Invariant measures for stochastic heat equations with unbounded coefficients

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Abstract

The paper deals with the Cauchy problem in $\mathbb{R}^d$ of a stochastic heat equation $\frac{\partial u}{\partial t} = \lambda \Delta u + f(u) + g(u) \dot{W}$. The locally Lipschitz drift coefficient $f$ can have polynomial growth while the diffusion coefficient $g$ is supposed to be Lipschitz but not necessarily bounded. Of course, for the existence of a solution alone, a certain dissipativity of $f$ is needed. Applying the comparison method, a condition on the strength of this dissipativity is derived even ensuring the existence of an invariant measure.

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1. Introduction

We deal with the long-time behaviour of stochastic heat equations of the form

\begin{equation}
\frac{\partial}{\partial t} u(t,x) = \lambda \Delta u(t,x) + f(u(t,x)) + \sigma(u(t,x)) \cdot \dot{W}(t,x), \quad t > 0, \quad x \in \mathbb{R}^d,
\end{equation}

\begin{equation}
u(0,x) = \theta(x), \quad x \in \mathbb{R}^d,
\end{equation}

where $\lambda > 0$ is a constant, $\Delta$ denotes the Laplacian\textsuperscript{1} in the unbounded domain $\mathbb{R}^d$, $f, \sigma$ are measurable real functions and the Gaussian noise $\dot{W}(t,x)$ is white in time.

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\textsuperscript{1}That is why this class of equations refers to the heat equation.
resp. white or coloured in space. Such equations can describe dynamical systems in physics and mathematical biology, for example, and even in these sciences there is a strong need to know the ergodic behaviour of dynamical systems. Of course, no ergodic behaviour without an invariant measure for the corresponding dynamical system; hence one at first has to clarify the existence of invariant measures.

In their paper, Da Prato et al. (1992) established the following procedure of how to prove the existence of an invariant measure for those stochastic partial differential equations (SPDEs) of which the partial differential operator generates a semigroup of operators being somehow compact:

1° Show existence of a Markovian solution to the corresponding SPDE in a certain function space where its transition semigroup is Feller.

2° Show that some solution starting from a suitable initial condition is bounded in probability with respect to this or a related function space.

While Step 1° is not so restrictive to the defining properties of the SPDE and can be verified in many cases of interest (cf. Cerrai, 2001b; Maslowski and Seidler, 1999 and the references therein); Step 2° is known to demand further assumptions on the coefficients as well as the boundary conditions of the SPDE. Both together, the differential operator and the drift given by $f$, have to be dissipative enough in order to control the diffusion part governed by $\sigma$ and the covariance of the driving noise. So, differential operators in bounded domains—in order to strengthen the contraction properties of the corresponding semigroups—and bounded diffusion coefficients have been studied intensively in the literature; we refer to Brzezniak and Gatarek (1999) and Cerrai (2001a) and the references therein.

If the differential operator is given by the Laplacian in a bounded subinterval of $\mathbb{R}$ with Dirichlet or Neumann boundary conditions, then the contraction properties of the corresponding heat semigroup are so strong that, even in case of a space-time white noise, for the existence of an infinitesimally invariant measure, $f$ basically (locally lipschitz+polynomially bounded) needs to satisfy

$$uf(u) \leq C + \varepsilon |u|^2, \quad u \in \mathbb{R}$$

for a sufficiently small $\varepsilon > 0$. This remarkable result was recently obtained in Bogachev and Röckner (2001) by applying a method based on Lyapunov functions which is different from the above procedure $1° + 2°$. That method also allows to consider certain diffusion coefficients which can even be unbounded. However, the existence of an infinitesimally invariant measure does not yet mean that there really is a stationary solution to the corresponding equation having this measure as an invariant state. For further needed conditions we refer to Stannat (1999).

