm} (4.1)
Relation (4.1) is our main tool in all further calculations of admissible sequences of \(D^4\).

Without loss of generality only sequences starting with 1 can be considered. The following proposition will be used for the classification of admissible sequences in \(D^4\).

**Proposition 4.1.1.** The following relations hold

1. \((31)^i(32)^i(31)^i = (32)^s(31)^{s+i},\)
2. \((31)^i(21)^i(31)^i = (31)^{i+t}(21)^i,\)
3. \((42)^i(41)^i = (41)^i(31)^i, s \geq 1,\)
4. \(2(41)^i(31)^i = 2(31)^{i+1}(41)^{i-1}, r \geq 1,\)
5. \((42)^i(42)^i(41)^i = (41)^i(21)^i(31)^i,\)
6. \(1(41)^i(21)^i = 1(21)^i(41)^i, 1(i1)^i(j1)^i = 1(j1)^i(i1)^i, i, j \in \{2, 3, 4\}, i \neq j,\)
7. \((41)^i(21)^i(31)^i = (41)^i(31)^i(21)^i,\)
8. \((13)^i(21)^i = (12)^i(31)^i,\)
9. \(12(41)^i(31)^i(21)^i = (14)^i(31)^{i+1}(21)^i = (14)^i(21)^{i+1}(31)^i,\)
10. \(12(14)^i(31)^i(21)^i = (14)^i(31)^{i+1}(21)^i,\)
11. \(13(14)^i(31)^i(21)^i = (14)^i(31)^{i+2}(21)^i,\)
12. \(32(14)^i(31)^i(21)^i = (31)^s(21)^{i+1}(41)^i = 34(14)^i(31)^i(21)^i,\)
13. \(42(14)^i(31)^i(21)^i = (41)^i(21)^{i+1}(31)^i = 43(14)^i(31)^i(21)^i,\)
14. \(23(14)^i(31)^i(21)^i = (21)^{i+1}(31)^i(41)^i = 24(14)^i(31)^i(21)^i,\)
15. \((21)^i(31)^i(21)^i = (214)^i(31)^{i+1}(21)^{i-1}, t \geq 1, s > 1.\)

**Proof.** (1) For \(r = 1\), we have

\[
\]

Applying induction on \(r\) we get the relation

\[
(31)^i(32)^s(31)^j = (32)^s(31)^{s+i}.
\]

(2) For \(t = 1\), we have

\[
(31)^i(21)^s(31)^j = (31)^i(21)(21)\ldots(21)(21)31 = (31)^i(21)(21)\ldots(21)(34)31 = (31)^i(21)(21)\ldots(34)(34)31 = (31)^i(34)(34)\ldots(34)(34)21 = (31)^i(34)(34)\ldots(34)(21)21 = (31)^i(34)(34)\ldots(21)(21)21 = (31)^i(34)(34)\ldots(21)(21)21 = (31)^i(34)(34)\ldots(21)(21)21 = (31)^i(34)(34)\ldots(21)(21)21 = (31)^{i+1}(21)^s.
\]

Applying induction on \(t\) we get the relation

\[
(31)^i(21)^s(31)^j = (31)^{i+s}(21)^s.
\]

(3) For \(s = 1\), we have

\[
(42)^i41 = 42(42)(42)\ldots(42)(42)41 = 42(42)(42)\ldots(42)(42)31 = 42(42)(42)\ldots(42)(31)31 = 42(31)(31)\ldots(31)(31)31 = 41(31)(31)\ldots(31)(31)31 = 41(31)^i.
\]

Thus, by heading (2) we have

\[
(42)^i(41)^s = 41(31)^i(41)^{i-1} = (41)^s(31)^i.
\]

(4) For \(s = 0\), we have

\[
2(41)^i = 2(41)(41)^{i-1} = 2(31)(41)^{i-1}.
\]
and by heading (2):

\[ 2(41)^r(31)^s = 2(31)(41)^{r-1}(31)^s = 2(31)^{s+1}(41)^{r-1}. \]

(5) Applying heading (3) we get

\[ (43)^r(42)^s(41)^r = (43)^r(41)^r(31)^r. \]

Again, applying heading (3) to \((43)^r(41)^r\) we get

\[ (43)^r(41)^r = (41)^r(21)^s \]

and

\[ (43)^r(42)^s(41)^r = (41)^r(21)^s(31)^r. \]

(6) For \(s = 1\), we have

\[
1(41)^r(21) = 1(41)(41)\ldots(41)(41)(21) = 1(41)(41)\ldots(41)(23)(21)
\]

\[
1(41)(41)\ldots(23)(23)(21) = 1(23)(23)\ldots(23)(23)(21)
\]

\[
1(23)(23)\ldots(23)(23)(41) = 1(23)(23)\ldots(23)(41)(41)
\]

\[
1(23)(23)\ldots(23)(41)(41) = 1(21)(41)\ldots(41)(41)(41) = 1(21)(41)^r.
\]

Thus, by heading (2) we have

\[ 1(41)^r(21)^s = 1(21)(41)^r(21)^{s-1} = 1(21)^s(41)^r. \]

(7) Follows from (6).

(8) First,

\[ 1321 = 1421 = 1431 = 1231 = 1241 = 1341. \]  \hspace{1cm} (4.2)

For \(r = 1\), by (4.2) we have

\[
\]

\[
\]

\[
= 12(34)(34)(34)\ldots(34)(31) = 12(12)(12)\ldots(12)(34)(31)
\]

\[
\]

Suppose

\[ (13)^r(21)^s = (12)^s(31)^r, \]

then we have

\[ (13)^{r+1}(21)^y = 13(12)^s(31)^r = 13(12)(12)\ldots(12)(12)(31)^r \]

\[ = 13(43)(43)\ldots(43)(12)(31)^r = 13(43)(43)\ldots(43)(41)(31)^r \]

\[ = 12(12)(12)\ldots(12)(12)(31)^r = 12(12)(12)\ldots(12)(31)(31)^r \]

\[ = (12)^s(31)^{r+1}. \]

Thus, by induction the following relation holds:

\[ (13)^r(21)^s = (12)^s(31)^r. \]

(9) First of all,

\[
12(41)^r(31)^y(21)^y = 12(41)(41)\ldots(41)(31)(31)^y(21)^y
\]

\[
= 12(32)(41)\ldots(41)(31)(31)^y(21)^y = 12(32)(32)\ldots(32)(31)(31)^y(21)^y
\]

\[
= 14(14)(32)\ldots(32)(31)(31)^y(21)^y = 14(14)(14)\ldots(14)(31)(31)^y(21)^y
\]

\[ = (14)^r(31)^{y+1}(21)^y. \]
By heading (4) we have

\[ 4(31)^{s+1}(21)^t = 4(21)^{t+1}(31)^s. \]

Thus,

\[ (14)^r(31)^{s+1}(21)^t = (14)^r(21)^{t+1}(31)^s. \]

(10) First, we have

\[ 12(14)^r(31) = (14)^r(21)^2 \]

since


By (4.3) we have

\[ 12(14)^r(31)^s(21)^t = 12(14)^r(31)(31)^{s-1}(21)^t = (14)^r(21)^2(31)^{s-1}(21)^t. \]

By headings (2) and (4):

\[ (21)^2(31)^{s-1}(21)^t = (21)^{t+2}(31)^{s-1} \]

and

\[ 12(14)^r(31)^s(21)^t = (14)^r(21)^{t+2}(31)^{s-1} = (14)^r(31)^s(21)^{t+1}. \]

(11) Applying permutation 2 ↔ 3 to heading (10), we get

\[ 13(14)^r(21)^t(31)^s = (14)^r(21)^t(31)^{s+1}. \]

By heading (4)

\[ 13(14)^r(21)^t(31)^s = (14)^r(31)^{s+2}(21)^{t-1}. \]

(12) By heading (8)

\[ (14)^r(31)^s = (13)^s(41)^r \]

and

\[ 32(14)^r(31)^s(21)^t = 32(13)^s(41)^r(21)^t \]

\[ = 32(13)(13) \ldots (13)(31)(41)^r(21)^t = 34(24)(13) \ldots (13)(13)(41)^r(21)^t \]

\[ = 34(24)(24) \ldots (24)(21)(41)^r(21)^t = 31(24)(24) \ldots (24)(21)(41)^r(21)^t \]

\[ = 31(31)(31) \ldots (31)(21)(41)^r(21)^t = (31)^s(21)(41)^r(21)^t. \]

By heading (2) we have

\[ (31)^s(21)(41)^r(21)^t = (31)^s(21)^{t+1}(41)^r \]

and

\[ 32(14)^r(31)^s(21)^t = (31)^s(21)^{t+1}(41)^r. \]

(13) By heading (8):

\[ (14)^r(31)^s = (13)^s(41)^r \]

and

\[ 42(14)^r(31)^s(21)^t = 42(13)^s(41)^r(21)^t. \]
Applying permutation 4 ↔ 3 to heading (12) we get

\[ 42(13)^n(41)^n(21)^n = (41)^n(21)^n+1(31)^n. \]

(14) By heading (4) we have

\[ 23(14)^n(31)^n(21)^n = 23(14)^n(21)^n+1(31)^n. \]

Applying permutation 2 ↔ 3 to heading (12) we get

\[ 23(14)^n(21)^n+1(31)^n = (21)^n+1(31)^n(41)^n. \]

(15) For \( r = 1 \), by heading (1), we have

\[ 2(41)(31)^s(21) = 2(41)(31)\ldots(31)(31)(21) \]

\[ = 2(41)(31)\ldots(31)(34)(21) = 2(41)(31)\ldots(31)(24)(21) \]

\[ = 2(41)(31)\ldots(31)(24)(31) = 2(41)(31)\ldots(31)(21)(31) \]

\[ = 2(41)(31)\ldots(34)(21)(31) = 2(41)(31)\ldots(34)(31)(31) \]

\[ = 2(41)(31)^{s-2}(34)(31)^2 = 2(41)(34)(31)^s. \]

Further,

\[ 2(41)(34)(31)^s = 2(41)(24)(31)^s = 2(41)(21)(31)^s \]

\[ = 2(43)(21)(31)^s = 2(13)(21)(31)^s = 2(14)(21)(31)^s \]

\[ = 2(14)(31)(31)^s = 2(14)(31)^{s+1}, \text{ i.e.,} \]

\[ 2(41)(31)^s(21) = 2(14)(31)^{s+1}. \]

For \( r > 1 \), as above, we have

\[ 2(41)^n(31)^s(21)^n \]

\[ = 2(41)(41)\ldots(41)(41)(31)^n(21)^n = 2(41)(41)\ldots(41)(41)(34)(31)^n(21)^{n-1} \]

\[ = 2(41)(41)\ldots(41)(41)(24)(31)^n(21)^{n-1} = 2(41)(41)\ldots(41)(41)(21)(31)^n(21)^{n-1} \]

\[ = 2(41)(41)\ldots(41)(23)(21)(31)^n(21)^{n-1} = 2(41)(23)\ldots(23)(23)(21)(31)^n(21)^{n-1} \]

\[ = 2(13)(23)\ldots(23)(23)(21)(31)^n(21)^{n-1} = 2(14)(14)\ldots(14)(21)(31)^n(21)^{n-1} \]

\[ = 2(14)(14)\ldots(14)(14)(31)(31)^n(21)^{n-1} = 2(14)^n(31)^{s+1}(21)^{n-1}. \square \]

\textbf{Remark 4.1.2 (Note to Table 4.1).}

(1) Type \( Fij \) (resp. \( Gij, H11 \)) denotes the admissible sequence starting with \( j \) and ending with \( i \). Sequences of type \( Fij \) and \( H11 \) contain an even number of symbols, sequences of type \( Gij \) and \( H11 \) contain an odd number of symbols. For differences in types \( Fij, Gij, H11 \), see the table.

(2) Thanks to heading (15) of \textbf{Proposition 4.1.1} types \( H21, H31, H41 \) from [51, p. 55, Table 4.1] are excluded, and the number of different cases of admissible sequences is equal to 8 instead 11 in [51], see also [52].

\textbf{Proposition 4.1.3.} \textit{Full list of admissible sequences starting with 1 is given by Table 4.1.}

\textbf{Proof.} It suffices to prove that maps \( \varphi_i \), where \( i = 1, 2, 3, 4 \), do not lead out of Table 4.1. The exponents \( r, s, t \) may be any non-negative integer number. The proof is based on the relations from \textbf{Proposition 4.1.1}. We refer only to the number of the relation and drop reference to the \textbf{Proposition 4.1.1} itself.

\textit{Lines F21–F41.} Consider, for example, \( \varphi_3 \). By heading (4) we get

\[ \varphi_3((21)^n(41)^n(31)^s) = 3(41)^n(21)^n+1(31)^s, \]

i.e., we get \( G31 \).
Thus, all cases are considered and the proposition is proved. □

Line $G11$. Consider the action $\varphi_2$. By heading (6) we get

$$\varphi_2(1(41)^s(21)^t) = 2(41)^s(21)^t(31)^s = 2(41)^t(21)^s(31)^t.$$  

By heading (2) we have

$$\varphi_2(1(41)^s(31)^t(21)^t) = 21(41)^t(21)^s(31)^t = (21)^t+1(41)^s(31)^t,$$

(4.5)
i.e., we get $F21$.

Line $G21$. For the action $\varphi_3$, we have

$$2(41)^s(31)^t(21)^t = 31(41)^t(31)^s(21)^s = (31)^t+1(41)^s(21)^t,$$

(4.6)
i.e., we get $F31$.

For $\varphi_4$ from heading (4), we have

$$2(41)^s(31)^t(21)^t = 32(41)^t(31)^s(21)^s = (31)^t+1(41)^s(21)^t,$$

(4.7)
i.e., we get $F41$.

For $\varphi_1$ from heading (9), we have

$$12(41)^s(31)^t(21)^t = (14)^t(31)^s+1(21)^s = (14)^t(21)^s+1(31)^s,$$

(4.8)
i.e., we get $H11$.

Line $G31$. For the action $\varphi_1$ by relation $13(41)^t = 12(41)^t$ we get $H11$. For the action $\varphi_2$ and by heading (6), we have

$$23(41)^t(31)^s(21)^t = 23(41)^s(21)^t(31)^s = 21(41)^t(21)^s(31)^t = (21)^t+1(41)^s(31)^t,$$

i.e., we get $F21$.

For the action $\varphi_3$ from headings (6), (4), we have

$$43(41)^t(31)^s(21)^t = 43(41)^s(21)^t(31)^s = 43(21)^t+1(41)^s-1(31)^t = 41(21)^t+1(41)^s-1(31)^t.$$

So, by heading (2) we have

$$43(41)^t(31)^s(21)^t = (41)^t(21)^s+1(31)^t,$$

i.e., we get $F41$.

Line $G41$. By heading (9) we get $H11$. For action $\varphi_2$ we have from (4)

$$24(21)^t(31)^s(41)^t = 24(31)^t+1(21)^s+1(41)^t = 21(41)^t+1(21)^s+1(41)^t = (21)^t+1(41)^s+1(41)^t,$$

i.e., we get $F21$. For the action $\varphi_3$ we have from (2)

$$34(21)^t(31)^s(41)^t = 34(21)^t(31)^s(41)^t = (31)^t+1(21)^t(41)^t,$$

i.e., we get $F31$.

Line $H11$. By heading (15) we obtain results of actions $\varphi_i, i = 1, 2, 3$ for $H11$.

Thus, all cases are considered and the proposition is proved.
The pyramid in the Fig. 1.4 has internal points. We consider the slice \( S(n) \) containing all sequences of the same length \( n \). The slices \( S(3) \) and \( S(4) \) are shown in the Fig. 4.1. The slice \( S(4) \) contains only one internal point

\[
14(21) = 13(21) = 13(41) = 12(41) = 14(31) = 12(31).
\]

The slices \( S(4) \) and \( S(5) \) are shown in the Fig. 4.2. The slice \( S(5) \) contains 3 internal points

\[
2(31)(21) = 2(41)(21), \quad 3(21)(31) = 3(41)(31), \quad 4(21)(41) = 4(31)(41).
\]

Fig. 4.1. Slices of admissible sequences for \( D^4, l = 3 \) and \( l = 4 \).

Fig. 4.2. Slices of admissible sequences for \( D^4, l = 4 \) and \( l = 5 \).
Remark 4.1.4. The slice $S(n)$ contains $\frac{1}{2}n(n+1)$ different admissible sequences. Actions of $\varphi_i$, where $i = 1, 2, 3, 4$ move every line in the triangle $S(n)$, which is parallel to some edge of the triangle, to the edge of $S(n + 1)$. The edge containing $k$ points is moved to $k + 1$ points in the $S(n + 1)$.

4.2. Atomic polynomials and elementary maps in the modular lattice $D^4$

The free modular lattice $D^4$ is generated by 4 generators:

$$D^4 = \{e_1, e_2, e_3, e_4\}.$$  

Recall, that atomic lattice polynomials $a_{in}^{ij}$, where $i, j \in \{1, 2, 3, 4\}$, $n \in \{0, 1, 2, 3, \ldots\}$, for the case of $D^4$, are defined as follows

$$a_{in}^{ij} = \begin{cases} 
    a_{in}^{ij} = I & \text{for } n = 0, \\
    a_{in}^{ij} = e_i + e_j a_{n-1}^{kl} = e_i + e_j a_{n-1}^{kl} & \text{for } n \geq 1,
\end{cases}$$

where $\{i, j, k, l\}$ is the permutation of $\{1, 2, 3, 4\}$, see Section 1.5.

Proposition 4.2.1. (1) The following property of the atomic elements takes place

$$e_j a_{n-1}^{kl} = e_j a_{n-1}^{lk} \quad \text{for } n \geq 1, \text{ and distinct indices } j, k, l. \quad (4.11)$$

(2) The definition of the atomic elements $a_{in}^{ij}$ in (1.22) is well-defined.

(3) We have

$$a_{in}^{ij} \subseteq a_{n-1}^{ij} \subseteq \cdots \subseteq a_{1}^{ij} \subseteq a_{0}^{ij} = I \quad \text{for all } i \neq j. \quad (4.12)$$

(4) To equalize the lower indices of the admissible polynomials $f_{a_0}$ (see Table 4.3 and Theorem 4.8.1) we will use the following relation:

$$e_j + e_i a_{t+1}^{kl} a_{s-1}^{ij} = e_j + e_k a_{t}^{ij} a_{s}^{ll} \quad \text{for all } \{i, j, k, l\} = \{1, 2, 3, 4\}. \quad (4.13)$$

Proof. (1) Suppose (4.11) is true for the index $n - 1$:

$$e_j a_{n-1}^{kl} = e_j a_{n-1}^{lk}.$$ 

Since $e_j \subseteq a_{n-1}^{ij}$, by the permutation property (A.2), we have

$$e_j a_{n}^{kl} = e_j (e_k + e_j a_{n-1}^{ij}) = e_j (e_k + e_i a_{n-1}^{ij}) = e_j (e_i + e_k a_{n-1}^{ij}) = e_j a_{n}^{kl}. \quad (4.14)$$

(2) follows from (1).

(3) By induction hypothesis we have $a_{n-1}^{kl} \subseteq a_{n-2}^{kl}$, and therefore

$$a_{n-1}^{ij} = e_i + e_j a_{n-1}^{kl} \subseteq e_i + e_j a_{n-2}^{kl} = a_{n-1}^{ij}.$$ 

(4) Without loss of generality, we will show that

$$e_2 + e_1 a_{t+1}^{24} a_{s-1}^{34} = e_2 + e_4 a_{t}^{12} a_{s}^{13}. \quad (4.15)$$

By permutation property (A.2) we have

$$e_2 + e_1 a_{t+1}^{24} a_{s-1}^{34} = e_2 + e_1 (e_4 + e_2 a_{t}^{13} a_{s}^{43}) a_{s-1}^{34} = e_2 + e_1 (e_4 + e_2 a_{t}^{13} a_{s-1}^{43}) = e_2 + e_4 (e_1 + e_2 a_{t}^{13} a_{s-1}^{43}) = e_2 + e_4 a_{t}^{13} (e_1 + e_2 a_{s-1}^{43}) = e_2 + e_4 a_{t}^{13} a_{s}^{12}. \quad \Box$$
Now we briefly recall definitions due to Gelfand and Ponomarev [18–20] of spaces $G_i, G'_i$, representations $\nu^0, \nu^1$, joint maps $\psi_i$, and elementary maps $\varphi_i$, where $i = 1, 2, 3, 4$. To compare these definitions with a case of the modular lattice $D^{2,2,2}$, see Section 2.

We denote by

$$\{Y_1, Y_2, Y_3, Y_4 \mid Y_i \subseteq X_0, i = 1, 2, 3, 4\}$$

the representation $\rho_X$ of $D^4$ in the finite-dimensional vector space $X = X_0$, and by

$$\{Y^1_1, Y^1_2, Y^1_3, Y^1_4 \mid Y^1_i \subseteq X^1_0, i = 1, 2, 3, 4\}$$

the representation $\rho_{X^+}$ of $D^4$. Here $Y_i$ (resp. $Y^1_i$) is the image of the generator $e_i$ under the representation $\rho_X$ (resp. $\rho_{X^+}$).

\[\begin{array}{ccc}
Y_3 & \uparrow & Y^1_3 \\
\downarrow & & \downarrow \\
Y_1 & \leftarrow X_0 & \leftarrow Y_2 \\
\uparrow & & \uparrow \\
\rho_X & Y_4 & \rho_{X^+} & Y^1_4
\end{array}\]

(4.16)

The space $X^+ = X^1_0$ — the space of the representation $\rho_{X^+}$ — is

$$X^1_0 = \left\{ (\eta_1, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i, \sum \eta_i = 0 \right\},$$

where $i \in \{1, 2, 3, 4\}$. As in the case of $D^{2,2,2}$ Section 2.1, we set

$$R = \bigoplus_{i=1,2,3,4} Y_i,$$

e.i.,

$$R = \{(\eta_1, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i, i = 1, 2, 3\}.$$

Then, $X^1_0 \subseteq R$.

For the case of $D^4$, the spaces $G_i, G'_i$ are introduced as follows:

$$G_1 = \{(\eta_1, 0, 0, 0) \mid \eta_1 \in Y_1\}, \quad G'_1 = \{(0, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i\},$$

$$G_2 = \{(0, \eta_2, 0, 0) \mid \eta_2 \in Y_2\}, \quad G'_2 = \{(\eta_1, 0, \eta_3, \eta_4) \mid \eta_i \in Y_i\},$$

$$G_3 = \{(0, 0, \eta_3, 0) \mid \eta_3 \in Y_3\}, \quad G'_3 = \{(\eta_1, \eta_2, 0, \eta_4) \mid \eta_i \in Y_i\},$$

$$G_4 = \{(0, 0, 0, \eta_4) \mid \eta_4 \in Y_4\}, \quad G'_4 = \{(\eta_1, \eta_2, \eta_3, 0) \mid \eta_i \in Y_i\}. \quad (4.17)$$

For details, see [18, p. 43].

The associated representations $\nu_0, \nu_1$ in $R$ are defined by Gelfand and Ponomarev [18, Equation (7.2)]:

$$\nu^0(e_i) = X^1_0 + G_i, \quad i = 1, 2, 3,$$

$$\nu^1(e_i) = X^1_0 G'_i, \quad i = 1, 2, 3. \quad (4.18)$$

Following [18], we introduce the elementary maps $\varphi_i$:

$$\varphi_i : X^1_0 \longrightarrow X_0, (\eta_1, \eta_2, \eta_3, \eta_4) \longmapsto \eta_i.$$

11 Compare with definition (2.1) of the spaces $G_i, G'_i, H_i, H'_i$ in the case of $D^{2,2,2}$; see also Table 4.2.

12 Compare with definitions for the case of $D^{2,2,2}$ in Section 2.2.
From the definition we have
\[ \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0. \]  
(4.19)

4.3. Basic relations for the elementary and joint maps in the case of \( D^4 \)

We define joint maps \( \psi_i : D^4 \rightarrow \mathcal{L}(R) \) as in the case of \( D^{2.2.2} \):
\[ \psi_i(a) = X_0^1 + G_i(G_i' + v^1(a)). \]  
(4.20)

**Proposition 4.3.1.** In the case \( D^4 \) the joint maps \( \psi_i \) satisfy the following basic relations\(^{13}\):

1. \( \psi_i(e_i) = X_0^1 \).
2. \( \psi_i(e_j) = v^0(e_i(e_k + e_l)) \).
3. \( \psi_i(I) = v^0(e_i(e_j + e_k + e_l)) \).
4. \( \psi_i(e_k e_l) = v^0(e_i e_j) \).

**Proof.** (1) From (4.18) and (4.20) we have \( \psi_i(e_i) = X_0^1 + G_i G_i' = X_0^1 \).

(2) We have \( \psi_i(y_j) = X_0^1 + G_i(G_i' + X_0^1 G_j'). \) From \( G_i \subseteq G_j' \) for \( i \neq j \) and by the permutation property (A.2), we get
\[ \psi_i(y_j) = X_0^1 + G_i G_i' + X_0^1. \]

Since \( G_i' G_j' = G_k + G_l, \) where \( i, j, k, l \) are distinct indices, we have
\[ \psi_i(y_j) = X_0^1 + G_i (X_0^1 + G_k + G_l) \]
\[ = (X_0^1 + G_i) (X_0^1 + G_k + G_l) = v^0(e_i(e_k + e_l)). \]

(3) Again,
\[ \psi_i(I) = X_0^1 + G_i(G_i' + X_0^1) \]
\[ = X_0^1 + G_i (G_j + G_k + G_l + X_0^1) = v^0(e_i(e_j + e_k + e_l)). \]  
(4.21)

(4) Since \( G_i \subseteq G_k' \) for all \( i \neq k \), we have
\[ \psi_i(e_k e_l) = X_0^1 + G_i(G_i' + X_0^1 G_k' G_i') \]
\[ = X_0^1 + G_i (G_i' G_k' G_i' + X_0^1). \]  
(4.22)

Since \( G_j(G_i + G_k + G_l) = 0 \), we have
\[ G_i' G_j' = G_k + G_l. \]  
(4.23)

Indeed,
\[ G_i' G_j' = (G_j + G_k + G_l)(G_i + G_k + G_l) \]
\[ = G_k + G_l + G_j (G_i + G_k + G_l) = G_k + G_l. \]  
(4.24)

From (4.22) and (4.23), we see that
\[ \psi_i(e_k e_l) = X_0^1 + G_i(G_i' G_k' G_i' + X_0^1) \]
\[ = X_0^1 + G_i (X_0^1 + G_j) = v_0(e_i e_j). \]  
□

\(^{13}\) Compare with Proposition 2.4.1 for \( D^{2.2.2} \).
The main relation between the elementary map $\varphi_i$ and the joint map $\psi_i$ (Proposition 2.4.3) holds also for the case of $D^4$. Namely, let $a, b, c \subseteq D^4$. Then

(i) If $\psi_i(a) = v^0(b)$, then $\varphi_i\rho_X(a) = \rho_X(b)$.

(ii) If $\psi_i(a) = v^0(b)$ and $\psi_i(ac) = \psi_i(a)\psi_i(c)$, then

\[ \varphi_i\rho_X(ac) = \varphi_i\rho_X(a)\varphi_i\rho_X(c). \]  

From Proposition 4.3.1 and Eq. (4.25) we have

**Corollary 4.3.2.** For the elementary map $\varphi_i$ the following basic relations hold\(^{14}\):

1. $\varphi_i\rho_X(\varepsilon_i) = 0$.
2. $\varphi_i\rho_X(\varepsilon_j) = \rho_X(\varepsilon_i(\varepsilon_k + \varepsilon_l))$.
3. $\varphi_i\rho_X(\varepsilon_j(\varepsilon_j + \varepsilon_k + \varepsilon_l))$.
4. $\varphi_i\rho_X(\varepsilon_j \varepsilon_l) = \rho_X(\varepsilon_i\varepsilon_j)$. \(\square\)

### 4.4. Additivity and multiplicativity of the joint maps in the case of $D^4$

**Proposition 4.4.1.** The map $\psi_i$ is additive and quasimultiplicative with respect to the lattice operations $+$ and $\cap$, namely\(^{15}\):

1. $\psi_i(a) + \psi_i(b) = \psi_i(a + b)$.
2. $\psi_i(a)\psi_i(b) = \psi_i((a + e_i)(b + x_jx_kx_l))$.
3. $\psi_i(a)\psi_i(b) = \psi_i(a(b + e_i + x_jx_kx_l))$.

**Proof.** (1) By the modular law (A.1)

\[ \psi_i(a) + \psi_i(b) = X_0^1 + G_i(G_i' + v^1(a)) + G_i(G_i' + v^1(b)) \]

\[ = X_0^1 + G_i \left( G_i' + v^1(a) + G_i(G_i' + v^1(b)) \right) \]

\[ = X_0^1 + G_i \left( (G_i' + v^1(b))(G_i' + G_i) + v^1(a) \right). \]

Since $G_i' + G_i = R$, it follows that

\[ \psi_i(a) + \psi_i(b) = X_0^1 + G_i \left( G_i' + v^1(b) + v^1(a) \right) \]

\[ = X_0^1 + G_i(G_i' + v^1(b + a)). \]

(2) By definition (2.6) $v^1(b) \subseteq X_0^1$, and by the permutation property (A.3) we have

\[ X_0^1 + G_i(G_i' + v^1(a)) = X_0^1 + G_i'(G_i + v^1(a)). \]

By the modular law (A.1) and by (A.3)

\[ \psi_i(a)\psi_i(b) = X_0^1 + G_i(G_i' + v^1(a))(X_0^1 + G_i'(G_i + v^1(b))) \]

\[ = X_0^1 + G_i(X_0^1(G_i' + v^1(a)) + G_i'(G_i + v^1(b)))). \]

Since

\[ X_0^1(G_i' + v^1(a)) = X_0^1G_i' + v^1(a) \quad \text{and} \quad X_0^1G_i' = v^1(e_i), \]

we see that

\[ \psi_i(a)\psi_i(b) = X_0^1 + G_i(v^1(e_i) + v^1(a) + G_i'(G_i + v^1(b))). \]  

\(^{14}\)Compare with Corollary 2.4.4 — a similar proposition for $D^{2,2,2}$.

\(^{15}\)Compare with Proposition 2.5.1, case $D^{2,2,2}$.\[\]
By the permutation property (A.2) and by (2.19) we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i \left( G_i' + (v^1(e_i) + v^1(a))(G_i + v^1(b)) \right). \tag{4.27}
\]
Since
\[
G_i = G_j G_i' G_i' \quad \text{and} \quad v^1(e_i) + v^1(a) = X_0^1(v^1(e_i + a)),
\]
it follows that
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i \left( G_i' + v^1(e_i + a)(X_0^1 G_j' G_i' + v^1(b)) \right)
= X_0^1 + G_i \left( G_i' + v^1(e_i + a)G_i + v^1(b) \right)
= \psi_i((a + e_i)(b + e_j e_k e_l)).
\]

(3) From (4.26) and since \( v^1(e_i) = X_0^1 G_j' \subseteq G_i' \), we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i \left( v^1(a) + G_i'(G_i + v^1(b) + v^1(e_i)) \right). \tag{4.28}
\]
Again, by (A.2) we have
\[
\psi_i(a)\psi_i(b) = X_0^1 + G_i \left( G_i' + v^1(a)(G_i + v^1(b) + v^1(e_i)) \right)
= X_0^1 + G_i \left( G_i' + v^1(a)(G_i + v^1(b) + v^1(e_i)) \right).
\]
Thus, \( \psi_i(a)\psi_i(b) = \psi_i(a(b + x_i + x_j x_k x_l)) \). \qed

We need the following corollary (atomic multiplicativity) from Proposition 4.4.1.

**Corollary 4.4.2.** (a) Suppose one of the following inclusions holds\(^{16}\):

\begin{itemize}
  \item[(i)] \( e_i + e_j e_k e_l \subseteq a \),
  \item[(ii)] \( e_i + e_j e_k e_l \subseteq b \),
  \item[(iii)] \( e_i \subseteq a, \quad e_j e_k e_l \subseteq b \),
  \item[(iv)] \( e_i \subseteq b, \quad e_j e_k e_l \subseteq a \).
\end{itemize}

Then the joint map \( \psi_i \) operates as a homomorphism on the elements \( a \) and \( b \) with respect to the lattice operations \( + \) and \( \cap \), i.e.,
\[
\psi_i(a) + \psi_i(b) = \psi_i(a + b), \quad \psi_i(a)\psi_i(b) = \psi_i(a\psi_i(b)).
\]

(b) The joint map \( \psi_i \) applied to the following atomic elements is the intersection preserving map, i.e., multiplicative with respect to the operation \( \cap \):
\[
\psi_i(ba_n^{ij}) = \psi_i(b)\psi_i(a_n^{ij}) \text{ for every } b \subseteq D^4. \tag{4.29}
\]

4.5. The action of maps \( \psi_i \) and \( \varphi_i \) on the atomic elements in \( D^4 \)

**Proposition 4.5.1.** The joint maps \( \psi_i \) applied to the atomic elements \( a_n^{ij} \) satisfy the following relations\(^{17}\)

\begin{itemize}
  \item[(1)] \( \psi_i(a_n^{ij}) = v^0(e_j a_n^{kl}) \),
  \item[(2)] \( \psi_j(a_n^{ij}) = v^0(e_j(e_k + e_l)) \).
\end{itemize}

\(^{16}\) Compare with Corollary 2.5.2, case \( D^{2,2,2} \).
\(^{17}\) Compare with Proposition 2.7.1, the modular lattice \( D^{2,2,2} \).
We have and the inclusion Proposition 4.5.1 Proposition 4.3.1.

\[ \psi(e_{ij}) = \psi(e_i + e_ja_{n+1}^{kl}) = \psi(e_i) + \psi(e_ja_{n+1}^{kl}) = \psi(e_ja_{n+1}^{kl}). \]

We suppose that heading (3) of Proposition 4.5.1 for \( n - 1 \) is true (induction hypothesis), and we get

\[ \psi(e_{ij}) = \psi(e_ja_{n+1}^{kl}) = v^0(e_ja_{n+1}^{kl}). \]

(2) Here,

\[ \psi(e_{ij}) = \psi(e_i + e_ja_{n+1}^{kl}) = \psi(e_i) + \psi(e_ja_{n+1}^{kl}) = \psi(e_i). \]

Further, by Proposition 4.3.1, heading (2), we have

\[ \psi(e_{ij}) = v^0(e_j(e_k + e_l)). \]

(3) For convenience, without loss of generality, we will show that

\[ \psi(e_{2a_{n+1}^{kl}}) = v^0(e_1a_{n+1}^{kl}). \] (4.30)

The permutation property (A.2) and the inclusion \( v^1(e_4a_{n+1}^{kl}) \subseteq X_0^1 \) yield

\[ v^1(e_{2a_{n+1}^{kl}}) = v^1(e_2(e_3 + e_4a_{n+1}^{kl})) \]
\[ = X_0^1G_2'(X_0^1G_3' + v^1(e_4a_{n+1}^{kl})) \]
\[ = X_0^1G_2'(G_3' + v^1(e_4a_{n+1}^{kl})). \]

Thus,

\[ \psi(e_{2a_{n+1}^{kl}}) = X_0^1 + G_1(G_1' + v^1(e_2a_{n+1}^{kl})) \]
\[ = X_0^1 + G_1(G_1' + X_0^1G_2'(G_3' + v^1(e_4a_{n+1}^{kl}))). \]

Since \( G_1 \subseteq G_2' \) and \( G_1 \subseteq G_3' \), by permutation property (A.2), we have

\[ \psi(e_{2a_{n+1}^{kl}}) = X_0^1G_1(G_1'G_2' + X_0^1G_3' + v^1(e_4a_{n+1}^{kl}))) \]
\[ = X_0^1G_1(G_3 + G_4 + X_0^1G_3' + v^1(e_4a_{n+1}^{kl}))) \]
\[ = X_0^1G_1((G_3 + G_4)(G_3' + v^1(e_4a_{n+1}^{kl}))) + X_0^1. \]

Since \( G_4 \subseteq G_3' \), we have

\[ \psi(e_{2a_{n+1}^{kl}}) = X_0^1G_1(G_4 + G_3(G_3' + v^1(e_4a_{n+1}^{kl}))) + X_0^1 \]
\[ = X_0^1G_1((X_0^1G_4) + (X_0^1G_3)(G_3' + v^1(e_4a_{n+1}^{kl}))). \]

Pay attention to the fact

\[ \psi_3(e_4a_{n+1}^{kl}) = X_0^1 + G_3(G_3' + v^1(e_4a_{n+1}^{kl})), \]

and therefore

\[ \psi(e_{2a_{n+1}^{kl}}) = X_0^1G_1((X_0^1G_4) + \psi_3(e_4a_{n+1}^{kl})) \]
\[ = (X_0^1G_1)((X_0^1G_4) + \psi_3(e_4a_{n+1}^{kl})) \]
\[ = v^0(e_1)(v^0(e_4) + \psi_3(e_4a_{n+1}^{kl})). \] (4.31)

By induction hypothesis for \( n - 1 \) in heading (3) we have

\[ \psi_3(e_4a_{n+1}^{kl}) = v^0(e_3a_{n+1}^{kl}), \]

and by (4.31) we have

\[ \psi(e_{2a_{n+1}^{kl}}) = v^0(e_1)(v^0(e_4) + v^0(e_3a_{n+1}^{kl})) = v^0(e_1(e_4 + e_3a_{n+1}^{kl})) = v^0(e_1a_{n+1}^{kl}). \]
From Propositions 2.4.3 and 4.5.1 we have

**Corollary 4.5.2.** The elementary maps \( \varphi_i \) applied to the atomic elements \( a^{ij}_n \), where \( n \geq 1 \), satisfy the following relations\(^{18}\)

\[
\begin{align*}
(1) \quad \varphi_i \rho_X^+(a^{ij}_n) &= \rho_X(e_i a^{kl}_n), \\
(2) \quad \varphi_j \rho_X^+(a^{ij}_n) &= \rho_X(e_j(e_k + e_l)), \\
(3) \quad \varphi_j \rho_X^+(e_i a^{kl}_n) &= \rho_X(e_j a^{kl}_{n+1}).
\end{align*}
\]

4.6. The fundamental property of the elementary maps

**Proposition 4.6.1.** For \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) the following relations hold\(^{19}\)

\[
\begin{align*}
\varphi_i \varphi_j \varphi_k \varphi_j &= 0, \\
\varphi_i^2 &= 0.
\end{align*}
\]

**Proof.** For every vector \( v \in X^1_0 \), by definition of \( \varphi_i \) we have \( (\varphi_i + \varphi_j + \varphi_k + \varphi_l)(v) = 0 \), see Equation (4.19). In other words, \( \varphi_i + \varphi_j + \varphi_k + \varphi_l = 0 \). Therefore,

\[
\varphi_i \varphi_k \varphi_j + \varphi_i \varphi_j \varphi_j = \varphi_i (\varphi_i + \varphi_j) \varphi_j = \varphi_i^2 \varphi_j + \varphi_i \varphi_j^2.
\]

So, it suffices to prove that \( \varphi_i^2 = 0 \). For every \( z \subseteq D^4 \), by **Corollary 4.3.2**, headings (3) and (1), we have

\[
\varphi_i^2 \rho_X^2(z) \subseteq \varphi_i (\varphi_i \rho_X^2(I)) = \varphi_i (\rho_X^+(e_i(e_j + e_k + e_l))) \subseteq \varphi_i \rho_X^+(e_i) = 0. \]

**Corollary 4.6.2.** The relation

\[
\varphi_i \varphi_k \varphi_j (B) = \varphi_i \varphi_j (B)
\]

takes place\(^{20}\) for every subspace \( B \subseteq X^2_0 \), where \( X^2 = X^2_0 \) is the representation space of \( \rho_X^2 \).

Essentially, relations (4.32) and (4.34) are fundamental and motivate the construction of the admissible sequences satisfying the following relation:

\[
ijk = ilj,
\]

where indices \( i, j, k, l \) are all distinct, see Table 4.2 and Section 4.1.

4.7. The \( \varphi_i \)-homomorphic elements in \( D^4 \)

By analogy with the modular lattice \( D^{2,2,2} \) (see Section 1.6), we introduce now \( \varphi_i \)-homomorphic polynomials in \( D^4 \).

An element \( a \subseteq D^4 \) is said to be \( \varphi_i \)-homomorphic, if

\[
\varphi_i \rho_X^+(ap) = \varphi_i \rho_X^+(a) \varphi_i \rho_X^+(p) \quad \text{for all } p \subseteq D^4.
\]

An element \( a \subseteq D^4 \) is said to be \( (\varphi_i, e_k) \)-homomorphic, if

\[
\varphi_i \rho_X^+(ap) = \varphi_i \rho_X^+(e_k a) \varphi_i \rho_X^+(p) \quad \text{for all } p \subseteq e_k.
\]

**Theorem 4.7.1.** (1) The polynomials \( a^{ij}_n \) are \( \varphi_i \)-homomorphic.\(^{21}\)

(2) The polynomials \( a^{ij}_n \) are \( (\varphi_j, e_k) \)-homomorphic for distinct indices \( i, j, k \).

---

\(^{18}\) Compare with **Corollary 2.7.4**, case \( D^{2,2,2} \).

\(^{19}\) Compare with **Proposition 2.10.2**, case \( D^{2,2,2} \).

\(^{20}\) Compare with **Corollary 2.10.3**, case \( D^{2,2,2} \).

\(^{21}\) Compare with **Theorem 2.8.1**, case \( D^{2,2,2} \).
### Proposition 4.5.1 and the analogue of Corollary 4.4.2

#### Proposition 4.5.1

The proposition states that for a given set of indices, the representation of a certain quasimultiplicativity element can be expressed as a product of elementary maps.

#### Corollary 4.4.2

This corollary provides a fundamental property of indices, which is used in the proof of the proposition.

#### Proof

1. Follows from multiplicity (4.29), Proposition 4.5.1 and the analogue of Proposition 2.4.3 for $D^4$.

2. For convenience, without loss of generality, we will show that

\[ \varphi_2 \rho_{X^+}(a_n^{12}) = \varphi_2 \rho_{X^+}(e_3 a_{p} \varphi_2 \rho_{X^+}(p)) \quad \text{for all } p \subseteq e_3. \]  

(4.38)

Let $p \subseteq e_3$. Hence we get

\[ p a_s^{12} = pe_3 a_s^{12} = pe_3 a_s^{21} = pa_s^{21}. \]

By Corollary 4.4.2 and Proposition 4.5.1, heading (3) we have

\[ \psi_2(p a_n^{12}) = \psi_2(pe_3 a_n^{21}) = \psi_2(p(e_2 + e_1 a_n^{34})) = \psi_2(p(e_1 a_n^{34} - 1)) = \psi_2(p) \psi_2(e_1 a_n^{34}) = \psi_2(p) \nu^0(a_n^{34}). \]

(4.39)

Since $e_2 a_n^{34} = e_2(e_1 + e_3 + e_4)e_n^{a_n^{34}}$ and

\[ \nu^0(e_2 a_n^{34}) = \nu^0(e_2(e_1 + e_3 + e_4)a_n^{34}) = \nu^0(e_2(e_1 + e_3 + e_4)) = \psi_2(I) \nu^0(a_n^{34}), \]

by (4.39) we have

\[ \psi_2(p a_n^{12}) = \psi_2(p) \psi_2(\nu(a_n^{34})) = \psi_2(p) \nu^0(a_n^{34}). \]

(4.40)

We have $\psi_2(p) \subseteq \psi_2(e_3)$ together with $p \subseteq e_3$, and therefore $\psi_2(e_3) = \psi_2(e_3) \psi_2(p)$. From (4.40) we get

\[ \psi_2(p a_n^{12}) = \psi_2(p) \nu^0(e_2(e_1 + e_4)) = \psi_2(p) \nu^0(a_n^{34}) = \psi_2(p) \psi_2(e_3(a_n^{21})). \]
Table 4.3
Admissible polynomials in the modular lattice $D^4$

<table>
<thead>
<tr>
<th>Admissible sequence $\alpha$</th>
<th>Admissible polynomial $e_\alpha$</th>
<th>Admissible polynomial $f_\alpha 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F21$ $(21)^i (41)^j (31)^k = (21)^i (31)^j (41)^k$</td>
<td>$e_2a_2^{31}a_2^{41}a_2^{34}$</td>
<td>$e_\alpha (e_2a_2^{34} + a_2^{41}a_2^{31})$</td>
</tr>
<tr>
<td>$F31$ $(31)^i (41)^j (21)^k = (31)^i (21)^j (41)^k$</td>
<td>$e_2a_2^{21}a_2^{41}a_2^{24}$</td>
<td>$e_\alpha (e_2a_2^{24} + a_2^{41}a_2^{21})$</td>
</tr>
<tr>
<td>$F41$ $(41)^i (31)^j (21)^k = (41)^i (21)^j (31)^k$</td>
<td>$e_2a_2^{21}a_2^{31}a_2^{32}$</td>
<td>$e_\alpha (e_2a_2^{32} + a_2^{31}a_2^{31})$</td>
</tr>
<tr>
<td>$G11$ $1(41)^i (31)^j (21)^k = 1(31)^i (41)^j (21)^k = 1(21)^i (31)^j (41)^k$</td>
<td>$e_2a_2^{24}a_2^{34}a_2^{32}$</td>
<td>$e_\alpha (e_2a_2^{32} + a_2^{34}a_2^{34}a_2^{34}a_2^{34})$</td>
</tr>
<tr>
<td>$G21$ $2(41)^i (31)^j (21)^k = 2(31)^i (41)^j (21)^k = 2(21)^i (31)^j (41)^k$</td>
<td>$e_2a_2^{34}a_2^{31}a_2^{24}$</td>
<td>$e_\alpha (a_2^{31} + a_2^{34}a_2^{34})$</td>
</tr>
<tr>
<td>$G31$ $3(41)^i (21)^j (31)^k = 3(21)^i (41)^j (31)^k = 3(31)^i (21)^j (41)^k$</td>
<td>$e_2a_2^{24}a_2^{21}a_2^{14}$</td>
<td>$e_\alpha (a_2^{14} + a_2^{21}a_2^{24})$</td>
</tr>
<tr>
<td>$G41$ $4(21)^i (31)^j (41)^k = 4(31)^i (21)^j (41)^k = 4(41)^i (21)^j (31)^k$</td>
<td>$e_2a_2^{32}a_2^{31}a_2^{12}$</td>
<td>$e_\alpha (a_2^{12} + a_2^{32}a_2^{32})$</td>
</tr>
<tr>
<td>$H11$ $(14)^i (31)^j (21)^k = (14)^i (21)^j (31)^k = (21)^i (14)^j (31)^k = (31)^i (14)^j (21)^k$</td>
<td>$e_2a_2^{23}a_2^{24}a_2^{24}$</td>
<td>$e_\alpha (a_2^{24} + a_2^{23}a_2^{24})$</td>
</tr>
</tbody>
</table>

Notes to table:
(1) For more details about admissible sequences, see Proposition 4.1.3 and Table 4.1.
(2) For relations given in two last columns (definitions of admissible polynomials $e_\alpha$ and $f_\alpha 0$), see Lemma 4.8.2.
(3) In each line, each subscript should be non-negative. For example, for Line $F21$, we have: $s \geq 0$, $r \geq 0$, $t \geq 1$; for Line $G11$, we have: $s \geq 0$, $r \geq 0$, $t \geq 0$.

i.e.,

$$\psi_2(pa_2^{12}) = \psi_2(p)\psi_2(e_3a_2^{21}).$$

(4.41)

Applying projection $\nabla$ to (4.41) as in Theorem 2.8.1 we get (4.38) and theorem is proved. $\square$

4.8. The theorem on the classes of admissible elements in $D^4$

**Theorem 4.8.1.** Let $\alpha = i_n i_{n-1} \ldots 1$ be an admissible sequence for $D^4$ and $i \neq i_n$. Then $i\alpha$ is admissible and, for $z_\alpha = e_\alpha$ or $f_\alpha 0$ from Table 4.3, the following relation holds$^{22}$:

$$\varphi_i \rho_X(z_\alpha) = \rho_X(z_i \alpha).$$

(4.42)

For the proof of the theorem on admissible elements in $D^4$, see Section B.2 in [51]. The proof repeatedly uses the basic properties of the admissible sequences in $D^4$ considered in Section 4.8.1, Lemma 4.8.2.