As far as we know, Tessitore and Zabczyk (1998) is the only paper which deals with invariant measures for SPDEs of which the differential operator is defined in an unbounded domain as well as the diffusion coefficient is unbounded. Indeed, the authors consider the Cauchy problem ($\star$) in the case of $\lambda = \frac{1}{2}$, $f \equiv 0$ and $\sigma$ globally lipschitz. They follow the procedure $1° + 2°$ and, as already mentioned, the verification of $2°$ is the hard part. Their method based on properties of the heat semigroup in
weighted $L^2_0(\mathbb{R}^d)$-spaces pulls them into $d \geq 3$ space dimensions where the driving Gaussian noise at least has to be coloured in space as long as $\sigma$ really occurs to be nonlinear since the heat semigroup is not smoothing enough. For this purpose, a spatially homogeneous Wiener process was chosen; the main condition, ensuring the boundedness in probability of a special solution, estimates this Wiener process’ covariance against the lipschitz constant of $\sigma$.

We now generalize the above result by showing that Step 2o can even be done if $f$ is a polynomially bounded locally lipschitz continuous drift satisfying the condition (f3) in the paper’s main Theorem 3 below. Improving the idea in the proof of Theorem 3.3, Tessitore and Zabczyk (1998), our crucial estimation is based on a comparison theorem for SPDEs (cf. Manthey and Zausinger, 1999 for example). Compared with (1), the condition (f3) actually expresses and somehow needs to express more dissipativity because both the weaker dissipativity of the Laplacian in the whole space as well as the stronger increase of the diffusion (un bounded) have to be compensated. The possible polynomial growth of $f$ forces us to consider the solutions of $(\star)$ in weighted $L^p_0(\mathbb{R}^d)$-spaces for $p \geq 2$. As a consequence, (f3) does not only depend on the covariance of the driving Gaussian noise, the lipschitz constant $\sigma$ and the parameter $\mu$, but also on $p$.

We have to mention that, in case of a coloured noise in space, we choose a $Q$-Wiener process (cf. Da Prato and Zabczyk, 1992) instead of a spatially homogeneous Wiener process because we could not find an analogous theorem for Theorem 1 in the setting of spatially homogeneous Wiener processes. However, the different choice of the driving noise only results in a different way of expressing the influence of its covariance on the main condition. In this sense, our Corollary 1 is only another version of Theorem 3.3 in Tessitore and Zabczyk (1998).

2. Preliminaries and main result

Assume that the stochastic source of Eq. $(\star)$ is given by an independent sequence of $\mathbb{F}$-adapted one-dimensional Wiener processes $B_1(t), B_2(t), \ldots$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ denotes a right-continuous filtration of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ already contains all sets of $\mathbb{P}$-measure zero and that the increments $B_k(t) - B_k(s)$ are independent of $\mathcal{F}_s$ for all $t \geq s \geq 0, k = 1, 2, \ldots$.

Now, fix an orthonormal basis $(e_k)_{k=1}^{\infty}$ in $L^2(\mathbb{R}^d)$ which is uniformly bounded, i.e.

$$\sup_k \sup_{x \in \mathbb{R}^d} |e_k(x)| < \infty,$$

and, depending on the space dimension $d \geq 1$, we define the infinite-dimensional Wiener process $W(t)$ driving Eq. $(\star)$ as follows:

- **cylindrical case** ($d = 1$)

$$W(t) = \sum_{k=1}^{\infty} B_k(t)e_k,$$
• **nuclear case** $(d \geq 1)$

$$W(t) = \sum_{k=1}^{\infty} \sqrt{a_k} B_k(t)e_k,$$

where $(a_k)_{k=1}^{\infty}$ is a sequence of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k < \infty.$$

For later use in the nuclear case, we also introduce the finite number

$$a = \sum_{k=1}^{\infty} a_k \|e_k\|_\infty^2.$$

In what follows, for an arbitrary but fixed $\lambda > 0$,

$$G(t,x) = (4\lambda \pi t)^{-d/2} \exp \left\{ -\frac{|x|^2}{4\lambda t} \right\}, \quad t \geq 0, \ x \in \mathbb{R}^d$$

always denotes the fundamental solution to the Cauchy problem $\partial_t \varphi - \lambda \varphi = 0$ in the unbounded domain $\mathbb{R}^d$. Furthermore, for $p \geq 2$ and $q > d$, we consider the weighted Banach spaces $L^p_{\tilde{q}}(\mathbb{R}^d)$ of Borel measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|u\|_{p,q}^p = \int_{\mathbb{R}^d} |u(x)|^p (1 + |x|^2)^{-q/2} \, dx < \infty.$$