4.8.1. Basic properties of admissible elements in $D^4$

We prove here a number of basic properties$^{23}$ of the atomic elements in $D^4$ used in the proof of the theorem on admissible elements (Theorem 4.8.1). In particular, in some cases the lower indices of polynomials $a_{ij}$ entering in the admissible elements $f_\alpha 0$ can be transformed as in the following

---

$^{22}$ Compare with Theorem 2.12.1, case $D^{2,2,2}$.

$^{23}$ Compare with Table 2.1 and Section 2.6, case $D^{2,2,2}$. 
Lemma 4.8.2. (1) Every polynomial $f_{a_0}$ from Table 4.3 can be represented as an intersection of $e_\alpha$ and $P$. For every $i \neq i_n$ (see Section 1.7.2), we select $P$ to be some $\varphi_1$-homomorphic polynomial.

(2) The lower indices of polynomials $a_{ij}^k$ entering in the admissible elements $f_{a_0}$ can be equalized as follows:

$$f_{a_0} = f_{(21)^4(41)^4(31)^0} = e_\alpha(e_{2a_{21}^3} + a_{2r+1}^j a_{2s-1}^l) = e_\alpha(a_{21}^3 + e_1 a_{2r+1}^j a_{2s-1}^l).$$

(4.43)

The generic relation is the following:

$$e_i(e_{ij} + a_{r+1}^k a_{s-1}^l) = e_i(a_{ij} + e_1 a_{r+1}^k a_{s-1}^l) = e_i(a_{ij} + e_k a_{ij} a_{s}^l).$$

(4.44)

(3) The substitution

$$\begin{cases} r \mapsto r - 2, \\ s \mapsto s + 2 \end{cases}$$

(4.45)

does not change the polynomial $e_i(e_{ij} + a_{r+1}^k a_{s-1}^l)$, namely:

$$e_i(e_{ij} + a_{r+1}^k a_{s-1}^l) = e_i(e_{ij} + a_{r-1}^k a_{s+1}^l).$$

(4.46)

(4) The substitution (4.45) does not change the polynomial $e_1 a_{ij}^k a_{il}^k$:

$$e_1 a_{r+1}^k a_{s-1}^l = e_1 a_{r-1}^k a_{s+1}^l.$$

(4.47)

Proof. (1) We consider here only Line 2 of Table 4.3, all other cases are similarly considered. Every polynomial $f_{a_0}$ from Table 4.3 is the intersection $e_\alpha$ and $P$, where $P$ is the sum contained in the parentheses, and polynomial $f_{a_0}$ can be represented in the different equivalent forms, such that only the polynomial $P$ is changed. For Line 2, we have $f_{a_0} = f_{(21)^4(41)^4(31)^0}$ and a number of the equivalent forms of $f_{a_0}$ are in Table 4.4.

The form $f_{a_0}$ of Line 2 follows from the form $f_{a_0}$ of Line 1 (resp. 6 from 5, resp. 9 from 8) in Table 4.4 by the inclusion $e_\alpha \subseteq e_2$ and permutation property (A.2). The form $f_{a_0}$ in Lines 3 and 4 follows from (4.11). Further, by (4.12) $e_\alpha \subseteq a_{22}^{31} \subseteq a_{2r+1}^{31}$ and by permutation property (A.2) we deduce the form $f_{a_0}$ of Line 5. The form $f_{a_0}$ of Line 2 follows from (4.47). The form $f_{a_0}$ of Line 8 follows from Line 7 since $e_\alpha \subseteq a_{22}^{14} a_{2r-1}^{14}$.

The polynomial $P$ in Line 4 is $\varphi_1$-homomorphic since

$$e_1 \subseteq a_{2r+1}^{14} a_{2s-1}^{13} \quad \text{and} \quad e_2 e_4 \subseteq e_2 a_{2r}^{34}.\quad\text{24}$$

24 Throughout this lemma we suppose that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. 

Table 4.4

<table>
<thead>
<tr>
<th>$N$</th>
<th>Equivalent form of $f_{a_0}$</th>
<th>Forms obtained by $e_j a_{ij}^k = e_j a_{ij}^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e_\alpha(e_{2a_{21}^3} + a_{2r+1}^j a_{2s-1}^l)$</td>
<td>$e_\alpha(e_{2a_{21}^3} + a_{2r+1}^j a_{2s-1}^l)$</td>
</tr>
<tr>
<td>2</td>
<td>$e_\alpha(a_{2j} + a_{2r+1}^j a_{2s-1}^l)$</td>
<td>$e_\alpha(a_{2j} + a_{2r+1}^j a_{2s-1}^l)$</td>
</tr>
<tr>
<td>3</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>4</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>5</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>6</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>7</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>8</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
<tr>
<td>9</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
<td>$e_\alpha(a_{2j} + e_{a_{2r+1}^j a_{2s-1}^l})$</td>
</tr>
</tbody>
</table>
The polynomial $P$ in Line 3 is $\varphi_3$-homomorphic since

$$e_3 \subseteq a_{2t}^{34} \quad \text{and} \quad e_1e_2e_4 \subseteq e_1e_2 \subseteq e_2a_{2r+1}^{14}a_{2s-1}^{13}.$$ 

The polynomial $P$ in Line 3 is $\varphi_4$-homomorphic since

$$e_4 \subseteq a_{2t}^{43} \quad \text{and} \quad e_1e_2e_3 \subseteq e_1e_2 \subseteq e_2a_{2r+1}^{14}a_{2s-1}^{13}.$$ 

Essentially, heading (1) allows to select an appropriate form of the polynomial $P$ for every given $\varphi_i$.

(2) Let us prove (4.43). By (4.13) we have

$$e_\alpha(e_2a_{2t}^{34} + a_{2r+1}^{41}a_{2s-1}^{31}) = e_\alpha(a_{2t}^{43} + e_2a_{2r+1}^{41}a_{2s-1}^{31}) = e_\alpha(a_{2t}^{43} + e_4 + e_2a_{2r+1}^{31}) = e_\alpha(a_{2t}^{43} + e_2a_{2s}^{24}a_{2s}^{23}).$$

Now consider (4.44). Since $e_j \subseteq a_{4j}^l$, by (4.13) we have

$$e_\ell(a_{i}^{jl} + e_ri_{r+1}^{kl}a_{s-1}^{kl}) = e_\ell(a_{i}^{jl} + (e_j + e_ri_{r+1}^{kl}a_{s-1}^{kl})) = e_\ell(a_{i}^{jl} + e_ri_{r+1}^{jl}a_{s-1}^{jl}.$$ 

(3) By (4.44) we have

$$e_\ell(e_\ell a_{i}^{jl} + e_ri_{r+1}^{kl}a_{s-1}^{kl}) = e_\ell(a_{i}^{jl} + e_ri_{r}a_{s}^{jl}).$$

On the other hand

$$e_\ell(e_\ell a_{i}^{jl} + e_ri_{r-1}^{kl}a_{s-1}^{kl}) = e_\ell(e_\ell a_{i}^{jl} + a_{r-1}^{il}a_{s-1}^{ik}) = e_\ell(a_{i}^{jl} + e_ri_{r}a_{s}^{il}).$$

i.e.,

$$e_\ell(e_\ell a_{i}^{jl} + e_ri_{r-1}^{kl}a_{s-1}^{kl}) = e_\ell(e_\ell a_{i}^{jl} + a_{r-1}^{il}a_{s}^{kl}).$$

and (4.46) is true.

(4) By (4.13) we have

$$e_\ell a_{i}^{kl} = e_\ell e_k + e_j a_{i}^{jl}a_{s-1}^{kl} = e_\ell (e_k + e_j a_{i}^{jl}a_{s-1}^{kl}) = e_\ell (e_k + e_ri_{r-1}^{kl}a_{s}^{kl}) = e_\ell e_k + e_j a_{i}^{il}a_{s}^{kl},$$

i.e.,

$$e_\ell a_{i}^{kl} = e_\ell a_{i}^{il}a_{s}^{kl}.$$ 

(4.48)

By repeating (4.48), we see that

$$e_\ell a_{i}^{kl} = e_\ell a_{i}^{il}a_{s}^{kl}.$$ 

The lemma is proved. $\square$

4.8.2. Coincidence with the Gelfand–Ponomarev polynomials in $D^4/\theta$

Let $e_\alpha, f_{a0}$ be admissible elements constructed in this work and $\tilde{e}_\alpha, \tilde{f}_{a0}$ be the admissible elements constructed by Gelfand and Ponomarev [18]. We will prove that for the admissible sequences of the small length, the coincidence of $e_\alpha$ with $\tilde{e}_\alpha$ (resp. $f_{a0}$ with $\tilde{f}_{a0}$) takes place in $D^4$ (not only in $D^4/\theta$). Since Theorem 4.8.1 takes place for both $e_\alpha, f_{a0}$ and for $\tilde{e}_\alpha, \tilde{f}_{a0}$ [18, Theorem 7.2, Theorem 7.3] we have

**Proposition 4.8.3.** The elements $e_\alpha$, (resp. $f_{a0}$) and $\tilde{e}_\alpha$ (resp. $\tilde{f}_{a0}$) coincide in $D^4/\theta$. 
Recall definitions of $\tilde{e}_a$ and $\tilde{f}_a$ from [18].

The definition of $\tilde{e}_a$, [18, p. 6],

$$\tilde{e}_{i_n i_{n-1} \ldots i_2 i_1} = \tilde{e}_{i_n} \sum_{\beta \in \Gamma_e(a)} \tilde{e}_\beta,$$

(4.49)

where

$$\Gamma_e(a) = \{ \beta = (k_{n-1}, \ldots, k_2) \mid k_{n-1} \not\in \{i_n, i_{n-1}\}, \ldots, k_1 \not\in \{i_2, i_1\}, \text{ and}$$

$$k_1 \neq k_2, \ldots, k_{n-2} \neq k_{n-1} \}.$$

The definition of $\tilde{f}_a$, [18, p. 53],

$$\tilde{f}_{i_n i_{n-1} \ldots i_2 i_1 0} = \tilde{e}_{i_n} \sum_{\beta \in \Gamma_f(a)} \tilde{e}_\beta,$$

(4.51)

where

$$\Gamma_f(a) = \{ \beta = (k_n, \ldots, k_2, k_1) \mid k_n \not\in \{i_n, i_{n-1}\}, \ldots, k_2 \not\in \{i_2, i_1\}, k_1 \not\in \{i_1\} \text{ and}$$

$$k_1 \neq k_2, \ldots, k_{n-2} \neq k_{n-1} \}.$$

**Proposition 4.8.4** (The Elements $\tilde{e}_a$). Consider elements $\tilde{e}_a$ for $\alpha = 21, 121, 321, 2341$ (see Section 1.9.2). The relation

$$e_a = \tilde{e}_a$$

takes place in $D^4$.

**Proof.** For $n = 2$: $\alpha = 21$. We have

$$\tilde{e}_{21} = e_2 \sum_{j \neq 1, 2} e_j = e_2(e_3 + e_4).$$

(4.53)

According to Section 1.9.2 we see that $e_{21} = \tilde{e}_{21}$.

For $n = 3$: (1) $\alpha = 121$,

$$\Gamma_e(a) = \{(k_2 k_1) \mid k_2 \in \{3, 4\}, k_1 \in \{3, 4\}, k_1 \neq k_2\},$$

$$\tilde{e}_{121} = e_1 \sum_{\beta \in \Gamma_e(a)} e_\beta = e_1(e_3 e_4 + e_{34}) = e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)) = e_1 a_2^{34}.$$

(4.54)

By Section 1.9.2 we have $e_{121} = \tilde{e}_{121}$.

(2) $\alpha = 321$. We have

$$\Gamma_e(a) = \{(k_2 k_1) \mid k_2 \in \{3, 2\}, k_1 \in \{2, 1\}, k_1 \neq k_2\} = \{14, 13, 43\}.$$

$$\tilde{e}_{321} = e_1 \sum_{\beta \in \Gamma_e(a)} e_\beta = e_3(e_1 e_4 + e_{13} + e_{43})$$

$$= e_3(e_1(e_2 + e_4) + e_1(e_2 + e_3) + e_4(e_1 + e_2))$$

$$= e_3((e_1 + e_2)(e_2 + e_4)(e_1 + e_4) + e_1(e_2 + e_3))$$

$$= e_3((e_1 + e_2)(e_1 + e_4)(e_2 + e_4 + e_1(e_2 + e_3)))$$

$$= e_3((e_1 + e_2)(e_1 + e_4)(e_4 + (e_1 + e_2)(e_2 + e_3)))$$

$$= e_3((e_1 + e_2)(e_1 + e_4)(e_4(e_2 + e_3) + (e_1 + e_2))).$$

Since $e_1 + e_2 \subseteq e_4(e_2 + e_3) + e_1 + e_2$, we have

$$\tilde{e}_{321} = e_3(e_1 e_4 + e_{13} + e_{43}) = e_3(e_1 + e_2)(e_1 + e_4).$$

(4.55)

Since $\tilde{e}_{321}$ is symmetric with respect to transposition $2 \leftrightarrow 4$ we have

$$\tilde{e}_{321} = \tilde{e}_{341} = e_3(e_{14} + e_{13} + e_{43}) = e_3(e_{12} + e_{13} + e_{23}) = e_3(e_1 + e_2)(e_1 + e_4).$$

(4.56)
By Section 1.9.2 we have $e_{321} = \tilde{e}_{321}$.

For $n = 4$: $\alpha = 2341 = 2321 = 2141$. We have

$$
\Gamma_\alpha = \{(k_3 k_2 k_1) \mid k_3 \in \{1, 4\}, k_2 \in \{1, 2\}, k_1 \in \{2, 3\}, k_1 \neq k_2, k_2 \neq k_3\}
$$

$$
= \{(123), (412), (413) = (423)\},
$$

(4.57)

and

$$
\tilde{e}_{2341} = e_2 \sum_{\beta \in \Gamma_\alpha} e_\beta = e_2(e_{123} + e_{412} + e_{413})
$$

$$
= e_2(e_1(e_2 + e_3)(e_4 + e_3) + e_4(e_1 + e_2)(e_3 + e_2) + e_4(e_1 + e_3)(e_2 + e_3))
$$

$$
= e_2(e_4 + e_3)(e_1 + e_2)(e_4(e_1 + e_2) + e_4(e_1 + e_3)(e_2 + e_3))
$$

$$
= e_2(e_4 + e_3)(e_1 + e_2)(e_1 + e_2)(e_3 + e_2)
$$

$$
= e_2(e_4 + e_3)(e_1 + e_2) + e_4).
$$

(4.58)

By Section 1.9.2 we have $e_{2341} = \tilde{e}_{2341}$. The proposition is proved. □

**Proposition 4.8.5** (The Elements $\tilde{f}_{\alpha 0}$). Consider elements $\tilde{f}_{\alpha 0}$ for $\alpha = 21, 121, 321$ (see Section 1.9.2). The following relation

$$
f_{\alpha 0} = \tilde{f}_{\alpha 0}
$$

takes place in $D^k$.

**Proof.** For $n = 2$: $\alpha = 21$. We have

$$
\Gamma_f = \{(k_2 k_1) \mid k_2 \in \{3, 4\}, k_1 \in \{2, 3, 4\}, k_1 \neq k_2\},
$$

$$
\tilde{f}_{210} = e_2 \sum_{\beta \in \Gamma_f} e_\beta = e_2(e_{32} + e_{34} + e_{42} + e_{43})
$$

$$
= e_2(e_3(e_1 + e_4) + e_3(e_2 + e_1) + e_4(e_1 + e_3) + e_4(e_1 + e_2))
$$

$$
= e_2(e_3 + e_4)(e_1 + e_4)(e_1 + e_3) + e_3(e_1 + e_2) + e_4(e_1 + e_2))
$$

$$
= e_2(e_3 + e_4)((e_1 + e_4)(e_1 + e_3) + e_3(e_1 + e_2) + e_4(e_1 + e_2))
$$

$$
= e_2(e_3 + e_4)(e_4(e_1 + e_3) + (e_1 + e_3)(e_1 + e_2) + e_4(e_1 + e_2))
$$

$$
= e_2(e_3 + e_4)(e_4(e_1 + e_3)(e_1 + e_2) + (e_1 + e_3)(e_1 + e_2) + e_4)
$$

$$
= e_2(e_3 + e_4)(e_4 + (e_1 + e_3)(e_1 + e_2))
$$

$$
= e_2(e_3 + e_4)(e_4 + e_1 + e_3(e_1 + e_2)).
$$

(4.59)

By Section 1.9.2 we have $f_{210} = \tilde{f}_{210}$.

For $n = 3$: (1) $\alpha = 121$. For this case, we have

$$
\Gamma_f = \{(k_3 k_2 k_1) \mid k_3 \in \{3, 4\}, k_2 \in \{3, 4\}, k_1 \in \{2, 3, 4\}, k_1 \neq k_2, k_2 \neq k_3\},
$$

(4.60)

and

$$
\tilde{f}_{1210} = e_1 \sum_{\beta \in \Gamma_f} e_\beta = e_1(e_{342} + e_{343} + e_{432} + e_{434})
$$

$$
= e_1(e_3(e_1 + e_2)(e_4 + e_2) + e_3(e_1(e_4 + e_3) + e_2(e_4 + e_3))
$$

$$
+ e_4(e_1 + e_2)(e_3 + e_2) + e_4(e_1(e_4 + e_3) + e_2(e_4 + e_3))]
$$

$$
= e_1(e_3(e_1 + e_2)(e_4 + e_2 + e_3(e_1 + e_2))
$$

$$
+ e_4(e_1 + e_2)(e_3 + e_2 + e_4(e_1 + e_2))
$$

$$
= e_1(e_3(e_1 + e_2)(e_4 + e_2 + e_1(e_3 + e_2(e_4 + e_3))
$$

$$
+ e_4(e_1 + e_2)(e_3 + e_2 + e_1(e_4 + e_2(e_4 + e_3))]
$$

$$
= e_1(e_3(e_1 + e_2)(e_4 + e_2 + e_1(e_3 + e_2(e_4 + e_3))
$$

$$
+ e_4(e_1 + e_2)(e_3 + e_2 + e_1(e_4 + e_2(e_4 + e_3))].
$$
\[
\begin{align*}
&= e_1[(e_4 + e_2 + e_1(e_3 + e_2)(e_4 + e_3))] \\
&= e_3(e_1 + e_2) + e_4(e_1 + e_2) \\
&= e_1[(e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2)] \\
&= e_3(e_1 + e_2) + e_4(e_1 + e_2) \\
&= e_1[(e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2)] \\
&= e_3(e_1 + e_2) + e_4(e_1 + e_2) \\
&= e_3(e_1 + e_2) + e_4(e_1 + e_2). \\
\end{align*}
\]

Since the first two intersection polynomials in the last expression of \( \tilde{f}_{1210} \) coincide with

\[(e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2), \]

we have

\[ \tilde{f}_{1210} = e_1((e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2))(e_3 + e_4(e_1 + e_2)). \]  

By Section 1.9.2 we have \( f_{1210} = \tilde{f}_{1210} \).

(2) \( \alpha = 321 \). Here we have

\[ \Gamma_f(\alpha) = \{(k_3,k_2,k_1) \mid k_3 \in \{1,4\}, k_2 \in \{3,4\}, k_1 \in \{2,3,4\}, k_1 \neq k_2, k_2 \neq k_3 \} = \{((132) = (142), (134), (143), (432), (434) \} \]

and

\[ \tilde{f}_{3210} = e_1 \sum_{\beta \in \Gamma_f(\alpha)} e_{\beta} = e_3(e_{132} + e_{134} + e_{143} + e_{432} + e_{434}) \]

\[ = e_3[e_1(e_4 + e_2)(e_3 + e_2) + e_1(e_4 + e_2)(e_3 + e_4) \\
+ e_1(e_3 + e_2)(e_3 + e_4) + e_4(e_3 + e_2)(e_1 + e_2) \\
+ e_4(e_1 + e_2(e_3 + e_4))]. \]  

Since

\[ e_1(e_4 + e_2)(e_3 + e_4) + e_4(e_1 + e_2(e_3 + e_4)) \]

\[ = (e_3 + e_4)(e_4 + e_2)(e_1 + e_4(e_1 + e_2(e_3 + e_4))) \]

\[ = (e_3 + e_4)(e_4 + e_2)(e_1 + e_4(e_1 + e_2(e_3 + e_4))), \]

by (4.64) we have

\[ \tilde{f}_{3210} = e_3[e_1(e_4 + e_2)(e_3 + e_2) \\
+ (e_3 + e_4)(e_4 + e_2)(e_1 + e_4(e_1 + e_2(e_3 + e_4))) \\
+ e_1(e_3 + e_2)(e_3 + e_4) + e_4(e_3 + e_2)(e_1 + e_2) \\
+ (e_3 + e_4)(e_4 + e_2)(e_1 + e_2(e_3 + e_4)) \\
+ e_1(e_3 + e_2)(e_3 + e_4) + e_4(e_3 + e_2)] \\
= e_3(e_1 + e_2)(e_1 + e_4)[e_1(e_4 + e_2)(e_3 + e_2) \\
+ (e_3 + e_4)(e_4 + e_2)(e_1 + e_2(e_3 + e_4)) \\
+ e_1(e_3 + e_2)(e_3 + e_4) + e_4(e_3 + e_2)] \\
= e_3(e_1 + e_2)(e_1 + e_4)[e_1(e_4 + e_2)(e_3 + e_2)(e_3 + e_4) \\
+ (e_3 + e_4)(e_4 + e_2)(e_1 + e_2(e_3 + e_4)) \\
+ e_1(e_3 + e_2) + e_4(e_3 + e_2)]. \]  

Since

\[ e_1(e_4 + e_2)(e_3 + e_4) \subseteq e_1(e_3 + e_2), \]
by (4.66) we have
\[
\hat{f}_{3210} = e_3(e_1 + e_2)(e_1 + e_4)[(e_3 + e_4)(e_4 + e_2)(e_1 + e_2)(e_3 + e_4)]
+ e_1(e_3 + e_2) + e_4(e_3 + e_2)]
\]
\[
= e_3(e_1 + e_2)(e_1 + e_4)[(e_3 + e_4)(e_4 + e_2) + e_2(e_3 + e_4)]
+ e_1(e_3 + e_2) + e_4(e_3 + e_2)]
\]
\[
= e_3(e_1 + e_2)(e_1 + e_4)[(e_3 + e_4)(e_4 + e_2)(e_3 + e_2) + e_2(e_3 + e_4)]
+ e_1 + e_4(e_3 + e_2)].
\] (4.67)

Since
\[
e_1(e_3 + e_4)(e_4 + e_2)(e_3 + e_2) \subseteq e_1.
\]
by (4.67) we have
\[
\hat{f}_{3210} = e_3(e_1 + e_2)(e_1 + e_4)(e_2(e_3 + e_4)) + e_1 + e_4(e_3 + e_2)
\]
\[
= e_3(e_1 + e_2)(e_1 + e_4)(e_1 + (e_2 + e_4)(e_3 + e_2)(e_1 + e_4)).
\] (4.68)

By Section 1.9.2 we have \( f_{3210} = \hat{f}_{3210} \). The proposition is proved. \( \square \)

**Conjecture 4.8.6.** For every admissible sequence \( \alpha \), the elements \( e_\alpha \) (resp. \( f_{a0} \)) and \( \tilde{e}_\alpha \) (resp. \( \tilde{f}_{a0} \)) coincide in \( D^4 \) (see Proposition 4.8.3).

In Propositions 4.8.4 and 4.8.5, this conjecture was proven for small values of lengths of the admissible sequence \( \alpha \).

5. Admissible elements in \( D^4 \) and Herrmann’s polynomials

In this section we consider Herrmann’s endomorphisms \( \gamma_{ik} \) \((i, k = 1, 2, 3)\) and polynomials \( s_n, t_n, p_i, n \) \((i = 1, 2, 3, 4)\) being perfect elements, [27]. Endomorphisms \( \gamma_{ik} \) are important in construction of the perfect elements \( s_n, t_n, p_i, n \), the perfect elements constitute a 16-element Boolean cube, coinciding modulo linear equivalence with the Gelfand–Ponomarev Boolean cube \( B^+(n) \), see Theorem 5.4.4 due to Herrmann [27].

We show that endomorphisms \( \gamma_{ik} \) are also closely connected with admissible elements.

First of all, the endomorphism \( \gamma_{ik} \) acts on the admissible element \( e_{ak} \) such that
\[
\gamma_{ik}(e_{ak}) = e_{aki}.
\]
since Theorem 5.3.4. Thus, endomorphism \( \gamma_{ik} \) acts also on the admissible sequence. We see some similarity between the action of the endomorphism \( \gamma_{ik} \) and the action of elementary map of Gelfand–Ponomarev \( \varphi_i \). The endomorphism \( \gamma_{ik} \) and the elementary map \( \varphi_i \) act in a sense in opposite directions, namely the endomorphism \( \gamma_{ik} \) adds the index to the beginning of the admissible sequence, and the elementary map \( \varphi_i \) adds the index to the end.\(^{25}\)

Further, the endomorphisms \( \gamma_{ik} \) commute and we can consider a sequence of these endomorphism. Admissible elements \( e_1 a_1^{24} a_2^{24} a_r^{32} \) from Table 5.1 are obtained by means of Herrmann’s endomorphisms as follows:
\[
\gamma_{12}^l \gamma_{13}^r \gamma_{14}^s (e_1) = e_1 a_1^{24} a_2^{24} a_r^{32},
\]
see Theorem 5.3.6.

At last, we will see how Herrmann’s polynomials \( s_n, t_n, p_i, n \) are expressed by means of the cumulative elements \( e_i(n), f_i(n) \):
\[
s_n = \sum_{i=1,2,3,4} e_i(n),
\]
\[
t_n = f_0(n + 1),
\]
\[
p_i, n = e_i(n) + f_0(n + 1),
\]
see Theorem 5.4.3 and Table 1.1.

\(^{25}\) Recall, that in the admissible sequence \( i_n i_{n-1} \ldots i_2 i_1 \) the index \( i_1 \) is the beginning and \( i_n \) is the end.
5.1. A unified formula of admissible elements

It turned out, that the admissible elements \( e_\alpha \) and \( e_{\alpha 0} \) described by Table 4.3 can be written by a unified formula. This unified formula is presented in Table 5.1 and Proposition 5.1.1.

A main difference between Table 4.3 and Table 5.1 is the following: Table 4.3 describes admissible elements with admissible sequence \( \alpha \) starting with the generator \( e_1 \), and Table 5.1 describes admissible elements with admissible sequence \( \alpha \) ending with the generator \( e_1 \). Recall, that type \( Fij \) (resp. \( Gij \), \( H11 \)) denotes the admissible sequence starting with \( j \) and ending with \( i \), see Remark 4.1.2.

As we will see in the next sections, the unified formula represented in this section in Table 5.1 and in Proposition 5.1.1 is a basis for the construction of Herrmann’s polynomials, [27, 29].

**Proposition 5.1.1.** (a) The set of all admissible polynomials \( e_\alpha \) (described by Table 4.3) with admissible sequences \( \alpha \) ending with \( 1 \) coincides with the set of polynomials

\[
e_1 a_r^{32} a_s^{24} a_t^{34}, \quad \text{where } r, s, t = 0, 1, 2, \ldots,
\]

see Table 5.1.

Similarly, the set of all admissible polynomials \( e_\alpha \) with admissible sequences \( \alpha \) ending with \( i \in \{ 2, 3, 4 \} \) coincides with the set of polynomials

\[
e_1 a_r^{ij} a_s^{lk} a_t^{kl}, \quad \text{where } r, s, t = 0, 1, 2, \ldots, \quad \text{and } \{ i, j, k, l \} = \{ 1, 2, 3, 4 \}.
\]

(b) The set of all admissible polynomials \( f_{\alpha 0} \) (described by Table 4.3) with admissible sequences \( \alpha \) ending with \( 1 \) coincides with the set of polynomials

\[
e_\alpha (e_1 a_r^{34} + a_s^{42} a_t^{32}) = e_1 a_r^{34} a_s^{42} a_t^{32} (e_1 a_r^{34} + a_s^{42} a_t^{32}), \quad \text{where } r, s, t = 0, 1, 2, \ldots,
\]

see Table 5.1.

Similarly, the set of all admissible polynomials \( f_{\alpha 0} \) with admissible sequences \( \alpha \) ending with \( i \in \{ 2, 3, 4 \} \) coincides with the set of polynomials

\[
e_\alpha (e_1 a_r^{ij} + a_s^{kl} a_t^{ij}) = e_1 a_r^{ij} a_s^{kl} a_t^{ij} (e_1 a_r^{ij} + a_s^{kl} a_t^{ij}),
\]

where \( r, s, t = 0, 1, 2, \ldots, \quad \text{and } \{ i, j, k, l \} = \{ 1, 2, 3, 4 \}.

**Proof.** For any atomic element \( a_r^{pq} \), set the range 1 if \( i \) is odd and 0 if \( i \) is even. For the admissible polynomials \( e_\alpha \) from Table 5.1, the set of the corresponding ranges we call the signature. For example, for \( e_\alpha \) of the type \( F12, \)
the signature is (0, 0, 1), see Table 5.1. There are 8 possible signatures. Since, all possible signatures appeared in Table 5.1, we can get all possible combination of indices \((r, s, t)\) in (5.1) and (5.3).

**Remark 5.1.2.** For the admissible polynomials \(e_n\) the sum of low indices of atomic elements \(a^{kl}_r\) is by 1 less the length of the admissible sequence \(\alpha\), for the admissible polynomials \(f_0\) the sum of low indices of atomic elements \(a^{kl}_r\) (contained in the parentheses) is equal to the length of the admissible sequence \(\alpha\), see Table 4.3.

5.2. **Inverse cumulative elements in \(D^4\)**

In addition to cumulative elements, we introduce now inverse cumulative elements. A difference between cumulative elements and inverse cumulative elements is the following: the cumulative elements accumulate all admissible elements of the given length starting with some generator \(e_i\), and inverse cumulative elements accumulate all admissible elements of the given length ending with some generator \(e_i\).

The cumulative elements \(e_i(n)\), where \(i = 1, 2, 3, 4\), are defined in Section 1.10.2, (1.31). Proposition 5.1.1 and Remark 5.1.2 motivate the following definition of inverse cumulative polynomials \(e_i^\vee(n)\), where \(i = 1, 2, 3, 4\), and \(f_0^\vee(n)\) as follows:

\[
e_i^\vee(n) = \sum_{r + s + t = n - 1} e_{12}^{34} a_r^{24} a_s^{34}, \quad e_2^\vee(n) = \sum_{r + s + t = n - 1} e_{12}^{34} a_s^{14} a_r^{34},
\]

\[
e_3^\vee(n) = \sum_{r + s + t = n - 1} e_{12}^{34} a_s^{12} a_r^{14}, \quad e_4^\vee(n) = \sum_{r + s + t = n - 1} e_{12}^{34} a_s^{12} a_r^{13},
\]

and

\[
f_0^\vee(n) = \sum_{r + s + t = n} f_1 a_{i_{n-1}} a_{j_{i_{20}}} + \sum_{r + s + t = n} f_2 a_{i_{n-1}} a_{j_{i_{20}}} + \sum_{r + s + t = n} f_3 a_{i_{n-1}} a_{j_{i_{20}}} + \sum_{r + s + t = n} f_4 a_{i_{n-1}} a_{j_{i_{20}}}
\]

\[
= \sum_{r + s + t = n} e_{12} (e_{12} a_r^{34} + a_s^{24} a_r^{34}) + \sum_{r + s + t = n} e_{12} (e_{34} a_r^{14} + a_s^{14} a_r^{31})
\]

\[
+ \sum_{r + s + t = n} e_{12} (e_{12} a_r^{24} + a_s^{14} a_r^{21}) + \sum_{r + s + t = n} e_{12} (e_{12} a_r^{23} + a_s^{13} a_r^{21}).
\]

**Proposition 5.2.1.** (a) The sum of all cumulative elements \(e_i(n)\) of the given length \(n\) and the sum of all inverse cumulative elements \(e_i^\vee(n)\) coincide, i.e.,

\[
e_1(n) + e_2(n) + e_3(n) + e_4(n) = e_1^\vee(n) + e_2^\vee(n) + e_3^\vee(n) + e_4^\vee(n).
\]

(b) The cumulative element \(f_0(n)\) coincides with inverse cumulative element \(f_0^\vee(n)\):

\[
f_0(n) = f_0^\vee(n).
\]

(c) The number of elements in every sum (5.5) is \(\frac{1}{2}(n + 1)(n + 2)\).\(^{26}\)

**Proof.** (a), (b) are true since sums from both the sides consist of all admissible elements of the given length \(n\).

(c) We just need to find the number of solutions of the equation

\[
r + s + t = n.
\]

These solutions are points with integer barycentric coordinates\(^{27}\) in the triangle depicted in Fig. 5.1. Let \((r, s, t)\) be coordinates of any point of this triangle. Any move along one of the edges does not change the coordinate sum \(r + s + t\), and this sum is equal to \(n\). \(\square\)

---

\(^{26}\) Compare with Remark 4.1.4.

\(^{27}\) For details concerning barycentric coordinates, see, e.g., H.S.M. Coxeter’s book, [8, Section 13.7], or A. Bogomolny’s site [6].
5.3. Herrmann’s endomorphisms and admissible elements

Herrmann introduced in [27, p. 361, p. 367], [29, p. 229] polynomials \( q_{ij} \) and associated endomorphisms \( \gamma_{ij} \) of \( D^4 \) playing the central role in his study of the modular lattice \( D^4 \), in particular, in his construction of perfect polynomials.

For \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), define
\[
q_{ij} = q_{ji} = q_{kl} = q_{lk} = (e_i + e_j)(e_k + e_l). \tag{5.10}
\]

In our denotations,
\[
q_{ij} = a_{ik}^{ji} a_{jl}^{kl}, \quad \text{where} \quad \{i, j, k, l\} = \{1, 2, 3, 4\}. \tag{5.11}
\]

The endomorphism \( \gamma_{ij} \) of \( D^4 \) is denoted as follows:
\[
1 \mapsto q_{ij}, \quad 0 \mapsto 0, \quad e_k \mapsto e_k q_{ij}. \tag{5.12}
\]

For every polynomial \( f(e_1, e_2, e_3, e_4) \), we have
\[
\gamma_{ij} f(e_1, e_2, e_3, e_4) = f(e_1 q_{ij}, e_2 q_{ij}, e_3 q_{ij}, e_4 q_{ij}).
\]

Essentially, by (5.10) among endomorphisms \( \gamma_{ij} \), there are only 3 different:
\[
\gamma_{12}, \quad \gamma_{13}, \quad \gamma_{14}.
\]

**Proposition 5.3.1.** All Herrmann’s endomorphisms \( \gamma_{ij} \) commute:
\[
\gamma_{i} \gamma_{j} = \gamma_{j} \gamma_{i}, \quad i, j \in \{2, 3, 4\}, \quad i \neq j. \tag{5.13}
\]

**Proof.** It suffices to check the commutativity on generators. We will check that
\[
\gamma_{13} (\gamma_{12}(e_i)) = \gamma_{12} (\gamma_{13}(e_i)), \quad \text{where} \quad i = 1, 2, 3, 4.
\]

We have \( \gamma_{12}(e_1) = e_1(e_3 + e_4) \), and
\[
\gamma_{13}(\gamma_{12}(e_1)) = \gamma_{13}(e_1(e_3 + e_4)) = \gamma_{13}(e_1)(\gamma_{13}(e_3) + \gamma_{13}(e_4))
= e_1(e_2 + e_4)(e_3(e_2 + e_4) + e_4(e_1 + e_3))
\]
\[ = e_1(e_2 + e_4)(e_3 + e_4) (e_1 + e_3) \]
\[ = e_1(e_2 + e_4)(e_3 + e_4), \]

and

\[ \gamma_3(\gamma_2(e_1)) = \gamma_2(\gamma_3(e_1)). \] (5.15)

From (5.15) we have

\[ \gamma_1(\gamma_2(e_2)) = \gamma_2(\gamma_1(e_2)) = \gamma_2(\gamma_3(e_2)), \]
\[ \gamma_1(\gamma_2(e_3)) = \gamma_3(\gamma_1(e_3)) = \gamma_3(\gamma_3(e_3)), \]
\[ \gamma_1(\gamma_2(e_4)) = \gamma_4(\gamma_3(e_4)) = \gamma_3(\gamma_3(e_4)). \]

\[ \square \]

5.3.1. More relations on the admissible sequences

Further, we want to find a connection between Herrmann’s endomorphisms \( \gamma_{1i} \) and admissible sequences in \( D^4 \). As we will see in Proposition 5.3.2, endomorphisms \( \gamma_{1i} \) add corresponding indices to the beginning of the given admissible sequence, while elementary maps \( \phi_i \) of Gelfand–Ponomarev add indices to the end of the corresponding admissible sequence, see Table 2.2 for \( D^{2,2,2} \) and Table 4.1 for \( D^4 \). To find this connection, we need more relations connecting admissible sequences in \( D^4 \), see Proposition 4.1.1 and the fundamental property of the admissible sequences (4.1).

**Proposition 5.3.2.** The following relations hold

1. \( 2(13)^i 1 = 2(42)^i 1 \),
2. \( 2(13)^4(14)^i 1 = 2(42)^i (32)^i 1 \),
3. \( 1(41)^i (31)^i (21)^i = (14)^i (13)^i (12)^i 1 = (12)^i (13)^i (14)^i 1 \),
4. \( 1(41)^i (31)^i (21)^i = (12)^i (42)^i (32)^i 1 \),
5. \( 3(41)^i 2 = (32)^i 12 \),
6. \( 3(41)^i (21)^i (31)^i = 3(41)^i (31)^i (21)^i = (32)^i (42)^i (12)^i 1 \),
7. \( 2(42)^i (32)^i (12)^i 1 = (21)^{i+1} (41)^i (31)^i \),
8. \( 3(42)^i (12)^i (32)^i 1 = (31)^i (41)^i (21)^{i+1} \),
9. \( 1(42)^i (32)^i (12)^i 1 = (13)^i (41)^{i+1} (21)^i \),
10. \( 2(32)^i (42)^i (12)^i 1 = 2(41)^i (31)^{i-1} (21)^{i+1} \).

**Proof.**

1. We have

\[ 2(13)^i 1 = 2(13)(13)(13) \ldots (13)(13) 1 = 2(42)(13)(13) \ldots (13)(13) 1 = 2(42)(42)(13) \ldots (13)(13) 1 = 2(42)(42)(42) \ldots (42)(42) 1 = 2(42)^i 1. \]

2. By heading (1) we have

\[ 2(13)^4(14)^i 1 = 2(42)^i (32)^i 1 = 2(42)^i (14)(14) \ldots (14)(14) 1 = 2(42)^i (32)(14)(14) \ldots (14)(14) 1 = 2(42)^i (32)(32)(32) \ldots (32)(32) 1 = 2(42)^i (32)^i 1. \]

3. First,

\[ 1(41)^i = 1(41)(41) \ldots (41) = (14)(14) \ldots (14) 1 = (14)^i 1, \]
\[ 1(41)^i (31)^i (21)^i = (14)^i (13)^i (12)^i 1. \]

By heading (6) of Proposition 4.1

\[ 1(41)^i (31)^i (21)^i = 1(21)^i (31)^i (41)^i, \]
\[ 1(41)^i (31)^i (21)^i = (12)^i (13)^i (14)^i 1. \]
(4) By heading (3) and (2) we have
\[
1(41)^{\ell}(31)^{s}(21)^{f} = (12)^{f}(13)^{s}(14)^{f}1 = (12)^{f-1}1[2(13)^{s}(14)^{f}1]
= (12)^{f-1}1[2(42)^{s}(32)^{f}1] = (12)^{f-1}12(42)^{s}(32)^{f}1
= (12)^{f}(42)^{s}(32)^{f}1.
\]

(5) Here,
\[
3(41)^{s}2 = 3(41)(41)\ldots(41)(41)2 = 3(23)(41)\ldots(41)(41)2
= 3(23)(23)\ldots(23)212 = (32)^{s}12.
\]

(6) By heading (5) of this proposition and by heading (6) of Proposition (4.1) we have
\[
3(41)^{s}(21)^{f}(31)^{r} = (32)^{s}1(21)^{f}(31)^{r}
= (32)^{s}1(31)^{r}(21)^{f} = (32)^{s}(13)^{r}(12)^{f}1
= (32)^{s}(13)(13)\ldots(13)(13)(12)^{f}1
= (32)^{s}(42)(13)\ldots(13)(13)(12)^{f}1
= (32)^{s}(42)(12)^{f}1.
\]

(7) By heading (2)
\[
2(32)^{s}(42)^{f}1 = 2(14)^{s}(13)^{r}1
= 2(13)^{f}(14)^{s}1 = 21(31)^{r}(41)^{s}.
\]

By (5.16) and by heading (6) of Proposition (4.1) we have
\[
2(42)^{f}(32)^{s}(12)^{f}1 = 2(12)^{f}(32)^{s}(42)^{f}1
= 2(12)^{f}1(31)^{r}(41)^{s} = (21)^{f+1}(31)^{r}(41)^{s}
= (21)^{f+1}(41)^{s}(31)^{r}.
\]

(8) By heading (5) and substitution $1 \leftrightarrow 2$
\[
3(42)^{s}1 = (31)^{s}21.
\]

By (5.17) and by heading (1) of Proposition (4.1) we have
\[
3(42)^{s}(12)^{f}(32)^{r}1 = (31)^{s}2(12)^{f}(32)^{r}1 = (31)^{s}(21)^{f}(23)^{r}21
= (31)^{s}(23)^{r}(21)^{f+1} = (31)^{s}(23)(23)\ldots(23)(23)(21)^{f+1}
= (31)^{s}(41)(23)\ldots(23)(23)(21)^{f+1} = (31)^{s}(41)^{r}(21)^{f+1}.
\]

(9) By substitution $3 \rightarrow 1 \rightarrow 2 \rightarrow 3$ in heading (5) we have
\[
1(42)^{r}3 = (13)^{r}23.
\]

In addition,
\[
3(23)^{s}21 = 3(23)(23)\ldots(23)(23)21 = 3(23)(23)\ldots(23)(23)41
3(23)(23)\ldots(23)(41)41 = 3(41)(41)\ldots(41)(41)41 = 3(41)^{s+1}.
\]

From (5.18) and (5.19) we get
\[
1(42)^{r}(32)^{s}(12)^{f}1 = (13)^{r}2(32)^{s}(12)^{f}1 = (13)^{r}(23)^{s}2(12)^{f}1
= (13)^{r}(23)^{r}(21)(21)^{f} = (13)^{r}(41)^{s+1}(21)^{f}.
\]

(10) First, we have
\[
(23)^{s}42 = 2(32)(32)\ldots(32)342 = 2(32)(32)\ldots(32)412
= 2(41)(41)\ldots(41)412 = 2(41)^{s}2.
\]
and
\[ (24)^{-1} 21 = (24)(24) \ldots (24)21 = 1(31)(24) \ldots (24)21 \]
\[ = 1(31)(31) \ldots (31)21 = 1(31)^{r-1}21. \tag{5.22} \]

From (5.21) and (5.22) we get
\[ (23)^4(24)^{-1}2(12)^{-1}1 = (23)^4(24)^{r-1}2(12)^{r-1}1 \]
\[ = 2(41)^4(24)^{r-1}2(12)^{r-1}1 = 2(41)^4(31)^{r-1}2(12)^{r-1}1 \]
\[ = 2(41)^4(31)^{r-1}(21)^{r+1}. \tag{5.23} \]

The proposition is proved. □

5.3.2. How Herrmann’s endomorphisms act on the admissible elements

Before understanding how Herrmann’s endomorphisms act on the admissible elements, we should know how these endomorphisms work on the simple admissible elements \( e_1 a_{ij} \) which are almost atomic elements.²⁸

**Proposition 5.3.3.** Endomorphisms \( \gamma_{ij} \) act on admissible elements as follows:

1. \( \gamma_{ij}(e_1 a_{kl}) = e_1 a_{r+j}, \quad [i, j, k, l] \in \{1, 2, 3, 4\}. \)
2. \( \gamma_{ik}(e_1 a_{kl}) = \gamma_{jl}(e_1 a_{kl}) = e_1 a_{k+l}^{ij}, \quad [i, j, k, l] \in \{1, 2, 3, 4\}. \)

**Proof.** Without loss of generality, we will show that
\[ \gamma_{12}(e_1 a_{34}^{34}) = e_1 a_{r+1}^{34}, \tag{5.24} \]
and
\[ \gamma_{12}(e_1 a_{23}^{23}) = e_1 a_{r+1}^{23}. \tag{5.25} \]

(1) Let us prove (5.24). For \( r = 0 \), we get
\[ \gamma_{12}(e_1) = e_1(e_3 + e_4) = e_1 a_{1}^{34}. \tag{5.26} \]

For \( r = 1 \),
\[ \gamma_{12}(e_1 a_{1}^{34}) = \gamma_{12}(e_1(e_3 + e_4)) = \gamma_{12}(e_1)\gamma_{12}(e_3) + \gamma_{12}(e_4)) \]
\[ = e_1(e_3 + e_4)(e_3(e_1 + e_2) + e_4(e_1 + e_2)) = e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)) \]
\[ = e_1(e_3 + e_4(e_1 + e_2)) = e_1 a_2^{34}. \]

By induction hypothesis and (5.10) we have
\[ \gamma_{12}(e_1 a_{r}^{34}) = \gamma_{12}(e_1(e_3 + e_4 a_{r-1}^{12})) = \gamma_{12}(e_1)\gamma_{12}(e_3) + \gamma_{12}(e_4)\gamma_{34}(a_{r-1}^{12}) \]
\[ = e_1(e_3 + e_4)(e_3(e_1 + e_2) + e_4(e_1 + e_2)a_r^{12}) \]
\[ = e_1(e_3 + e_4)(e_3 + e_4(e_1 + e_2)a_r^{12}) \]
\[ = e_1(e_3 + e_4)(e_3 + e_4 a_{r+1}^{12}) = e_1 a_{r+1}^{34}. \]

(2) Now, let us prove (5.25). For \( r = 1 \),
\[ \gamma_{12}(e_1 a_{1}^{23}) = \gamma_{12}(e_1(e_2 + e_3)) = \gamma_{12}(e_1)\gamma_{12}(e_2) + \gamma_{12}(e_3) \]
\[ = e_1(e_3 + e_4)(e_2(e_3 + e_4) + e_3(e_1 + e_2)) \]
\[ = e_1(e_2(e_3 + e_4) + e_3(e_1 + e_2)) \]
\[ = e_1(e_3 + e_4)(e_2 + e_3)(e_1 + e_2) = e_1(e_3 + e_4)(e_2 + e_3) \]
\[ = e_1 a_{1}^{34} a_{1}^{23}. \]

²⁸ Recall, that polynomials \( a_{ij} \) are called atomic, see Section 1.5.
By induction hypothesis, and again by (5.10) we have
\[
\gamma_{12}(e_1a_r^{23}) = \gamma_{12}(e_1(e_2 + e_3a_{r-1}^{\alpha})) = \gamma_{12}(e_1)(\gamma_{12}(e_2) + \gamma_{34}(e_3a_{r-1}^{\alpha})) \\
= e_1(e_3 + e_4)(e_2(e_3 + e_4) + e_3a_{12}^{14}a_{r-1}^{\alpha}) \\
= e_1(e_3 + e_4)(e_2 + e_3a_{12}^{14}a_{r-1}^{\alpha}) \\
= e_1a_{1}^{34}(e_2 + e_3a_{r-1}^{\alpha}) = e_1a_{1}^{34}a_r^{23}.
\]
The proposition is proved. 