It is a matter of fact that

$$[S(t)u](x) := \int_{\mathbb{R}^d} G(t,x-y)u(y) \, dy, \quad t \geq 0$$

defines a strongly continuous semigroup on each of the Banach spaces $L^p_{\tilde{q}}(\mathbb{R}^d)$. Moreover we have that

$$S(t) : L^p_{\tilde{q}}(\mathbb{R}^d) \to L^p_{\tilde{q}}(\mathbb{R}^d)$$

is compact for each $t > 0$, if $\tilde{q} > 0$ is chosen such that $q - d > \tilde{q}$. Although this result is rather standard we sketch its proof in the appendix.

**Theorem 1** (Manthey and Zausinger, 1999, cf. Theorem 3.4.1). Assume that

1. $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies $|\sigma(x) - \sigma(y)| \leq c_\sigma |x - y|$, $x, y \in \mathbb{R}$ for some constant $c_\sigma > 0$;
2. $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq c_v |x - y|(1 + |x|^{\nu-1} + |y|^{\nu-1}), \quad x, y \in \mathbb{R}$$

for some constant $c_v > 0$, where $\nu \geq 1$ can be a real-valued exponent;
3. There exists a constant $\kappa > 0$ such that

$$uf(u) \leq \kappa(1 + |u|^2), \quad u \in \mathbb{R}.$$
If the initial condition \( \theta \) belongs to \( L^p_0(\mathbb{R}^d) \), where \( p \) depends on the drift function’s polynomial growth as

\[
p = 2 \vee v
\]
then there exists a pathwise unique \( \mathbb{F} \)-adapted continuous solution \( u(t,\cdot) \in L^p_0(\mathbb{R}^d) \), \( t \geq 0 \), of the mild form

\[
(Eq) \quad u(t,\cdot) = S(t)\theta + \int_0^t S(t-s)f(u(s,\cdot)) \, ds + \int_0^t S(t-s)\sigma(u(s,\cdot)) \, dW(s)
\]
of equation (\( \star \)) such that for all \( q \geq p \)

\[
\sup_{t \in [0,T]} \mathbb{E}\|u(t,\cdot)\|_{p,q}^q \leq c_T(1 + \|\theta\|_{p,q}^q), \quad \forall T > 0.
\]

**Remark 1.** (a) The equality in (Eq) is seen as usual in the sense of equivalence classes.

(b) The first integral on the right-hand side of (Eq) is understood as a Bochner integral while the second one is a function space-valued stochastic integral (cf. Da Prato and Zabczyk, 1992 for example). Because of the properties of the heat semigroup \( S(t)_{t \geq 0} \), for the existence of the stochastic integral in our nonlinear situation, we have to reduce the space dimension to \( d = 1 \) in the cylindrical case. In the nuclear case, the driving Wiener process is a \( Q \)-Wiener process with nuclear covariance \( Q \) defined by

\[
Qe_k = a_k e_k, \quad k = 1,2,\ldots.
\]

(c) Condition (f1) especially implies that the function \( f \) is locally lipschitz with at most polynomial growth

\[
|f(x)| \leq \tilde{c}_x (1 + |x|^v), \quad x \in \mathbb{R}.
\]

It goes back to Theorem 2.2.1 in Manthey (2001) ensuring the pathwise uniqueness of the solution. Though this theorem is proven for other state spaces, the proof easily applies in our situation, too.

(d) Condition (f2) which is also called a “one-sided linear growth condition” expresses the “almost necessary” dissipativity of \( f \) for the existence of a solution. Here, we dare to say “almost necessary” since we do not know a weaker condition; it even appears in Gyöngy’s important paper (Gyöngy, 1995). In case of locally bounded functions \( f \), this condition is obviously equivalent to

\[
f(u) \geq \tilde{\kappa}(u - 1), \quad u \leq 0
\]
and

\[
f(u) \leq \tilde{\kappa}(u + 1), \quad u \geq 0,
\]
where \( \tilde{\kappa} > 0 \) is another constant maybe different from \( \kappa \). Indeed, dividing both sides of the inequality in (f2) by \( u \), we obtain

\[
f(u) \geq \kappa \left( u - \frac{1}{|u|} \right), \quad u < 0
\]
and

\[
f(u) \leq \kappa \left( u + \frac{1}{|u|} \right), \quad u > 0,
\]
which yields the above condition for a certain constant \( \tilde{m} \) because \( f \) is locally bounded. We have to emphasize that Manthey and Zausinger (1999) preferred to use this equivalent condition.