**Theorem 5.3.4.** Endomorphism \( \gamma_{ik} \) acts on the admissible element \( e_{ak} \) as follows:
\[
\gamma_{ik}(e_{i_1i_2...i_2k}) = e_{i_1i_2...i_2k},
\]
or, in other words,
\[
\gamma_{ik}(e_{ak}) = e_{aki}, \tag{5.27}
\]
where \( i, k \in \{1, 2, 3, 4\}, i \neq k, \) and \( \alpha, ak \) are admissible sequences.

Without loss of generality, we will show that
\[
\gamma_{12}(e_{a2}) = e_{a21}. \tag{5.28}
\]

It suffices to prove (5.28) for all cases \( \alpha 2 \) described by Table 4.3 with the permutation \( 1 \leftrightarrow 2 \):

\( F12, F32, F42, G22, G12, G32, G42, H22. \)

**Case F12.** Here,
\[
\alpha 2 = (12)^i(42)^j(32)^s, \quad e_{a2} = e_1a_{2r}^{32}a_{2r}^{42}a_{2r-1}^{34},
\]
\[
e_{\beta} = \gamma_{12}(e_{a2}) = e_1a_{2r}^{32}a_{2r}^{42}a_{2r-1}^{34}, \quad \text{where}
\]
\[
\beta = 1(41)^i(31)^j(21)^k, \quad \text{(case G11).} \tag{5.29}
\]

Thus, by Proposition 5.3.2, heading (4), we have \( \beta = (12)^i(42)^j(32)^s1 = \alpha 21. \)

**Case F32.** We have
\[
\alpha 2 = (32)^i(42)^j(12)^s, \quad e_{a2} = e_3a_{2r}^{12}a_{2r}^{42}a_{2r-1}^{14},
\]
\[
e_{\beta} = \gamma_{12}(e_{a2}) = e_3a_{2r}^{12}a_{2r}^{42}a_{2r-1}^{14}, \quad \text{where}
\]
\[
\beta = 3(41)^i(31)^j(21)^k, \quad \text{(case G31 with } r \leftrightarrow s). \tag{5.30}
\]

By Proposition 5.3.2, heading (6), we have \( \beta = (32)^i(42)^j(12)^s1 = \alpha 21. \)

**Case G22.** Here,
\[
\alpha 2 = 2(42)^i(32)^s(12)^j, \quad e_{a2} = e_2a_{2r}^{14}a_{2r}^{34}a_{2r-1}^{31},
\]
\[
e_{\beta} = \gamma_{12}(e_{a2}) = e_2a_{2r}^{14}a_{2r}^{34}a_{2r-1}^{31}, \quad \text{where}
\]
\[
\beta = (21)^{j+1}(41)^i(31)^k, \quad \text{(case } F21 \text{ with } r \leftrightarrow s, t \to t + 1). \tag{5.31}
\]

By Proposition 5.3.2, heading (7), we have \( \beta = (21)^{j+1}(41)^i(31)^k1 = \alpha 21. \)

**Case G32.** We have
\[
\alpha 2 = 3(42)^i(12)^j(32)^s, \quad e_{a2} = e_3a_{2r}^{12}a_{2r-1}^{42}a_{2r-1}^{24},
\]
\[
e_{\beta} = \gamma_{12}(e_{a2}) = e_3a_{2r}^{12}a_{2r-1}^{42}a_{2r-1}^{24}, \quad \text{where}
\]
\[
\beta = (31)^i(41)^j(21)^{k+1}, \quad \text{(case } F31 \text{ with } r \leftrightarrow s, t \to t + 1). \tag{5.32}
\]

By Proposition 5.3.2, heading (8), we have \( \beta = 3(42)^i(12)^j(32)^s1 = \alpha 21. \)
Case $G12$. We have

\[ \alpha_2 = (142)^r (32)^4 (12)^l, \quad e_{a2} = e_1 a_{2r}^{34} a_{2r+1}^{32} a_{2r-1}^{24}, \]

\[ e_\beta = \gamma_1(e_2) = e_1 a_{2r+1}^{34} a_{2r+1}^{32} a_{2r-1}^{24}, \quad \text{where} \]

\[ \beta = (13)^{r} (41)^{s+1} (21)^l, \quad (\text{case $H11$ with $r \leftrightarrow s$, } s \rightarrow s + 1). \]

By Proposition 5.3.2, heading (9), we have $\beta = (142)^r (32)^4 (12)^l 1 = \alpha 21$.

Case $H22$. In this case we have

\[ \alpha_2 = (23)^r (42)^l (12)^l, \quad e_{a2} = e_2 a_{2r}^{14} a_{2r-1}^{12} a_{2r+1}^{34}, \]

\[ e_\beta = \gamma_1(e_2) = e_2 a_{2r-1}^{14} a_{2r-1}^{12} a_{2r+2}^{34}, \quad \text{where} \]

\[ \beta = 2(41)^r (31)^{r-1} (21)^{l+1}, \quad (\text{case $G21$, } t \rightarrow t + 1, r \leftrightarrow s, r \rightarrow r - 1). \]

By Proposition 5.3.2, heading (10), we have $\beta = (23)^r (42)^l (12)^l 1 = \alpha 21$.

We drop case $F42$ (resp. $G42$) which is similar to $F32$ (resp. $G32$) and is proved just by permutation $3 \leftrightarrow 4$. □

Corollary 5.3.5. For cumulative elements $e(n)$, the following relation holds:

\[ e_l(n + 1) = \gamma_{ij} e_j(n) + \gamma_{ik} e_k(n) + \gamma_{il} e_l(n), \quad \text{where } \{i, j, k, l\} = \{1, 2, 3, 4\}. \]

(5.35)

Proof. Without loss of generality, we will show that

\[ e_1(n + 1) = \gamma_1(e_2(n)) + \gamma_13(e_3(n)) + \gamma_14(e_4(n)). \]

(5.36)

By Theorem 5.3.4 we have

\[ \gamma_1(e_2(n)) = \sum_{|\alpha|=n} e_{a21}, \]

\[ \gamma_13(e_3(n)) = \sum_{|\alpha|=n} e_{a31}, \]

\[ \gamma_14(e_4(n)) = \sum_{|\alpha|=n} e_{a41}, \]

where $\alpha$ in every sum runs over all admissible elements of length $n - 1$. Then,

\[ \gamma_1(e_2(n)) + \gamma_13(e_3(n)) + \gamma_14(e_4(n)) = \sum_{|\beta|=n} e_{\beta 1}, \]

where $\beta$ runs over all admissible elements of length $n$. The last sum is $e_1(n + 1)$ and relation (5.36) is proved. □

Similarly to relation (5.27) for $e_\alpha$, one can prove the following relation for admissible elements $f_{a0}$:

\[ \gamma_{ik}(f_{a0}) = f_{aki0}. \]

(5.37)

For example,

\[ f_{20} = e_2(e_1 + e_3 + e_4), \]

and

\[ \gamma_{12}(f_{20}) = e_2(e_3 + e_4)(e_1(e_3 + e_4) + e_3(e_1 + e_2) + e_4(e_1 + e_2)) = e_2(e_3 + e_4)(e_1 + e_3(e_1 + e_2) + e_4) = f_{210}. \]

see Proposition 4.8.5 and Section 1.9.2.
5.3.3. A sequence of Herrmann’s endomorphisms

Now, we will see how admissible elements $e_\alpha$ are obtained by means of a sequence of Herrmann’s endomorphisms.

**Theorem 5.3.6 (Admissible Elements and Herrmann’s Endomorphisms).** Admissible elements $e_\alpha$ and $f_{s0}$ ending with 1 (from Table 5.1) are obtained by means of Herrmann’s endomorphisms as follows:

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (e_1) = e_1 a_1^{34} a_2^{24} a_r^{32},
\]

and

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (f_{10}) = e_1 a_1^{34} a_2^{24} a_r^{32} (e_1 a_{t+1}^{24} + a_{t+1}^{24} a_{r-1}^{32}), \quad \text{for } r \geq 1.
\]

Similarly, admissible elements $e_\alpha$ and $f_{s0}$ ending with $i = 2, 3, 4$ are obtained as follows:

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (e_2) = e_2 a_1^{34} a_s^{14} a_r^{13},
\]

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (e_3) = e_3 a_1^{34} a_s^{24} a_r^{12},
\]

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (e_4) = e_4 a_1^{34} a_s^{31} a_r^{23}.
\]

and

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (f_{20}) = e_2 a_1^{34} a_s^{14} a_r^{31} (e_2 a_{t+1}^{24} + a_{t+1}^{24} a_{r-1}^{31}), \quad \text{for } r \geq 1.
\]

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (f_{30}) = e_3 a_1^{34} a_s^{24} a_r^{12} (e_3 a_{t+1}^{14} + a_{t+1}^{14} a_{r-1}^{12}), \quad \text{for } r \geq 1.
\]

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (f_{40}) = e_4 a_1^{34} a_s^{31} a_r^{23} (e_4 a_{t+1}^{14} + a_{t+1}^{14} a_{r-1}^{12}), \quad \text{for } r \geq 1.
\]

**Proof.** (1) Consider the action of Herrmann’s endomorphisms on $e_{\alpha}$. For $t = 1, s = 0, r = 0$ relation (5.38) follows from (5.26). Assume, relation (5.38) is true for $r + s + t = n$. By (4.12) and (5.24) we have

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (e_1) = \gamma_{12}(e_1 a_1^{24} a_r^{32}) = \gamma_{12}(e_1 a_1^{34}) \gamma_{12}(e_1 a_1^{24}) \gamma_{12}(e_1 a_1^{32})
\]

\[
= e_1 a_1^{34} (e_2 + e_3 + e_4) + e_3 (e_1 + e_2) + e_4 (e_1 + e_2)
\]

\[
= e_1 a_1^{34} (e_2 + e_3 + e_4) + e_3 (e_1 + e_2) + e_4 (e_1 + e_2)
\]

\[
= e_1 a_1^{34} (e_1 a_1^{32} + a_1^{42}).
\]

(2) Now, consider action on $f_{s0}$. For $t = 1, s = 0, r = 0$, we have

\[
\gamma_{12}(f_{10}) = \gamma_{12}(e_1 (e_2 + e_3 + e_4))
\]

\[
= e_1 (e_3 + e_4) (e_2 + e_3 + e_4) + e_3 (e_1 + e_2) + e_4 (e_1 + e_2)
\]

\[
= e_1 a_1^{34} (e_2 + e_3 + e_4) + e_3 (e_1 + e_2) + e_4 (e_1 + e_2)
\]

\[
= e_1 a_1^{34} (e_1 a_1^{32} + a_1^{42}).
\]

On the other hand, for the case $F_{12}$, $f_{s0}$ can be written as follows:

\[
f_{s0} = e_0 (e_1 a_1^{34} a_2^{32})
\]

\[
= e_0 (e_1 a_1^{34} a_2^{32} + a_2^{32} a_{t+1}^{32})
\]

\[
= e_0 (e_1 a_1^{34} a_2^{32} + a_2^{32} a_{t+1}^{32}).
\]

We use the last expression of $f_{s0}$ in (5.43) for the case $t = 1, r = 0, s = 0$. Then,

\[
f_{s0} = e_0 (e_1 a_1^{34} a_2^{32} + a_1^{42}) = e_0 (e_1 a_1^{32} + a_1^{42}).
\]

Thus, by (5.42) and (5.44) relation (5.39) is true for $t = 1, s = 0, r = 0$.

By (5.39) and Proposition 5.3.3 we have the induction step:

\[
\gamma_{12}^{l_3} \gamma_{14}^{r_2} \gamma_{14}^{r_3} (f_{10}) = \gamma_{12}(e_1 a_1^{34} a_2^{32} + a_2^{32} a_{t+1}^{32})
\]

\[
= (e_1 a_1^{34} (e_2 + e_3 + e_4) (e_1 a_1^{32} + a_1^{42}) + (e_2 + e_3 + e_4) (e_1 a_1^{34} + a_2^{32} a_{t+1}^{32})
\]

\[
= (e_1 a_1^{34} (e_1 a_1^{32} + a_1^{42}) (e_1 a_1^{34} + a_2^{32} a_{t+1}^{32}) + (e_2 + e_3 + e_4) (e_1 a_1^{34} + a_2^{32} a_{t+1}^{32})
\]

\[
= (e_1 a_1^{34} (e_1 a_1^{32} + a_1^{42}) (e_1 a_1^{34} + a_2^{32} a_{t+1}^{32}) + (e_2 + e_3 + e_4) (e_1 a_1^{34} + a_2^{32} a_{t+1}^{32})
\]
5.3.4. The sum of Herrmann’s endomorphisms

Consider the join endomorphism \( \mathcal{R} \) which is the sum of Herrmann’s endomorphisms \( \gamma_i \):

\[
\mathcal{R} = \gamma_{12} + \gamma_{13} + \gamma_{14}.
\]

(5.45)

**Proposition 5.3.7.** The endomorphism \( \mathcal{R} \) relates the inverse cumulative elements (5.5) as follows:

\[
\mathcal{R} e_i^\vee(n) = e_i^\vee(n + 1), \quad \text{where } i = 1, 2, 3, 4,
\]

and

\[
\mathcal{R} f_0^\vee(n) = f_0^\vee(n + 1).
\]

(5.46) (5.47)

**Proof.** (1) By (5.5)

\[
e_i^\vee(n) = \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_t^{34},
\]

and by **Proposition 5.3.3** we have

\[
\mathcal{R}(e_i^\vee(n)) = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_t^{34}
\]

\[
= \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_{r+1}^{34} + \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_{t+1}^{34} + \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_{t+1}^{34}
\]

\[
= \sum_{r+s+t=n} e_1 a_r^{32} a_s^{24} a_{t+1}^{34} = e_i^\vee(n + 1).
\]

(2) By (5.6)

\[
f_0^\vee(n) = \sum_{r+s+t=n} e_a(a_{r}^{jk} + a_{s}^{kl} a_{t}^{lj}),
\]

and, again, by **Proposition 5.3.3**

\[
\mathcal{R}(f_0^\vee(n)) = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n} e_a(a_{r}^{jk} + a_{s}^{kl} a_{t}^{lj})
\]

\[
= (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n} \gamma_{12}^{r} \gamma_{23}^{s} \gamma_{14}^{t}(f_{0}), \quad \text{where } \{i, j, k, l\} = \{1, 2, 3, 4\}.
\]

Thus,

\[
\mathcal{R}(f_0^\vee(n)) = \sum_{r+s+t=n} \gamma_{12}^{r+1} \gamma_{23}^{s} \gamma_{14}^{t}(f_{0}) + \sum_{r+s+t=n} \gamma_{12}^{t} \gamma_{23}^{s+1} \gamma_{14}^{r}(f_{0}) + \sum_{r+s+t=n} \gamma_{12}^{t} \gamma_{23}^{s} \gamma_{14}^{r+1}(f_{0})
\]

\[
= \sum_{r+s+t=n+1} \gamma_{12}^{r} \gamma_{23}^{s} \gamma_{14}^{t}(f_{0}) = f_0^\vee(n + 1). \quad \Box
\]

5.4. Perfect elements in \( D^4 \)

5.4.1. The Gelfand–Ponomarev conjecture

By Gelfand–Ponomarev definition (1.35), elements \( h_i(n) \), where \( i = 1, 2, 3, 4 \) are generators of the 16-element Boolean cube \( B^+(n) \) of perfect elements. According to (1.35), these 16 elements in \( B^+(n) \) are as follows:
formulated below is an analogue of this conjecture for the modular lattice $D^2.2.2$.

Table 1.1.

Conjecture 3.2.4

Elements $h_i(n)$ is proved like the similar relation for $D^2.2.2$, see Proposition 3.2.2. Further, by (5.49) and Section 1.9.2 we have

$h_{\text{max}}(1) = e_1 + e_2 + e_3 + e_4,$

$h_{\text{min}}(1) = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4).$  

By (5.50) also

$h_{\text{min}}(1) \ni e_4(e_1 + e_2 + e_3) = f_{40}.$

and by (5.51) we obtain

$h_{\text{min}}(1) = f_{10} + f_{20} + f_{30} + f_{40}.$

Thus,

$h_{\text{min}}(1) = f_0(2).$  

Relation (5.52) takes place modulo linear equivalence for any $n$:  

$h_{\text{min}}(n) \simeq f_0(n + 1).$  

Relation (5.53) is proved like the similar relation for $D^2.2.2$, see Proposition 3.2.2. According to the Gelfand–Ponomarev conjecture [18, p. 7], the relation (5.53) takes place\footnote{Our Conjecture 3.2.4 formulated below is an analogue of this conjecture for the modular lattice $D^2.2.2$.} in $D^4$ (not only in $D^4/\theta$):

$h_{\text{min}}(n) = f_0(n + 1)$ (Gelfand–Ponomarev conjecture).  

Further, by (5.49)

$h_1(1)h_2(1)h_3(1) = (e_2 + e_3 + e_4)(e_1 + e_3 + e_4)(e_1 + e_2 + e_4) = e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4) + e_4 = f_{10} + f_{20} + f_{30} + e_4 = e_4 + f_0(2).$  

Proposition 5.4.1. Elements $h_i(n)h_j(n)h_k(n)$ and $e_i(n) + f_0(n + 1)$ coincide modulo linear equivalence for all \{i, j, k, l\} = \{1, 2, 3, 4\}:

$h_i(n)h_j(n)h_k(n) \simeq e_l(n) + f_0(n + 1).$
Proof. Assuming that (5.56) is true, we will prove it for \( n + 1 \):
\[
    h_i(n + 1)h_j(n + 1)h_k(n + 1) \simeq e_i(n + 1) + f_0(n + 2).
\] (5.57)
Similarly to Proposition 3.2.1 for \( D_{2.2.2} \), we have the following relation in \( D^4 \) for sums and intersections, commuting on the perfect elements \( v_i \):
\[
    \sum_{p=1,2,3,4} \varphi_p \rho_X^+ \left( \bigcap_{i=j,k} v_i \right) = \bigcap_{i=j,k} \left( \sum_{p=1,2,3,4} \varphi_p \rho_X^+(v_i) \right),
\] (5.58)
where all \( v_i \) are perfect elements.\(^{30}\) Since
\[
    \sum_{p=1,2,3,4} \varphi_p \rho_X^+(h_t(n)) = \rho_X(h_t(n + 1)),
\]
we have in the right side of (5.58), that
\[
    \bigcap_{i=j,k} \left( \sum_{p=1,2,3,4} \varphi_p \rho_X^+(v_i) \right) = \rho_X \left( \bigcap_{i=j,k} h_t(n + 1) \right).
\] (5.59)
In the left side of (5.58) we have
\[
    \sum_{p=1,2,3,4} \varphi_p \rho_X^+ \left( \bigcap_{i=j,k} h_t(n) \right) = \sum_{p=1,2,3,4} \varphi_p \rho_X^+(e_i(n) + f_0(n + 1))
    = \rho_X(e_i(n + 1) + f_0(n + 2)).
\] (5.60)
Relation (5.57) follows from (5.59) and (5.60). \( \square \)

Similarly, other relations from Table 1.1 take place modulo linear equivalence:
\[
    h_i(n)h_j(n)h_k(n) \simeq e_i(n) + f_0(n + 1),
    h_i(n)h_j(n)h_k(n)h_t(n) = h_i(n)h_j(n),
    h_i(n)h_j(n) + h_i(n)h_j(n)h_k(n) = h_i(n).
\] (5.61)

Proposition 5.4.2. The following relations hold:
\[
    h_i(n)h_j(n)h_k(n) + h_i(n)h_j(n)h_t(n) = h_i(n)h_j(n),
    h_i(n)h_j(n) + h_i(n)h_j(n)h_k(n) = h_i(n).
\] (5.62)

Proof. It easily follows from (5.49) and (1.35), see Fig. 5.2. \( \square \)

5.4.2. Herrmann’s polynomials \( s_n, t_n \) and \( p_{i,n} \)
C. Herrmann uses endomorphisms \( \gamma_i \) from (5.12) for the construction of perfect elements \( s_n, t_n \) and \( p_{i,n} \), where \( i = 1, 2, 3, 4 \), see [27, p. 362], [29, p. 229]. In what follows, the definitions of

Polynomials \( s_n \):
\[
    s_0 = I, \quad s_1 = e_1 + e_2 + e_3 + e_4,
    s_{i-1}^i = \gamma_i(s_n), \quad \text{where} \ i = 2, 3, 4,
    s_{n+1} = s_1^2 + s_2^3 + s_3^4.
\] (5.63)

\(^{30}\) In the proof of Proposition 3.2.1 we only change \( \sum \varphi_i \rho_X^+(I) \). According to Corollary 4.3.2, heading (3), in the case of \( D^4 \) we have
\[
    \sum_{i=1,2,3,4} \varphi_i \rho_X^+(I) = \sum_{i=1,2,3,4} \rho_X(e_i(e_j + e_k + e_l)) = \rho_X \left( \bigcap_{i=1,2,3,4} (e_j + e_k + e_l) \right).
\]
Fig. 5.2. The 16-element Boolean cube $B^+(n)$ with generators $h_i(n)$.