(e) In Manthey and Zausinger (1999, Theorem 3.4.1), the authors formulated their result in an \( L^2 \)-setting for random initial conditions and for more general SPDEs. We only consider deterministic initial conditions which simplifies the choice of the state space. So, we reformulated the theorem by using an \( L^p \)-setting with \( p \) given above. Of course, a lower bound for \( p \) is the polynomial growth of \( f \). We should also mention that, in our notation, Manthey and Zausinger (1999) would have formulated

\[
\sup_{t \in [0,T]} \mathbb{E} \|u(t, \cdot)\|_{L^p} < \infty
\]

but the proof of the theorem makes clear that our above estimation actually holds true.

In what follows, we denote the solution of Eq. (Eq) which exists under the conditions of Theorem 1 by \( u(t, \cdot, \cdot) = (u(\theta, t, \cdot))_{t \geq 0} \) in order to show the dependency on the initial condition. It is a classical result for uniquely solvable diffusion equations (cf. Ikeda and Watanabe, 1981, Theorem IV.5.1, for example) that

\[
P_t \phi(\theta) = \mathbb{E} \phi(u(\theta, t, \cdot)), \quad t \geq 0, \quad \phi \in \mathcal{B}_b
\]

defines a Markovian transition semigroup on the space \( \mathcal{B}_b \) of all bounded measurable functions on \( L^p_\theta(\mathbb{R}^d) \). We want to mention that this transition semigroup is stochastically continuous and possesses the Feller property, i.e.

\[
P_t : C_b(L^p_\theta(\mathbb{R}^d)) \to C_b(L^p_\theta(\mathbb{R}^d)).
\]

Indeed, by Dynkin’s theorem (cf. Dynkin, 1965), the above semigroup \( (P_t)_{t \geq 0} \) is stochastically continuous if

\[
\lim_{t \to 0} P_t \phi(\theta) = \phi(\theta)
\]

for arbitrary \( \phi \in C_b(L^p_\theta(\mathbb{R}^d)) \) and \( \theta \in L^p_\theta(\mathbb{R}^d) \), and this property easily follows from the continuity of the solution processes. The proof of the Feller property is only a simple modification of the proof of Theorem 5.10 (Brzezniak and Gatarek, 1999), where a nice method within a general Banach space setting is presented.

Based on the standard Krylov–Bogoliubov technique, we now give the following theorem for the existence of an invariant measure for \( (P_t)_{t \geq 0} \).

**Theorem 2.** Under the conditions of Theorem 1, let \( (P_t)_{t \geq 0} \) denote the stochastically continuous Feller semigroup in \( L^p_\theta(\mathbb{R}^d) \) associated with the solutions \( (u(\theta, t, \cdot))_{t \geq 0} \), \( \theta \in L^p_\theta(\mathbb{R}^d) \), of Eq. (Eq).

If \( q > \tilde{q} + d \) for some \( \tilde{q} > d \), and if \( (u(\theta_0, t, \cdot))_{t \geq 0} \) is bounded in probability in \( L^p_\theta(\mathbb{R}^d) \) for a suitable initial condition \( \theta_0 : \mathbb{R}^d \to \mathbb{R} \), i.e.

\[
\forall \varepsilon > 0 \exists R > 0 \forall t > 0 : P\left( \left\{ \|u(\theta_0, t, \cdot)\|_{L^p} \geq R \right\} \right) < \varepsilon,
\]

then there exists an invariant measure for \( (P_t)_{t \geq 0} \) on \( L^p_\theta(\mathbb{R}^d) \).

The method of the proof which originally goes back to Da Prato et al. (1992) is well known and has frequently been modified in the literature; cf. Brzezniak and
Gatarek (1999) and the references therein. In the appendix, we almost copy the proof of Theorem 6.1.2 in Da Prato and Zabczyk (1996), only pointing out in more detail those parts which have to be changed because of our different assumptions.