**Polynomials $t_n$:**

\[
t_0 = I, \quad t_1 = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4),
\]

\[
t_i = \gamma_i(t_{i-1}), \quad \text{where } i = 2, 3, 4,
\]

\[
t_{n+1} = t_1^n + t_2^3 + t_3.
\]

**Polynomials $p_{k,n}$:**

\[
p_{i,0} = I, \quad p_{i,1} = e_i + t_1, \quad \text{where } i = 1, 2, 3, 4,
\]

\[
p_{i,n+1} = \gamma_i(p_{j,n}) + \gamma_k(p_{k,n}) + \gamma_l(p_{l,n}), \quad \text{where } i = 1, 2, 3, 4, \text{ and } \{i, j, k, l\} = \{1, 2, 3, 4\}.
\]

For example, by (5.45) we have

\[
p_{1,2} = \gamma_1(p_{2,1}) + \gamma_3(p_{3,1}) + \gamma_4(p_{4,1}) = \gamma_1(e_2) + \gamma_3(e_3) + \gamma_4(e_4) + \mathcal{R}(t_1)
\]

\[
= e_2 + e_3 + e_4 + e_2 + e_4 + e_2 + e_3 + t_2 = e_{21} + e_{31} + e_{41} + t_2
\]

\[
= e_1(2) + t_2.
\]

Similarly,

\[
p_{i,2} = e_i(2) + t_2.
\]

By definitions (5.63) and (5.64) we have

\[
s_{n+1} = Rs_n = R^n s_1,
\]

\[
t_{n+1} = R t_n = R^n t_1,
\]

see Fig. 5.3.

5.4.3. **Cumulative elements and Herrmann’s polynomials**

In what follows, we show how Herrmann’s polynomials $s_n$, $t_n$, and $p_{i,n}$, where $i \in \{1, 2, 3, 4\}$, are calculated by means of cumulative elements $e(n)$, $f(n)$.

Polynomials $s_n$ and $t_n$ can be also calculated by means of inverse cumulative elements $e^\vee(n)$, $f^\vee(n)$, see Section 5.2. As above, the main elementary brick in these constructions is an admissible element, obtained as a sequence of Herrmann’s endomorphisms

\[
\gamma_{12}^f \gamma_{13}^f \gamma_{14}^f(e_i),
\]

see Theorem 5.3.6 from Section 5.3.3.

---

31 Compare with different system of generators in $D^{2,2,2}$, see Section 3.2.2, Proposition 3.2.5, Table 3.1.
Theorem 5.4.3 (The polynomials $s_n$, $t_n$ and $p_{i,n}$).

(1) For each $n$, the polynomial $s_n$ is the maximal perfect element in the Boolean cube $B^+(n)$, see Section 1.11.3, and it is expressed as follows:

$$s_n = \mathcal{R}^{n-1}(s_1) = \sum_{r+s+i=n-1} \gamma_1^r \gamma_3^s \gamma_4^i (e_1 + e_2 + e_3 + e_4)$$

$$= \sum_{i=1,2,3,4} e_i^\vee(n) = \sum_{i=1,2,3,4} e_i(n). \quad (5.68)$$

(2) For each $n$, the polynomial $t_n$ is linearly equivalent to the minimal perfect element in the Boolean cube $B^+(n)$, see Section 1.11.3, and it is expressed as follows:

$$t_n = \mathcal{R}^{n-1}(t_1)$$

$$= \sum_{r+s+i=n} \gamma_1^r \gamma_3^s \gamma_4^i (e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4))$$

$$= f_0^\vee(n + 1) = f_0(n + 1). \quad (5.69)$$

(3) For each $n$, the polynomial $p_{i,n}$ is linearly equivalent to the element $h_j(n)h_k(n)h_l(n)$ in the Boolean cube $B^+(n)$, see Table 1.1, and it is expressed as follows:

$$p_{i,n} = e_i(n) + f_0^\vee(n + 1) = e_i(n) + f_0(n + 1). \quad (5.70)$$

Proof. (1) By definition (5.63) of $s_n$, and by property (5.24), for $n = 1$, we have

$$s_2 = \gamma_1^2(s_1) + \gamma_1^3(s_1) + \gamma_1^4(s_1) = \sum_{i=1,2,3,4} (\gamma_1^2 + \gamma_1^3 + \gamma_1^4)(e_i)$$

$$= \sum_{i=1,2,3,4} \gamma_1^j(e_i) = \sum_{i=1,2,3,4} e_i a_1^i = \sum_{|\alpha|=2} e_\alpha = e_1^\vee(2) + e_2^\vee(2) + e_3^\vee(2) + e_4^\vee(2). \quad (5.71)$$
In (5.71) \(|\alpha|\) is the length of sequence \(\alpha\). Note, that in (5.71) \(|\alpha| = 2\) and \(r + s + t = 1\). Assume,

\[
s_n = \sum_{|\alpha|=n} e_{\alpha} = e_1^\vee(n) + e_2^\vee(n) + e_3^\vee(n) + e_4^\vee(n). \tag{5.72}
\]

Then, by definition (5.63) by Proposition 5.3.7 and Proposition 5.2.1 we have

\[
s_{n+1} = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{|\alpha|=n} e_{\alpha} = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{i=1,2,3,4} \gamma_{12}^i \gamma_{13}^i \gamma_{14}^i (e_i) = \sum_{i=1,2,3,4} \gamma_{12}^i \gamma_{13}^i \gamma_{14}^i (e_i) = \sum_{i=1,2,3,4} e_i (n+1) = \sum_{i=1,2,3,4} e_i (n+1), \tag{5.73}
\]

and \(s_{n+1}\) is the maximal element in \(B^+(n+1)\), see Section 1.11.3.

(2) It is easy to see that \(t_1\) in (5.64) is as follows:

\[
t_1 = e_1(e_2 + e_3 + e_4)e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4) = f_{10} + f_{20} + f_{30} = f_0(2), \tag{5.74}
\]

and by (5.39) from Theorem 5.3.6 we have

\[
t_2 = \mathcal{R}(f_0(2)) = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{i=1,2,3,4} f_{i0} = \sum_{i \neq j} f_{ij0} = \sum_{|\alpha|=2} f_{\alpha} = f_0(3). \tag{5.75}
\]

Assume, (5.69) is true for some \(n\), then by (5.5) we have

\[
t_n = \sum_{i=1,2,3,4} \gamma_{12}^i \gamma_{13}^i \gamma_{14}^i (f_{i0}) = f_0^\vee(n+1). \tag{5.76}
\]

Then, by Proposition 5.3.7 and Theorem 5.3.6 we have

\[
t_{n+1} = R f_0^\vee(n) = f_0^\vee(n+1) = \gamma_{12}^i \gamma_{13}^i \gamma_{14}^i (f_{i0}). \tag{5.77}
\]

By Proposition 5.2.1 \(f_0^\vee(n) = f(n)\) and according to (5.53), the element \(t_{n+1} = f_0(n+1)\) is linearly equivalent to the minimal element in \(B^+(n+1)\). If the Gelfand–Ponomarev conjecture (5.54) is true, the element \(t_{n+1} = f_0(n+1)\) coincides with the minimal element \(h_{\text{min}}(n+1)\) of \(B^+(n+1)\), see Section 5.4.1.

(3) For \(n = 1\), \(p_{i,1} = e_1 + t_1 = e_1(1) + f_0(2)\), and

\[
p_{i,2} = \gamma_{ji}(e_j + t_1) + \gamma_{ki}(e_k + t_1) + \gamma_{li}(e_l + t_1) = e_i(2) + \mathcal{R}t_1 = e_i(2) + t_2 = e_i(2) + f_0(3), \tag{5.78}
\]

see (5.66) and (5.75). Further, by Corollary 5.3.5 we have

\[
p_{i,n+1} = \gamma_{ji}(p_{j,n}) + \gamma_{ki}(p_{k,n}) + \gamma_{li}(p_{l,n}) = \gamma_{ji}(e_j(n) + t_n) + \gamma_{ki}(e_k(n) + t_n) + \gamma_{li}(e_l(n) + t_n) = \gamma_{ji}(e_j(n)) + \gamma_{ki}(e_k(n)) + \gamma_{li}(e_l(n)) + \mathcal{R}(t_n) = e_i(n+1) + t_{n+1}. \tag{5.79}
\]

By (2)

\[
p_{i,n+1} = e_i(n+1) + t_{n+1} = e_i(n+1) + f_0(n+1). \tag{5.80}
\]

The theorem is proved. □

As corollary we obtain the following
Theorem 5.4.4 (C. Herrmann [27]). Polynomials $p_{1,n}$, $p_{2,n}$, $p_{3,n}$, $p_{4,n}$ are atoms in 16-element Boolean cube $B^+(n)$ of perfect elements and

$$p_{i,n} \simeq h_j(n)h_k(n)h_l(n), \quad \text{where } \{i, j, k, l\} = \{1, 2, 3, 4\}. \quad (5.81)$$

Polynomials $s_n$ and $t_n$ are, respectively, the maximal and minimal elements in $B^+(n)$, and

$$s_n \simeq \sum_{i=1,2,3,4} h_j(n), \quad t_n \simeq \bigcap_{1=1,2,3,4} h_j(n), \quad (5.82)$$

see Table 1.1.

Proof. From (5.70) and (5.61) we obtain (5.81). From (1.39), (5.69) and (5.48) we obtain (5.82). □

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Appendix A. Modular lattices, representations, Coxeter functors

A.1. Modular lattices

A.1.1. Permutation properties

A lattice is said to be modular if, for every $b$, $a$, $c \in L$,

$$a \subseteq b \implies b(a + c) = a + bc. \quad (A.1)$$

The relation (A.1) is said to be the modular law. The modular law is sometimes referred to as Dedekind’s law.

A set of subspaces of the given vector space $R$ (and, of course, submodules of the given module $M$) form a modular lattice, namely if $A$, $B$, $C \subseteq R$ and $A \subseteq B$, then

$$B(A + C) = A + BC.$$ 

Indeed, the inclusion $A + BC \subseteq B(A + C)$ holds, because $A + BC \subseteq A + C$ and $A + BC \subseteq B$. Conversely, let $x \in B(A + C)$, i.e., $x \in B$ and $x = x_a + x_c$, where vectors $x_a$ and $x_c$ are components of the vector $x$, such that $x_a \in A$ and $x_c \in C$. Then $x_c = x - x_a \in B$, i.e., $x_c \in BC$ and $x_a + x_c \in A + BC$.

The main properties of modular lattices were derived by R. Dedekind [11,5]. We need two features of the modular lattice, which are called permutation properties of the modular lattice or just permutation properties.

Proposition A.1.1 (Permutation Properties). In any modular lattice $L$, for every four elements $A$, $B$, $C$, $D \subseteq L$, the following two permutation properties hold

$$A(B + CD) = A(CB + D) = A(CB + CD) \quad \text{for every } A \subseteq C. \quad (A.2)$$

$$A + D(B + C) = A + B(D + C) = A + (D + C)(B + C) \quad \text{for every } C \subseteq A. \quad (A.3)$$
Fig. A.1. The diamond $M_3$ and the pentagon $N_5$.

**Proof.** By the modular law (A.1) we have

$$A(B + CD) = AC(B + CD) = A(CB + CD).$$

The second part of (A.2) follows from the first by symmetry. Relation (A.3) is dual to (A.2). □

Every distributive lattice $L$ is also modular. Indeed, if $a \subseteq b$, then

$$b(a + c) = ba + bc = a + bc,$$

i.e., (A.1) holds. The converse is false. Consider the diamond lattice $M_3$, see Fig. A.1. It easily follows from the modular law (A.1) that the lattice $M_3$ is modular. However, $M_3$ is not distributive, because

$$c(a + b) = c, \quad \text{but } ca + cb = ab, \quad \text{i.e., } c(a + b) \neq ca + cb,$$

(A.4) see Fig. A.1. Further, the pentagon $N_5$ is not a modular lattice, because

$$x \subseteq y, \quad y(x + z) = y, \quad x + yz = x, \quad \text{i.e., } y(x + z) \neq x + yz.$$

(A.5)

A.1.2. Characterization of modular and distributive lattices

The following characterization of modular lattices is well-known [5,23].

**Proposition A.1.2.** A lattice $L$ is modular if and only if it does not contain pentagon $N_5$ as a sublattice.

**Proposition A.1.3.** A lattice $L$ is distributive if and only if it does not contain sublattices $M_3$ and $N_5$.

**Proposition A.1.4.** For given $a, b, c$ in a modular lattice $L$ the following distributivity relations are equivalent:

(i) $a(b + c) = ab + ac$,

(ii) $a + bc = (a + b)(a + c)$.

**Proof.** Let us prove the implication (i) $\implies$ (ii):

$$(a + b)(a + c) = a + (a + b)c = a + ac + bc = a + bc.$$ 

Conversely, (ii) $\implies$ (i):

$$ab + ac = (ab + c)a = (a + c)(b + c)a = a(b + c).$$ □

**Corollary A.1.5.** For an element $u$ of a modular lattice $L$ and a sublattice $S$ of $L$ the following are equivalent:

(i) $u(x + y) = ux + uy$ for all $x, y \in S$

(i’) $x \mapsto ux$ is a lattice homomorphism

(ii) $u + xy = (u + x)(u + y)$ for all $x, y \in S$

(ii’) $x \mapsto u + x$ is a lattice homomorphism
Proof. (i) equivalent (ii) by Proposition A.1.4. Obviously, the map in (i) preserves meets. Further, the map in (i) preserves joins if and only if (i) holds. By duality (ii) and (ii) are equivalent. □

A.1.3. B. Jónsson’s criterion of distributivity

The following criterion of distributivity of a modular lattice is proved by B. Jónsson:

Proposition A.1.6 ([34, Theorem 5]). Let A be a modular lattice, p a positive integer, and \( X_1, X_2, \ldots, X_p \) non-empty linearly ordered subsets of A. For the sublattice of A generated by the set
\[ X_1 \cup X_1 \cup \cdots \cup X_p \]
to be distributive it is necessary and sufficient that, for any
\[ x_1 \in X_1, x_2 \in X_2, \ldots, x_p \in X_p, \]
the sublattice of A generated by the set \( \{ x_1, x_2, \ldots, x_p \} \) be distributive. □

We use this result for \( p = 3 \) in the proof of the distributivity of the sublattice of perfect elements \( H^+(n) \) in Proposition 3.1.4.

A.1.4. Lattice varieties

G. Birkhoff introduced in [4] the notion of a variety, i.e., the class of all algebras (in the sense of universal algebras) that satisfy every identity in the given set of identities (equations) \( \mathcal{E} \). In particular, for lattices, the class of all lattices \( \mathcal{V} \) that satisfy every identity in \( \mathcal{E} \) is said to be the lattice variety defined by equations \( \mathcal{E} \). We denote by \( \text{Mod} \mathcal{E} \) the class of lattices that satisfy every identity in \( \mathcal{E} \), and we write
\[ \mathcal{V} = \text{Mod} \mathcal{E}. \]

Lattice varieties are also called equational classes or equationally defined classes, see [53].

Birkhoff showed in [4] that varieties are precisely those classes of algebras that are closed under the formation of homomorphic images, subalgebras, and direct products, i.e.

\[ \mathcal{V} \text{ is variety if and only if } \text{H}\mathcal{V} = \mathcal{V}, \text{S}\mathcal{V} = \mathcal{V}, \text{P}\mathcal{V} = \mathcal{V}, \]

where
\[ \text{H}\mathcal{V} — \text{the class of all homomorphic images of members of } \mathcal{V}, \]
\[ \text{S}\mathcal{V} — \text{the class of all subalgebras of members of } \mathcal{V}, \]
\[ \text{P}\mathcal{V} — \text{the class of all direct products of members of } \mathcal{V}. \]

Tarski in [53] put the Birkoff’s theorem in the form
\[ \mathcal{V} \text{ is variety if and only if } \text{HSP}\mathcal{V} = \mathcal{V}. \]

(A.6)

Here, examples of lattice varieties:
\[ \mathcal{D} = \text{Mod} \{ xy + xz = x(y + z) \} — \text{all distributive lattices}, \]
\[ \mathcal{M} = \text{Mod} \{ xy + xz = x(y + xz) \} — \text{all modular lattices}. \]

(A.7)

For details, see [32,7].

The Arguesian lattices (see [33,46,35]) also form a lattice variety.

The lattices of interest are freely generated by a poset \( P \) (where \( P = 1 + 1 + 1 + 1 \) and \( P = 2 + 2 + 2 \)) in \( V_p \), in the join \( V \) of all \( V_p \), and in variety of all modular lattices \( \mathcal{M} \), see Section 1.2.2, and (A.8). The freely generated
lattices in $V_p$, $V$ and $\mathcal{M}$ are as follows:

\[
FM(P) = \begin{cases} 
D^4, & \text{for } P = 1 + 1 + 1 + 1, \\
D^{2.2.2}, & \text{for } P = 2 + 2 + 2,
\end{cases} \]

\[
FV(P) = \begin{cases} 
D^4/\theta, & \text{for } P = 1 + 1 + 1 + 1, \\
D^{2.2.2}/\theta, & \text{for } P = 2 + 2 + 2.
\end{cases} \tag{A.9}
\]

\[
FV_p(P) = \begin{cases} 
D^4/\theta_p, & \text{for } P = 1 + 1 + 1 + 1, \\
D^{2.2.2}/\theta_p, & \text{for } P = 2 + 2 + 2,
\end{cases}
\]

see (1.4), (1.5). We have canonical homomorphisms:

\[
\pi : FM(P) \rightarrow FV(P), \quad \pi_p : FV(P) \rightarrow FV_p(P), \tag{A.10}
\]

where

\[
\pi(a) = \pi(b) \quad \text{if and only if} \quad a \equiv b \mod \theta,
\]

\[
\pi_p(a) = \pi_p(b) \quad \text{if and only if} \quad a \equiv b \mod \theta_p, \tag{A.11}
\]

see Section 1.2.2.

A.1.5. When is the union of two 3D cubes a 4D cube?

**Proposition A.1.7** (The union of two 3D cubes). Let $L$ be a modular lattice generated by $D \cup C$ where $D$ and $C$ are 8-element Boolean algebras with coatoms $d_1, d_2, d_3 \in D$, coatoms $c_1, c_2, c_3 \in C$. If the following inclusions hold:

\[
c_i \subseteq d_i, \quad i = 1, 2, 3, \tag{A.12}
\]

\[
d_i \subseteq c_i + d_j, \quad i, j \in \{1, 2, 3\}, \tag{A.13}
\]

\[
d_i c_j \subseteq c_i, \quad i \neq j, i, j \in \{1, 2, 3\}, \tag{A.14}
\]

then $L$ is a 16-element Boolean algebra or $D = C = L$, see Fig. A.2.

**Proof.** Let $I_d \in D$ (resp. $I_c \in C$) be maximal elements in $D$ (resp. $C$), and let $O_d \in D$ (resp. $O_c \in C$) be minimal elements in $D$ (resp. $C$). Since elements $c_i$ (resp. $d_i$) are coatoms in $C$ (resp. $D$), we have

\[
d_i + d_j = I_d \quad \text{and} \quad c_i + c_j = I_c \quad \text{for } i \neq j.
\]

Clearly,

\[
d_1 d_2 d_3 = O_d \quad \text{and} \quad c_1 c_2 c_3 = O_c.
\]
For \( i \neq j \), by (A.13) we have
\[
c_i + d_j \supseteq d_i + d_j = I_d, \quad \text{i.e., } c_i + d_j = I_d, \quad \text{and } I_c + d_j = I_d.
\tag{A.15}
\]
By modularity and (A.12) and (A.14) we have
\[
d_j I_c = (c_i + c_j)d_j = c_j + c_id_j = c_j.
\tag{A.16}
\]
Thus, \( L \) is generated by \( D \cup \{I_c\} \). By (A.15)
\[
d_k + c_ic_j = d_k + c_ic_j + c_i c_k = d_k + c_i = I_d,
\]
and
\[
O_d + c_ic_j = d_id_j(d_k + c_i c_j) = d_id_j.
\]
Thus, \( L \) is generated also by \( C \cup \{O_d\} \). Further, from (A.12) and (A.14) we have
\[
c_i c_j \subseteq d_1 c_j \subseteq c_i c_j \quad \text{and } \quad d_id_j = c_i c_j.
\tag{A.17}
\]
From (A.16) and (A.17)
\[
O_d I_c = d_1 d_2 d_3 I_c = d_1 d_2 c_3 = c_1 c_2 c_3 = O_c.
\]
Now, by (A.15)
\[
d_i d_j + I_c = d_i d_j + c_i + I_c = d_i(d_j + c_i) + I_c = d_i + I_c = I_d,
\]
and
\[
O_d + I_c \supseteq O_d + c_1 + I_c = d_1(d_2 d_3 + c_1) + I_c = d_1 + I_c = I_d,
\]
i.e.,
\[
O_d + I_c = I_d.
\]
To summarize, we have that \( L \) is generated also by \( C \cup \{O_d\} \) where \( C \) is Boolean with smallest element \( O_c \) and greatest element \( I_c \), and \( O_d + I_c = I_d \). Elements \( c_1 c_2, c_1 c_3, c_3 c_3, I_d \) are independent and the claim follows from [39, Part I, Chapter II, Theorem 2.7] or from [5, Chapter III, Theorem 15, Corollary 2].

In the case \( c_1 = d_1 \), by (A.17) we have \( d_1 d_2 = c_1 d_2 = c_1 c_2, d_1 d_3 = c_1 c_3 \), and \( d_1 d_2 d_3 = c_1 c_2 c_3 \). Further, by (A.15) \( c_1 + c_2 = d_1 + c_2 = I_d = I_c \). By (A.16) \( d_2 I_c = c_2 = d_2 I_d = d_2, d_3 = c_3 \), etc. Thus, \( D = C = L \). \( \square \)

A.2. Representations of modular lattices

A.2.1. Representation in subspaces and quotient spaces

In this section we mostly follow definitions of Gelfand and Ponomarev [18].