We now come to the paper’s main theorem which presents a sufficient condition for the boundedness in probability demanded in Theorem 2.

**Theorem 3.** Assume $(\sigma)$, (f1) and choose $p = 2 \vee v$ as well as $q > d$. For $p > 2$, let $c(p)$ denote the universal constant in Burkholder–Davis–Gundy’s inequality such that

$$\|M_T\|_{L^p(\Omega)} \leq c(p)\|\langle M \rangle_T^{1/2}\|_{L^p(\Omega)}, \quad T > 0$$

for all continuous local martingales $(M_t)_{t \geq 0}$. In the case of $p = 2$ set $c(p) = 1$. If the following condition,

(f3) there exists a constant $c_{f,\kappa} > 0$ such that

$$uf(u) \leq c_{f,\kappa} - \kappa |u|^2, \quad u \in \mathbb{R},$$

where

$$\kappa > c(p)^4 \frac{\kappa_4}{16z} \text{ resp. } \kappa > c(p)^2 \frac{ac^2}{2}$$

in the cylindrical resp. nuclear case, is satisfied then

$$\sup_{t \geq 0} E\|u(\theta_0, t, \cdot)\|_{p,q}^p < \infty$$

for every bounded continuous function $\theta_0 : \mathbb{R}^d \rightarrow \mathbb{R}$.

**Remark 2.** (a) We have

$$c(p) \leq \left(\frac{p}{p-1}\right)^{p/2} \left[\frac{p}{2} (p - 1)\right]^{1/2}$$

by Itô’s formula. But, Hitczenko (1990) can even show that the growth rate of $c(p)$ as $p \rightarrow \infty$ is $p/\log p$.

(b) Choosing the same strength of dissipativity $-\kappa$, one easily verifies that the following condition,

there exists a constant $c_{f,\kappa} > 0$ such that

$$f(u) \geq -\kappa u - c_{f,\kappa}, \quad u \leq 0$$

and

$$f(u) \leq -\kappa u + c_{f,\kappa}, \quad u \geq 0,$$

is equivalent to (f3) (cf. Remark 1(d) above). Of course, the constants $c_{f,\kappa}$ mentioned here and in (f3), respectively, can differ from each other. By technical reason, in the below proof we rather apply this condition than (f3).

Because our proof of Theorem 3 improves the method presented by Tessitore and Zabczyk (1998), we easily find the following version of their Theorem 3.3 about boundedness in probability. The only difference is that we formulate the condition in terms of another covariance structure for the driving Wiener process.
Corollary 1. Assume \((\sigma)\) and choose \(q > d \geq 3\). If \(f \equiv 0\) and
\[
c^2 \left[ a + \sum_{k=1}^{\infty} a_k (\pi \lambda)^{-d/2} 2^{-(3/2)d-1} (d - 2) \right] < 1
\]
then
\[
\sup_{t \geq 0} E \|u(\theta_0, t, \cdot)\|_{L^q}^2 < \infty
\]
for every bounded continuous function \(\theta_0 : \mathbb{R}^d \to \mathbb{R}\).

As another conclusion, by combining Theorems 2 and 3, we immediately obtain the following result as a corollary.

Corollary 2. Assume \((\sigma), (f_1)\) and choose \(p = 2 \lor \nu\) as well as \(q > 2d\). If the following condition, there exists a constant \(c_{f, \kappa} > 0\) such that
\[
uf(u) \leq c_{f, \kappa} - \kappa |u|^2, \quad u \in \mathbb{R},
\]
where
\[
\kappa > c(p \nu)^4 \frac{c^4}{16\lambda} \quad \text{resp.} \quad \kappa > c(p \nu)^2 \frac{ac^2}{2}
\]
in the cylindrical resp. nuclear case, is satisfied then there exists an invariant measure for \((P_t)_{t \geq 0}\) on \(L_0^p(\mathbb{R}^d)\).

Though the strength of dissipativity needed for the existence of an invariant measure is negative, it only presents an asymptotic lower (for \(u < 0\)) resp. upper (for \(u > 0\)) linear bound on \(f(u)\), which leads to the following interesting application.