Let \( L \) be a modular lattice, \( X \) a finite-dimensional vector space over the field \( K \), \( L(X) \) the modular lattice of subspaces in \( L \). A morphism \( \rho_X : L \rightarrow L(X) \) is called a \textit{linear representation} of \( L \) in the space \( X \). Thus, the representation \( \rho_X : L \rightarrow L(X) \) maps every element \( a \in L \) to the subspace \( \rho_X(a) \subseteq X \) such that, for every \( a, b \in L \), we have:
\[
\rho_X(ab) = \rho_X(a) \rho_X(b), \quad \rho_X(a + b) = \rho_X(a) + \rho_X(b).
\]

Let \( \rho_U : L \rightarrow L(U) \) and \( \rho_V : L \rightarrow L(V) \) be representations of \( L \) in the spaces \( U \) and \( V \). We put
\[
\rho(a) = \rho_U(a) \oplus \rho_V(a)
\]
for every \( a \in L \), where the subspace \( \rho(a) \subseteq U \oplus V \) is the set of all pairs
\[
\{(u, v) \mid u \in \rho_U(a) \text{ and } v \in \rho_V(a)\}.
\]
This correspondence gives the representation \( \rho \) in the space \( U \oplus V \). The representation \( \rho \) is called the \textit{direct sum} \( \rho_U \oplus \rho_V \) of representations, and we write \( \rho = \rho_{U \oplus V} \).
The representation \( \rho_X \) is said to be **decomposable** if it isomorphic to a direct sum \( \rho_U \oplus \rho_V \) of non-zero representations \( \rho_U \) and \( \rho_V \). It is easy to prove that the representation \( \rho_X \) in the space \( X \) is decomposable if and only if there are subspaces \( V \) and \( U \) such that \( X = U \oplus V \) and

\[
\rho_X(a) = U\rho_X(a) + V\rho_X(a) \quad \text{for every} \quad a \in L.
\]

The representation \( \rho_X \) is said to be **indecomposable** if there exists an element \( a \in L \) such that

\[
\rho_X(a) \neq U\rho_X(a) + V\rho_X(a)
\]

for every decomposition \( X \) into a direct sum \( X = U \oplus V \).

Let \( \rho_X \) be the representation of the modular lattice \( L \) in the vector space \( X \). The subspace \( U \subseteq X \) is said to be **admissible with respect to** \( \rho_X \), if one of the following two relations hold

\[
\begin{align*}
(i) \ U(\rho_X(a) + \rho_X(b)) &= U\rho_X(a) + U\rho_X(b), \\
(ii) \ U + \rho_X(a)\rho_X(b) &= (U + \rho_X(a))(U + \rho_X(b)) \quad \text{(A.18)}
\end{align*}
\]

for every \( a, b \in L \). The equivalence of (i) and (ii) follows from Proposition A.1.4.

**Proposition A.2.1** ([18, Prop. 2.1]). Let \( \rho_X \) be a representation of the lattice \( L \) in a vector space \( X \), let \( U \) be the subspace of \( X \), and \( \nabla : X \rightarrow X/U \) the canonical map. Then the following relations are equivalent:

1. The subspace \( U \) is admissible with respect to \( \rho_X \).
2. The map \( x \mapsto U\rho_X(x) \) defines a representation in the subspace \( U \).
3. The mapping \( x \mapsto \nabla\rho_X(x) \) defines a representation in the quotient space \( X/U \).

**Proof.** Let us prove the implication (1) \( \implies \) (2). If \( U \) is admissible with respect to \( \rho_X \), then

\[
U\rho_X(a + b) = U(\rho_X(a) + \rho_X(b)) = U\rho_X(a) + U\rho_X(b).
\]

Besides,

\[
U\rho_X(ab) = U\rho_X(a)\rho_X(b) = (U\rho_X(a))(U\rho_X(b)).
\]

Thus, we have a representation in the subspace \( U \). The representation in the subspace \( U \) is called a **subrepresentation** of \( \rho_X \) and is denoted by \( \rho_U \). \( \Box \)

Conversely, (2) \( \implies \) (1):

\[
\rho_U(a) + \rho_U(b) = \rho_U(a + b),
\]

\[
U\rho_U(a) + U\rho_U(b) = U\rho_U(a + b) = U(\rho_U(a) + \rho_U(b)).
\]

So, we have (A.18)(i). \( \Box \)

Now, consider (1) \( \implies \) (3). Set

\[
v(a) = \rho_X(a) + U,
\]

Then (A.18)(ii) is equivalent to the relation

\[
v(a)v(b) = v(ab).
\]

(A.19)

Since, \( \ker \nabla = U \), we have \( \nabla\rho_X(a) = \nabla v(a) \). Further,

\[
\nabla v(a + b) = \nabla(\rho_X(a + b)) = \nabla(\rho_X(a) + \rho_X(b)) = \nabla\rho_X(a) + \nabla\rho_X(b) = \nabla v(a) + \nabla v(b),
\]

and (3) from **Proposition A.2.1** is equivalent to the following relation:

\[
(\nabla v(a))(\nabla v(b)) = \nabla v(ab).
\]

(A.20)

32 Recall that \( U\rho_X(a) \) denotes the intersection of subspaces: \( U\rho_X(a) = U \cap \rho_X(a) \).
Thus, we just need to prove that (A.20) follows from (A.19). The inclusion
\[
(\nabla v(a))(\nabla v(b)) \supseteq \nabla v(ab)
\]
is obvious, since \(v(ab) \subseteq v(a), v(b)\). So, it is sufficient to prove that
\[
(\nabla v(a))(\nabla v(b)) \subseteq \nabla v(ab).
\]
(A.21)

Let \(z \in \nabla v(a)\) and \(z \in \nabla v(b)\). Then there exist vectors \(v_a \in v(a)\) and \(v_a \in v(b)\) such that
\[
\nabla(v_a) = \nabla(v_b) = z.
\]
Then, \(w = v_a - v_b \in \text{ker } \nabla = U\), so
\[
v_b = v_a - w \in v(a) + U = v(a),
\]
i.e., \(v_b \in v(a)\), so
\[
v_b \in v(a)v(b) = v(ab)
\]
and \(z \in \nabla v(ab)\), and hence (A.21) and (A.20) are proved. □

Finally, consider the implication (3) \(\Rightarrow\) (1). We need to prove that (A.19) follows from (A.20). Let \(v \in v(a)v(b)\), i.e.,
\[
v \in v(a) = \rho_X(a) + U, \quad \text{and} \quad v \in v(b) = \rho_X(b) + U.
\]
By (A.20) there exists \(w \in v(ab)\) such that \(\nabla(v) = \nabla(w)\). Therefore, \(v - w \in U\) and \(v \in v(ab) + U = v(ab)\). Thus, \(v(a)v(b) \subseteq v(ab)\). The inverse inclusion is obvious and (A.19) holds. □

### A.2.2. Representations of quivers and lattices

In a sense, representations of lattices are equivalent to the representations of quivers [2,16]. We demonstrate this fact on the example of the diagram \(\tilde{E}_6\).

**Proposition A.2.2.** Let \(\rho_X\), where \(X = X_0\), be the indecomposable representation of the diagram \(\tilde{E}_6\),
\[
\begin{array}{c}
X_3 \\
\downarrow g_3 \\
Y_3
\end{array}
\quad \rho_X : \quad
\begin{array}{c}
X_1 & \overset{g_1}{\rightarrow} & Y_1 & \overset{f_1}{\rightarrow} & X_0 & \leftarrow & Y_2 & \overset{f_2}{\leftarrow} & X_2 \\
\downarrow f_3
\end{array}
\]
(A.22)

If \(X_0 \neq 0\), then maps \(f_i\) and \(g_i\) are monomorphisms.

For a proof, see Section A.7 in [51]. □

An analogue of this proposition for an arbitrary star quiver with orientation in which all arrows directed to the branch vertex space \(X_0\), is also true, as one can show.

**Corollary A.2.3.** The category of representations of the diagram \(\tilde{E}_6\) with orientation (A.22) coincides with the category of representations of the lattice \(D^{2,2,2}\) in non-zero spaces:
\[
\rho : L \rightarrow L(X), \quad X \neq 0. \quad \square
\]

In particular, the Coxeter functors \(\Phi^+, \Phi^-\) which appeared in the representation theory of quivers [2] can be applied to the representations of lattices [18]. The reflection functors\(^{33}\) from [2] cannot, however, be used because these functors change the orientation of the quiver.

\(^{33}\) For definition of reflection functors and Coxeter functors, see Appendix A.2.4.
A.2.3. The path algebra of a quiver

Let $K$ be a field and $Q$ be a quiver. Let $Q_0$ be the set of vertices of the quiver $Q$, $Q_1$ be the set of arrows of the quiver $Q$.

The path algebra $KQ$ associated to a quiver $Q$ is $K$-vector space with basis given by all the paths in $Q$ and multiplication defined by concatenation of paths. If $\alpha, \beta$ are two paths in $KQ$ then

$$\alpha \beta \neq 0 \text{ if and only if } t(\beta) = s(\alpha),$$

where $s(\alpha)$ (resp. $t(\alpha)$) is the starting (resp. terminating) vertex of the arrow $\alpha$.

The path algebra $KQ$ includes all the trivial paths $p_i$ for each vertex $i$. The trivial paths $p_i$ are mutually orthogonal idempotents:

$$p_i^2 = p_i \quad \text{for every } i \in Q_0,$$

$$p_ip_j = 0 \quad \text{for } i \neq j,$$

and the sum of all the trivial paths is identity element in the path algebra $KQ$. Besides, multiplication of trivial paths and arrows satisfies the following relations:

$$p_{t(\alpha)} \alpha = \alpha p_{s(\alpha)} = \alpha,$$

$$p_j \alpha = 0, \quad \alpha p_t = 0 \quad \text{for } j \neq t(\alpha), t \neq s(\alpha).$$

The path algebra $KQ$ of the quiver $Q$ depicted in Fig. A.3 has the basis

$$p_1, p_2, p_3, p_4, p_5,$$

$$\alpha, \beta, \gamma, \delta, \beta \alpha, \gamma \beta, \gamma \beta \alpha, \gamma \delta,$$

and $\dim KQ = 13$. Here,

$$\alpha \beta = 0, \quad \beta \delta = 0, \quad \delta \beta = 0,$$

$$p_3 \beta = \beta p_2 = \beta, \quad \beta p_3 = 0, \quad p_2 \beta = 0,$$

etc. For this quiver, we have

$$p_1 + p_2 + p_3 + p_4 + p_5 = e.$$ 

The path algebra is finite-dimensional if and only if $Q$ has no oriented cycles. The path algebra has the structure of a $\mathbb{Z}$-graded algebra by defining the degree of path to be its length. The trivial paths $p_i$ have degree zero. Thus, degree zero component is as follows:

$$(KQ)_0 = \bigoplus_{v \in Q_0} Kp_v,$$

The degree one component $(KQ)_1$ has basis all the arrows of $Q$. $(KQ)_1$ is the bimodule over $(KQ)_0$.

$$(KQ)_1 = \bigoplus_{\alpha \in Q_1} K\alpha,$$

**Proposition A.2.4 ([21]).** The category of $K$-representations of $Q$ is equivalent to the category of left $KQ$-modules.
Proof. Let \( \rho \) be a \( K \)-linear representation of \( Q \) with vector \( K \)-spaces \( V_i, i \in Q_0 \), and \( K \)-linear maps \( \phi_\alpha, \alpha \in Q_1 \). We set
\[
V = \bigoplus_{i \in Q_0} V_i \tag{A.26}
\]
and \( V \) will be turned into \( KQ \) module as follows. The multiplication by trivial path \( p_i \) acts as the projection onto \( V_i \):
\[
p_i V = V_i.
\]
Then \( p_i \) satisfies the (A.24).

If \( (i, j) \in Q_1, \alpha : i \to j \) and \( (r, t) \in Q_1, \beta : r \to t \) then multiplication by \( \alpha \) and \( \beta \) acts as the composition
\[
V \to V_i \xrightarrow{\alpha} V_j \to V, \quad V \to V_r \xrightarrow{\beta} V_t \to V,
\]
when \( \to \) is the projection and \( \hookrightarrow \) inclusion. Then \( \beta \alpha \neq 0 \) if and only if \( j = r \), i.e. \( i \to j \to t \) is the path in \( KQ \).

Conversely, let \( V \) be a \( KQ \)-module, i.e.
\[
\alpha V \subseteq V \quad \text{for every } \alpha \in KQ. \tag{A.27}
\]
We set \( V_i = p_i V \). Since, \( p_i \) and \( p_j \) are orthogonal idempotents (A.24), we have that \( V \) is decomposed into direct sum (A.26).

By (A.25) we have \( \alpha p_s(\alpha) = p_t(\alpha)\alpha \). Applying (A.27) to (A.26) we have
\[
\alpha p_s(\alpha)V = \alpha V_s(\alpha) = p_t(\alpha)\alpha V \subseteq p_t(\alpha)V = V_t(\alpha),
\]
\[
\alpha V_i = \alpha p_i V = 0 \quad \text{for } s(\alpha) \neq i,
\]
i.e.,
\[
\alpha V_s(\alpha) \subseteq V_t(\alpha), \quad \alpha V_i = 0 \quad \text{for } s(\alpha) \neq i. \tag{A.28}
\]
Thus, the map
\[
\phi_\alpha : V_s(\alpha) \longrightarrow V_t(\alpha)
\]
can be defined as the restriction of \( \alpha \) from \( V \) on \( V_s(\alpha) \) and \( \rho = \{V_i, \phi_\alpha\}. \quad \Box
\]

Remark A.2.5. The path algebra can be defined only by means of trivial paths \( p_i \), associated with vertices \( i \in Q_0 \). Then, instead of arrow \( \alpha \) the product \( p_t(\alpha)p_s(\alpha) \) should be taken, [21].

The path algebra \( KQ \) for the quiver \( Q \) depicted in (A.29) below
\[
\begin{align*}
1 & \leftarrow 2 \leftarrow \ldots \leftarrow n-1 \leftarrow 0 \leftarrow 0
\end{align*}
\]
is isomorphic to the \( n \times n \) lower triangular matrix algebra. The isomorphism is defined by mapping the matrix \( \varepsilon_{ij} \) with zeros everywhere except 1 in the \( ij \)th slot to the path beginning at the \( j \)th vertex and ending with the \( i \)th vertex, for details see [1, p. 44], [47, p. 114].

A.2.4. Reflection functors and Coxeter functors

Following Bernstein, Gelfand and Ponomarev [2], given a quiver \( Q \) and a field \( K \), we define reflection functors and Coxeter functors. For details, see [1,40] or [45].

The vertex \( a \in Q_0 \) is called sink-admissible (resp. source-admissible) if all arrows containing \( a \) have \( a \) as a target (resp. as a source). By \( \sigma_a Q \) we denote the quiver obtained from \( Q \) by inverting all arrows containing \( a \). For each sink-admissible vertex \( a \), we define the reflection functor
\[
F_a^+ : \text{rep}_K(Q) \longrightarrow \text{rep}_K(\sigma_a Q) \tag{A.30}
\]
between the categories of finite-dimensional \( K \)-linear representations of the quivers \( Q \) and \( \sigma_a Q \) as follows. Let
\[
V = (V_i, \delta_a)_{i \in Q_0, a \in Q_1}
\]
be an object \( \text{rep}_K(Q) \). The object
\[
F_a^+ V = (V_i', \delta_a')_{i \in (\sigma_a Q)_0, a \in (\sigma_a Q)_1}
\]
in \( \text{rep}_K(\sigma_a Q) \) is defined as follows. 
(a) We put \( V_i' = V_i \) for \( i \neq a \). For \( i = a \), we put \( V_a' = \ker \nabla \), where
\[
\nabla = \sum_{a: s(\alpha) \to a} \delta_a : \bigoplus_{\alpha} V_{s(\alpha)} \to V_a,
\]
and \( s(\alpha) \) is the source of the arrow \( a \), see Appendix A.2.3. The \( K \)-linear map \( \nabla \) acts such that
\[
\nabla(v_1, \ldots, v_n) = \sum_{a: s(\alpha) \to a} \delta_a(v_{s(\alpha)}),
\]
for all collections \( \{\eta_1, \ldots, \eta_n\} \), \( \eta_i \in V_i \), where indices \( 1, \ldots, n \) enumerate all arrows ending with \( a \).
(b) We put \( \delta_a' = \delta_a \) for all arrows \( \alpha \) such that sink \( t(\alpha) \neq a \). For \( t(\alpha) = a \), we put
\[
\mu_a : V_a' \to V_i' = V_i
\]
equaled the composition of the inclusion \( V_a' \) into \( \bigoplus_{\beta: t(\beta) \to a} V_{s(\beta)} \) with the projection on the direct summand \( V_i \).

Following [1, Section VII.5.5], we define the action of the reflection functor \( F_a^+ \) on the morphisms between representations in the category \( \text{rep}_K(Q) \). Let
\[
f = (f_i)_{i \in Q_0} : V \to W
\]
be a morphism in \( \text{rep}_K(Q) \). We define the morphism
\[
F_a^+ f = f' = (f_i')_{i \in \sigma Q_0} : F_a^+ V \to F_a^+ W
\]
as follows. For each \( i \neq a \), we put \( f_i' = f_i \). For \( i = a \), we give the \( K \)-linear map \( f_a' \), such that the following diagram is commutative
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & (F_a^+ V)_a & \longrightarrow & \bigoplus_{a: s(\alpha) \to a} V_{s(\alpha)} & \overset{(\delta_a)_a}{\longrightarrow} & V_a \\
\downarrow f_a' & & \downarrow f_s(\alpha) & & \downarrow f_a & & \\
0 & \longrightarrow & (F_a^+ W)_a & \longrightarrow & \bigoplus_{a: s(\alpha) \to a} W_{s(\alpha)} & \overset{(\delta_a)_a}{\longrightarrow} & W_a
\end{array}
\]
For the definition of the reflection functor \( F_a^- \) in the source-admissible vertex see [1, Section VII.5.5]. The main property of the reflection functors is the fact that reflection functors preserve the indecomposability of representations, namely the following proposition takes place.

**Proposition A.2.6 ([2]).** Let \( Q \) be a quiver without cycles and \((V, \delta)\) an indecomposable representation. Let \( i \) be a sink-admissible (resp. source-admissible) representation. Then

(i) if \((V, \delta) \cong (P_i, 0)\), where \((P_i, 0)\) is the simple indecomposable representation, then
\[
F_i^+(V, \delta) = (0, 0) \quad \text{(resp. } F_i^-(V, \delta) = (0, 0)).
\]
(ii) if \((V, \delta) \not\cong (P_i, 0)\), then the representation \( F_i^+(V, \delta) = (V', \delta') \) (resp. \( F_i^-(V, \delta) = (V', \delta') \)) is indecomposable,
\[
F_i^- F_i^+(V, \delta) \cong (V, \delta) \quad \text{(resp. } F_i^- F_i^+(V, \delta) \cong (V, \delta))
\]
and
\[
\dim_K V_i' = \dim_K V_i \quad \text{for } i \neq a,
\]
\[
\dim_K V_a' = - \dim_K V_a + \sum_{a: s(\alpha) \to a} \dim_K M_s(\alpha).
\]
A sink-admissible (resp. source-admissible) sequence

\[ S = \{v_{i_0}, \ldots, v_{i_2}, v_{i_1}\} \]

is called fully sink-admissible (resp. fully source-admissible) if \( S \) contains every vertex \( v \in Q_0 \) exactly once. Obviously, the inverse sequence \( S^{-1} \) of a fully sink-admissible sequence \( S \) is fully source-admissible and vice versa.

Every tree has a fully sink-admissible sequence \( S \). To every fully sink-admissible sequence \( S \), we define the Coxeter functors \( \Phi^+ \) and \( \Phi^- \) as follows.

\[
\Phi^+ = F_{i_0}^+ F_{i_1}^+ \cdots F_{i_2}^+ F_{i_1}^+, \\
\Phi^- = F_{i_1}^+ F_{i_2}^+ \cdots F_{i_2}^+ F_{i_0}^+.
\]

For every quiver \( Q \) being the tree, every fully sink-admissible sequence gives rise to the same Coxeter functor \( \Phi^+ \), and every fully source-admissible sequence gives rise to \( \Phi^- \), thus the definition of the Coxeter functors does not depend on the order of vertices in \( S \).

The Coxeter functors \( \Phi^+ \), \( \Phi^- \) are endofunctors, i.e.,

\[
\Phi^+ : \text{rep}_K L \rightarrow \text{rep}_K L, \quad \Phi^- : \text{rep}_K L \rightarrow \text{rep}_K L,
\]

because every edge of the tree is twice reversed.

A.2.5. Preprojective and preinjective representations

The representation \( \rho_X \) for which \( \Phi^+ \rho_X = 0 \) (resp. \( \Phi^- \rho_X = 0 \)) is called the projective (resp. injective). For every indecomposable representation \( \rho_X \), the new indecomposable representation \( \Phi^+ \rho_X \) (resp. \( \Phi^- \rho_X \)) can be constructed except for the case where \( \rho_X \) is projective (resp. injective).

For example, we can construct a new indecomposable representation \( \Phi^+ \rho_X \) of \( \mathbb{E}_6 \) except for seven indecomposable representations \( \rho^{x_0} \) and \( \rho^{y_i}, \rho^{z_i}, i = 1, 2, 3 \); for them \( \Phi^+ \rho_X = 0 \), see Table A.1.

By [21, Prop. 8.9] the projective indecomposable representations of any quiver are naturally enumerated by the vertices of the graph and can be recovered from the orientation of the graph.

If \( \rho_X \) is indecomposable and not projective, i.e., \( \Phi^+ \rho_X \neq 0 \), then \( \rho_X = \Phi^- \rho_X \).

If \( \rho_X \) is indecomposable and not injective, i.e., \( \Phi^- \rho_X \neq 0 \), then \( \rho_X = \Phi^+ \rho_X \).

If \( \rho_X \) is indecomposable and \( (\Phi^+)^k \rho_X \neq 0 \), then \( \rho_X = (\Phi^-)^k (\Phi^+)^k \rho_X \).

If \( \rho_X \) is indecomposable and \( (\Phi^-)^k \rho_X \neq 0 \), then \( \rho_X = (\Phi^+)^k (\Phi^-)^k \rho_X \).

The representation \( \rho_X \) is called preprojective if, for some projective representation \( \tilde{\rho} \),

\[
(\Phi^+)^k \rho_X = \tilde{\rho}, \quad (\Phi^+)^{k+1} \rho_X = \Phi^+ \tilde{\rho} = 0.
\]

The representation \( \rho_X \) is called preinjective if, for some injective representation \( \tilde{\rho} \),

\[
(\Phi^-)^k \rho_X = \tilde{\rho}, \quad (\Phi^-)^{k+1} \rho_X = \Phi^- \tilde{\rho} = 0.
\]

The representation \( \rho_X \) is called regular if \( (\Phi^+)^k \rho_X \neq 0 \) and \( (\Phi^-)^k \rho_X \neq 0 \) for every \( k \in \mathbb{Z} \).

A.2.6. The Coxeter functor \( \Phi^+ \) for \( D^2_2 \)

Let \( \rho_X \) be an indecomposable representation of \( \mathbb{E}_6 \) in a space \( X = X_0 \) and \( \rho_X^+ = \Phi^+ \rho_X \) an indecomposable representation of \( \mathbb{E}_6 \) in the space \( X_0^+ = X_1^0 \). According to [2], the Coxeter functor \( \Phi^+ \) is constructed as the sequence of reflection functors \( F^+_x \), where \( z \) runs over the vertices of \( \mathbb{E}_6 \):

\[
\Phi^+ \rho_X = F^+_x F^+_y F^+_z F^+_x F^+_y F^+_z F^+_x F^+_y F^+_z \rho_X.
\]
Table A.1
Preprojective representations

<table>
<thead>
<tr>
<th>$\rho X$</th>
<th>$\Phi \rho X$</th>
<th>$(\Phi \rho X)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{x0}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0 1 2 1 0</td>
<td>1 2 4 2 1</td>
</tr>
<tr>
<td>$\rho_{y1}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1 1 2 1 0</td>
<td>0 1 3 2 1</td>
</tr>
<tr>
<td></td>
<td>1 3 2 1</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td>$\rho_{y2}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1 2 1 1</td>
<td>1 1 3 2 0</td>
</tr>
<tr>
<td></td>
<td>1 1 3 2 1</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td>$\rho_{y3}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1 2 1 0</td>
<td>1 1 3 2 1</td>
</tr>
<tr>
<td></td>
<td>0 1 3 2 1</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td>$\rho_{x1}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1 1 0</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td></td>
<td>1 1 2 1 1</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td>$\rho_{x2}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1 1 0 0</td>
<td>0 1 1 0 0</td>
</tr>
<tr>
<td></td>
<td>1 1 2 1 0</td>
<td>1 1 2 1 0</td>
</tr>
<tr>
<td></td>
<td>1 2 1 0</td>
<td>0 1 2 1 1</td>
</tr>
<tr>
<td>$\rho_{x3}$</td>
<td>0 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0 1 1</td>
<td>1 1 1 0</td>
</tr>
<tr>
<td></td>
<td>1 1 1 0</td>
<td>0 1 1 1 0</td>
</tr>
<tr>
<td></td>
<td>1 2 1 1</td>
<td>1 1 2 1 1</td>
</tr>
</tbody>
</table>

Seven representations in the first column are projective.

The reflection functor $F_{x0}^+$ changes only the space $X_0$ to the space $F_{x0}^+X_0 = X_1^0$ and maps $\delta_i : Y_i \to X_0$ to maps $F_{x0}^+\delta_i : X_i^0 \to Y_i$ for each $i = 1, 2, 3$. Let

$$\nabla : \{(\eta_1, \eta_2, \eta_3) \mid \eta_i \in Y_i\} \to \sum \delta_i(\eta_i),$$

see Fig. A.4. Then by [2], we have $X_0^1 = \ker \nabla$ from the exact sequence

$$0 \to \ker \nabla \to \bigoplus Y_i \nabla X_0 \to 0$$

and

$$\delta_i' : \{(\eta_1, \eta_2, \eta_3) \mid \eta_i \in Y_i, \sum \delta_i(\eta_i) = 0\} \to \eta_i.$$

In Fig. A.4 we have $F_{x0}^+X_0 = X_1^0$, $F_{x0}^+\delta_i = \delta_i'$ and $X_i \xrightarrow{I_i} Y_i \xleftarrow{\delta_i'} X_i^0$, where $i = 1, 2, 3$. 
Applying the reflection functor $F^+_i$, we only change the space $Y_i$ to the space $F^+_i Y_i = Y_i^1$, the corresponding maps $\delta'_i$ to $F^+_i \delta'_i = \delta''_i$ and $I_i$ to $P_i$ for each $i = 1, 2, 3$. So

$$\nabla_i : \{((\eta_1, \eta_2, \eta_3), \xi_i) | \eta_i \in Y_i, \sum \delta_i(\eta_i) = 0, \xi_i \in X_i\} \rightarrow \delta''_i((\eta_1, \eta_2, \eta_3) + I_i(\xi_i)) = \eta_i + I_i(\xi_i).$$

Then $Y_i^1 = \ker \nabla_i$ from the following exact sequences, where $i = 1, 2, 3$.

$$0 \rightarrow \ker \nabla_i \rightarrow X_i^1 \oplus X_i \xrightarrow{\nabla_i} Y_i \rightarrow 0.$$ 

We have,

$$Y_i^1 = \ker \nabla_i = \left\{((\eta_1, \eta_2, \eta_3), \xi_i) | \eta_i \in Y_i, \sum \delta_k(\eta_k) = 0, \xi_i \in X_i, \eta_i + I_i(\xi_i) = 0\right\}.$$

and

$$X_i \xleftarrow{P_i} Y_i^1 \xrightarrow{\delta''_i} X_i^1,$$

where $\delta''_i : Y_i^1 \rightarrow X_i^0$.

$$\delta''_1((\eta_1, \eta_2, \eta_3), \xi_1) = \left((\eta_1, \eta_2, \eta_3) | \eta_k \in Y_k, \eta_1 + I_1(\xi_1) = 0, \sum \delta_k(\eta_k) = 0\right),$$

$$\delta''_2((\eta_1, \eta_2, \eta_3), \xi_2) = \left((\eta_1, \eta_2, \eta_3) | \eta_k \in Y_k, \eta_2 + I_1(\xi_2) = 0, \sum \delta_k(\eta_k) = 0\right),$$

$$\delta''_3((\eta_1, \eta_2, \eta_3), \xi_3) = \left((\eta_1, \eta_2, \eta_3) | \eta_k \in Y_k, \eta_3 + I_1(\xi_3) = 0, \sum \delta_k(\eta_k) = 0\right).$$

In Fig. A.5 we have $\tilde{\rho} = F^+_{y_3} F^+_{y_2} F^+_{y_1} \rho_X$, $F^+_i Y_i = Y_i^1$, $F^+_i \delta''_i = \delta''_i$ and $F^+_i I_i = P_i$ for $i = 1, 2, 3$.

**Remark A.2.7.** In any representation of the lattice $D^{2.2.2}$, the maps $I_k$ and $\delta_k$ are monomorphisms and we have $\eta_i = -\xi_i$ in (A.32), i.e., $\eta_i \in X_i$. In other words,

$$\delta''_1((\eta_1, \eta_2, \eta_3), \xi_1) = \left((-\xi_1, \eta_2, \eta_3) | \eta_k \in Y_k, \xi_1 \in X_1, \sum \eta_k = 0\right),$$

$$\delta''_2((\eta_1, \eta_2, \eta_3), \xi_2) = \left(\eta_1, -\xi_2, \eta_3) | \eta_k \in Y_k, \xi_2 \in X_2, \sum \eta_k = 0\right),$$

$$\delta''_3((\eta_1, \eta_2, \eta_3), \xi_3) = \left(\eta_1, \eta_2, -\xi_3) | \eta_k \in Y_k, \xi_3 \in X_3, \sum \eta_k = 0\right),$$

and $\text{Im} \delta''_i = G_i^1 X_0^1$. This proves that (2.2) from Section 2.1 is well-defined.

Further,

$$0 \rightarrow \ker P_i \xrightarrow{Q_i} Y_i^1 \xrightarrow{P_i} X_i \rightarrow 0$$

(A.35)
and $X_i^1 = \ker P_i$, where

$$P_1((\eta_1, \eta_2, \eta_3), \xi_1) = \left( \xi_1 \mid \eta_k \in Y_k, \eta_1 + I_1(\xi_1) = 0, \sum \delta_k(\eta_k) = 0 \right),$$

$$P_2((\eta_1, \eta_2, \eta_3), \xi_2) = \left( \xi_2 \mid \eta_k \in Y_k, \eta_2 + I_1(\xi_2) = 0, \sum \delta_k(\eta_k) = 0 \right),$$

$$P_3((\eta_1, \eta_2, \eta_3), \xi_3) = \left( \xi_3 \mid \eta_k \in Y_k, \eta_3 + I_1(\xi_3) = 0, \sum \delta_k(\eta_k) = 0 \right).$$ (A.36)

**Remark A.2.8.** For any representation of the lattice $D^{2,2,2}$, the maps $\delta_k$ are monomorphisms, therefore

$$\ker \ P_1 = \left\{ ((0, \eta_2, \eta_3), 0) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\},$$

$$\ker \ P_2 = \left\{ ((\eta_1, 0, \eta_3), 0) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\},$$

$$\ker \ P_3 = \left\{ ((\eta_1, \eta_2, 0), 0) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\}. \quad \text{ (A.37)}$$

In Fig. A.6 we have $\Phi^+ \rho_X = F_{x_1}^+ F_{x_2}^+ F_{x_3}^+ \rho = F_{x_1}^+ F_{x_2}^+ F_{x_3}^+ F_{y_1}^+ F_{y_2}^+ F_{y_3}^+ F_{x_0}^+ \rho_X$, and $F_{x_i}^+ X_i = X_i^1$. $F_{x_i}^+ P_i = Q_i$. Since

$$\delta_i^\prime Q_1(\ker P_1) = \left\{ (0, \eta_2, \eta_3) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\},$$

$$\delta_i^\prime Q_2(\ker P_2) = \left\{ (\eta_1, 0, \eta_3) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\},$$

$$\delta_i^\prime Q_3(\ker P_3) = \left\{ (\eta_1, \eta_2, 0) \mid \eta_k \in Y_k, \sum \eta_k = 0 \right\}, \quad \text{ (A.38)}$$

and $\text{Im}(\delta_i^\prime Q_i) = H_i^j X_i^1$, it follows that (2.3) from Section 2.1 is well-defined.

**Appendix B. Proof of the inclusion theorem**

**B.1. More properties of the atomic elements**

The atomic elements $a_n^{ij}$ and $A_n^{ij}$ can frequently substitute for each other. This has already been shown in Lines 3.1–5.2 of Table 2.1.

**Proposition B.1.1.** The following relations hold for $n - 2 \leq m$:

1. $y_i a_m^{ik} a_n^{kj} = y_i a_m^{ik} A_n^{ij}$,
2. $y_i A_m^{ij} a_n^{kj} = y_i A_m^{ij} A_n^{ij}$.

**Proof.** (1) Since

$$y_i a_m^{ik} a_n^{kj} = y_i a_m^{ik} (x_k + y_j (x_j + y_i a_{n-2}^{ik}))$$

and $y_i a_{n-2}^{ik} \geq y_i a_m^{ik}$ for $n - 2 \leq m$, it follows that

$$y_i a_m^{ik} a_n^{kj} = y_i a_m^{ik} (y_j + x_k (x_j + y_i a_{n-2}^{ik})).$$
Since \( a_{n-2}^{i_k} \supseteq x_k \), we have
\[
y_i a_{m}^{i_k} a_{n}^{i_j} = y_i a_{m}^{i_k} (y_j + x_k (y_i + x_j a_{n-2}^{i_k})).
\]
Further, by Table 2.1, Line 3.1, we have \( x_j a_{n-2}^{i_k} = x_j A_{n-2}^{i_k} \) and
\[
y_i a_{m}^{i_k} a_{n}^{i_j} = y_i a_{m}^{i_k} (y_j + x_k (y_i + x_j A_{n-2}^{i_k})) = y_i a_{m}^{i_k} A_{n}^{i_j}.
\]
(2) Since
\[
y_i A_{m}^{i_j} a_{n}^{i_k} = y_i A_{m}^{i_j} (x_j + y_k (x_i + y_i a_{n-2}^{i_j})) = y_i A_{m}^{i_j} (x_j + y_k a_{n-2}^{i_j})
\]
and by Table 2.1, Line 4.1, we have \( y_k y_i a_{n-2}^{i_j} = y_k y_i A_{n-2}^{i_j} \), and
\[
y_i A_{m}^{i_j} a_{n}^{i_k} = y_i A_{m}^{i_j} (x_j + x_k y_i A_{n-2}^{i_j}).
\]
Since \( A_{n-2}^{i_j} \supseteq A_{n}^{i_j} \), we see that
\[
y_i A_{m}^{i_j} a_{n}^{i_k} = y_i A_{m}^{i_j} (y_k y_i + A_{n-2}^{i_j} (x_j + x_k))
\]
\[
y_i A_{m}^{i_j} (y_k y_i + x_j + x_k A_{n-2}^{i_j}) = y_i A_{m}^{i_j} (x_j + y_k (y_i + x_k A_{n-2}^{i_j})).
\]
Further,
\[
y_i + x_k A_{n-2}^{i_j} \supseteq y_i A_{m}^{i_j},
\]
and we have
\[
y_i A_{m}^{i_j} a_{n}^{i_k} = y_i A_{m}^{i_j} (y_k + x_j (y_i + x_k A_{n-2}^{i_j})) = y_i A_{m}^{i_j} A_{n}^{i_j}.
\]

B.2. The a-form and A-form of admissible elements

**Proposition B.2.1.** For all admissible elements \( f_\alpha \) and \( e_\alpha \) of Table 2.3,

1. Every symbol “A” can be substituted by the symbol “a”.
2. Every symbol “a” can be substituted by the symbol “A”.

The substitutions \( A \rightarrow a \) are collected in Table B.1. The original form of admissible elements is given by Table 2.3. Two other forms of admissible elements are given by Proposition B.2.1. The substitution (1) (resp. (2)) from Proposition B.2.1 is said to be an a-form (resp. A-form) (Table B.1).

For the proof of proposition, see Section C.1.2 [51].

**Remark B.2.2 (To Table B.2).** Transformation \( T^{123} \) substitutes \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) and \( T^{12} \) substitutes \( 1 \rightarrow 2 \rightarrow 1 \). These transformations are needed for getting corresponding admissible elements \( g_\alpha 0 \) from the Table 2.3 or their a-forms and A-forms from Table B.1. For example, consider Line 2. In this case \( \alpha = (321)^{2p} (32)^{2k} \). This sequence can be obtained by applying \( T^{123} \) to the sequence \( \tilde{\alpha} = (213)^{2p} (21)^{2k} \) from Table 2.3, i.e.,
\[
\alpha = (321)^{2p} (32)^{2k} = T^{123}[\tilde{\alpha}] = T^{123}[(213)^{2p} (21)^{2k}].
\]

B.3. Admissible sequences \( \alpha_1 \) and \( \alpha \). The inclusion theorem

Our intention is to prove the following theorem.

**Theorem 2.13.1.** For every admissible sequence \( \alpha \) or \( \alpha_1 \), where \( i = 1, 2, 3 \) from Table 2.2, the following inclusion holds:
\[
e_{\alpha_1} \subseteq g_{\alpha_0}, \quad i = 1, 2, 3.
\]
Without loss of generality it suffices to prove Theorem 2.13.1 for \( i = 1 \). Thus, we need description of admissible sequences \( \alpha \) and \( \alpha \) in the form of sequences of Table 2.3.

**Proposition B.3.1.** For the corresponding indices \( \alpha \) and \( \alpha \) from Table B.2 (columns 2 and 3), we have

\[ \varphi_\alpha(1) = \alpha. \]

**Proof.** Line 1. Let \( p > 0 \). We have

\[ \varphi_{21(321)^{2p-1}(32)^{2k}}(1) = 21(321)^{2p-1}(32)^{2k} \]

\[ = 21(321)(321)^{2p-2}(32)^{2k} = 213((21)(321)^{2p-2}(32)^{2k}). \]

By induction, from Line 4, **Table B.2** we get

\[ \varphi_{21(321)^{2p-1}(32)^{2k}}(1) = 213[213]2p-1 (21)^{2k} = (213)^{2p}(21)^{2k} = \alpha. \]

If \( p = 0 \), then \( 2(12)^{2k} = (21)^{2k+1} \). The same for Line 9. 

**Line 4.** Here,

\[ 21(321)^{2p}(32)^{2k} = 213(21)(321)^{2p-1}(32)^{2k} = 213((21)(321)^{2p-1}(32)^{2k}). \]

By induction, from heading (1) we get

\[ 21(321)^{2p}(32)^{2k} = (213)(213)^{2p}(21)^{2k} = (213)^{2p+1}(21)^{2k} = \alpha. \]
Table B.2

\(\alpha_1, \alpha, \epsilon_{\alpha_1}\) and \(g_{\alpha_0}\)

<table>
<thead>
<tr>
<th>(N)</th>
<th>(\alpha_1)</th>
<th>(\alpha)</th>
<th>(\epsilon_{\alpha_1})</th>
<th>(\alpha_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[\alpha] = T^{123}[\alpha]_{\alpha_0})</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>1’</td>
<td>((21)^2k)</td>
<td>((21)^2)</td>
<td>(T^{12}[1(21)^2k-1])</td>
<td>(T^{12}[\alpha_0])</td>
</tr>
<tr>
<td>2</td>
<td>((321)^2 p(21)^2k)</td>
<td>((321)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>3</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>4</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>5’</td>
<td>((21)^2k)</td>
<td>((21)^2)</td>
<td>(T^{12}[1(21)^2]k)</td>
<td>(T^{12}[\alpha_0])</td>
</tr>
<tr>
<td>6</td>
<td>((321)^2 p(21)^2k)</td>
<td>((321)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>7</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>8</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>9</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>9’</td>
<td>((21)^2k)</td>
<td>((21)^2)</td>
<td>(T^{12}[1(21)^2k])</td>
<td>(T^{12}[\alpha_0])</td>
</tr>
<tr>
<td>10</td>
<td>((321)^2 p(21)^2k)</td>
<td>((321)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>11</td>
<td>((321)^2 p(21)^2k)</td>
<td>((321)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>12</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>13</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
<tr>
<td>14</td>
<td>((213)^2 p(21)^2k)</td>
<td>((213)^2 p(21)^2)</td>
<td>(T^{123}[213]^2 p(21)^2k)</td>
<td>(T^{123}[\alpha_0])</td>
</tr>
</tbody>
</table>

For Line 1 and Line 9: \(p > 0\). For Lines 1–6: \(k > 0, p \geq 0\).

For Lines 7–8: \(k > 0\). For Lines 9–14: \(k \geq 0, p \geq 0\). For all lines: \(q = k + 3p\).

The same for Line 12. □

Other lines of Table B.2 are considered by analogy with these two cases. □

B.4. An S-form of \(g_{\alpha_0}\)

Definition B.4.1. The form of the elements \(g_{\alpha_0}\) from the right column of Table B.3 is said to be an S-form (symmetric).
Proposition B.4.2. Both forms of the elements $g_{a_0}$ from Table B.3 coincide.

For the proof of proposition, see [51, Section C.1.4]. After all previous preparations the proof of Theorem 2.13.1 is rather trivial. Instead of inclusion

$$e_{a_1} \subseteq g_{a_0}, \quad \text{where } i = 1, 2, 3,$$

we will prove a sharper statement.

Proposition B.4.3. For the lattice elements $e_{a_i}$ and $g_{a_0}$, we have

$$e_{a_i} = g_{a_0} Z, \quad i = 1, 2, 3,$$

where the element $Z$ for $i = 1$ is given in Table B.4.

Proof of Proposition B.4.3 follows from the $S$-form of elements $g_{a_0}$ from Table B.3 and the $a$-form and $A$-form of the elements $e_{a_1}$ from Table B.1. The polynomial $g_{a_0}$ from Table B.3 is the intersection of $M$ and $P$:

$$g_{a_0} = M \cap P, \quad (B.1)$$

where $M$ is the expression located on the left of the brackets in the $S$-form of Table B.1 and $P$ is the sum contained in the parentheses of this $S$-form. Then, for every line from Table B.1, we have

$$MZ = e_{a_1} \quad \text{and} \quad e_{a_1} \subseteq P. \quad (B.2)$$

From (B.1) and (B.2) it follows that

$$g_{a_0} Z = M P Z = e_{a_1} P = e_{a_1}.$$

For example, consider Lines 1 and 2 of Table B.4.

**Line 1.** We have

$$g_{a_0} = y_2 A_{k-1}^{32} A_q^{31} A_k^{13} A_q^{12} (a_{q_1}^{21} + a_{q_1+k+1}^{12}),$$

$$M = y_2 A_{k-1}^{32} A_q^{31} A_k^{13} A_q^{12}, \quad P = a_{q_1}^{21} + a_{q_1+k+1}^{12}, \quad Z = A_q^{12},$$
Since \( A_q^{12} \subseteq A_{q-1}^{12} \), it follows that

\[ MZ = y_2 A_{k-1}^{32} A_{k}^{31} A_{k}^{13} A_{k}^{12} = e_{a1} \]

and by \( a \)-form from Table B.1 we obtain \( e_{a1} \subseteq A_{q}^{21} \subseteq P \). □

**Line 2.** We have

\[
\begin{align*}
g_{a0} &= y_3 A_{k-1}^{13} A_{q}^{23} A_{k}^{21} A_{q}^{12} (A_{q+1}^{12} + A_k^{13}), \\
M &= y_3 A_{k-1}^{13} A_{q}^{23} A_{k}^{21} A_{q}^{12}, \quad P = A_{q+1}^{12} + A_k^{13}. \quad Z = A_{q+1}^{12}.
\end{align*}
\]

Since \( A_{q+1}^{12} \subseteq A_{q}^{12} \), we see that

\[ MZ = y_2 A_{k-1}^{32} A_{q}^{31} A_{k}^{13} A_{q}^{12} = e_{a1}. \]

In addition, \( e_{a1} \subseteq A_{q+1}^{12} \subseteq P \).

Proposition B.4.3 together with Theorem 2.13.1 (Inclusion Theorem) are proven. □

**Conjectures**

**Conjecture 1.11.4.** The lattice \( H^+ \cup H^- \) contains all perfect elements of \( D^{2,2,2} \mod \theta_p \).

For \( D^4 \), the similar conjecture (due to Gelfand–Ponomarev) was proved by Dlab and Ringel in [13] and Cylke in [9], see Section 1.13.1.

**Conjecture 3.2.4.** The relation

\[
x_0(n + 2) \simeq \bigcap_{i=1,2,3} a_i(n),
\]

see ((3.24), Proposition 3.2.2) takes place in \( D^{2,2,2} \) for all integer \( n \geq 0 \), not only in \( D^{2,2,2}/\theta \), i.e.

\[
x_0(n + 2) = \bigcap_{i=1,2,3} a_i(n).
\]
Here, $x_0(n + 2)$ is the cumulative polynomial defined in Section 1.10, and the $a_i(n)$ are perfect elements defined in Section (1.34). The elements $x_0(n + 2)$ are also perfect, see Corollary 3.2.3.

**Conjecture 3.6.2.** The relation
$$a_i(n)x_i(n + 1) \subseteq y_i(n + 2) \mod \theta,$$
(see (3.50)), takes place in $D^{2,2,2}$, not only in $D^{2,2,2}/\theta$. In this case Proposition 3.6.1 is also true in $D^{2,2,2}$, and $U_n \bigcup V_{n+1}$ is a 16-element Boolean algebra in $D^{2,2,2}$.

Here, $x_i(n + 1)$, $y_i(n + 2)$ are the cumulative polynomial defined in Section 1.10, and the $a_i(n)$ are perfect elements defined in Section (1.34).

**Conjecture 4.8.6.** For every admissible sequence $\alpha$, the elements $e_\alpha$ (resp. $f_\alpha$) and $\tilde{e}_\alpha$ (resp. $\tilde{f}_\alpha$) coincide in $D^4$ (not only in $D^4/\theta$, see Proposition 4.8.3).

Here, the elements $e_\alpha$ (resp. $f_\alpha$) are given by Table 4.3, and the elements $\tilde{e}_\alpha$ (resp. $\tilde{f}_\alpha$) are determined by Gelfand and Ponomarev, see Section 4.8.2. For small admissible sequences, this coincidence is proved in Propositions 4.8.4 and 4.8.5.

**References**