Example. For arbitrary \(c > 0\) and \(x > 1\) define
\[
f(u) = -c \text{ sgn}(u)|u|^2, \quad u \in \mathbb{R},
\]
where \text{sgn} denotes the signum function. Then, for every \(\lambda > 0\) and every lipschitz continuous diffusion coefficient \(\sigma\), there is an invariant measure for the corresponding stochastic heat equation (\(\star\)) in any space dimension, as long as the driving Wiener process is one of those introduced before. Especially, the same holds true for all odd polynomials of degree greater than 1 with a negative leading coefficient which present important drift functions in physics and mathematical biology.

3. Proof of Theorem 3 and its corollaries

In what follows, we fix a bounded continuous function \(\theta_0\) and denote \(u(t, \cdot) = u(\theta_0, t, \cdot), \; t \geq 0\). From Manthey (2001) we know that \(u(t, \cdot) : \mathbb{R}^d \to \mathbb{R}\) is a continuous
function for all \( t \geq 0 \) \( \mathbb{P} \)-a.s., therefore we want to write Eq. (Eq) more suggestively as

\[
\begin{align*}
    u(t,x) &= \int_{\mathbb{R}^d} G(t,x-y) \theta_0(y) \, dy \\
    &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) \sigma(u(s,y)) \, dW_{sy} \\
    &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) f(u(s,y)) \, dy \, ds
\end{align*}
\]

in this section. The dependency on \( \omega \in \Omega \) is omitted by notational reason; we always work on the above-mentioned subset of \( \mathbb{P} \)-measure 1 where \( u(t, \cdot) \) is continuous on \( \mathbb{R}^d \).

Also, mild solutions of the equation

\[
(Eq[m]) \quad \frac{\partial}{\partial t} v = (\lambda A - m \cdot Id) v + F(v) + \sigma(v) \cdot \dot{W}
\]

starting from \( \theta_0 \) are denoted by the letter \( v \) in order to differ them from the solutions of Eq. (Eq) with fixed coefficients \( f, \sigma \) for which the letter \( u \) is reservated.

**Lemma 1.** Define

\[
h^*(u) = \begin{cases}
c_{f,\kappa}, & u \geq 0, \\
-\bar{m}u + \sup_{u \leq v \leq 0} (f(v) - f(0) + c_{f,\kappa}), & u \leq 0,
\end{cases}
\]

where \( \bar{m} \) is a sufficiently large number, and denote by \( v^*(t,x) \) the unique solution of \( (Eq[m]) \) with \( F = h^* \). If

\[
m > c(p)^4 \frac{e^4}{16 \lambda} \quad \text{resp.} \quad m > c(p)^2 \frac{ac^2_\sigma}{2}
\]

in the cylindrical resp. nuclear case, then

\[
\sup_{t \geq 0} \mathbb{E} \left\| v^*(t, \cdot) \right\|_{p,q}^p < \infty.
\]

**Proof.** Let \( v_0^* \equiv 0 \) and put

\[
\begin{align*}
v^*_n(t,x) &:= \int_{\mathbb{R}^d} e^{-mt} G(t,x-y) \theta_0(y) \, dy \\
    &+ \int_0^t \int_{\mathbb{R}^d} e^{-m(t-s)} G(t-s,x-y) \sigma(v^*_{n-1}(s,y)) \, dW_{sy} \\
    &+ \int_0^t \int_{\mathbb{R}^d} e^{-m(t-s)} G(t-s,x-y) h^*(v^*(s,y)) \, dy \, ds
\end{align*}
\]
for $n = 1, 2, \ldots$. At first, combining Burkholder–Davis–Gundy’s and Hölder’s inequality in the nuclear case, we get

$$E|v_2^*(t, x) - v_1^*(t, x)|^p \leq c(p) \sum_k a_k \int_0^t e^{2m(t-s)} \left( \int_{\mathbb{R}^d} G(t-s, x-y) \times [\sigma(v_1^*(s, y)) - \sigma(0)] e_k(y) dy \right) ds$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$. As a consequence, if

$$\sup_{s \geq 0, y \in \mathbb{R}^d} E|v_1^*(s, y)|^p = c_1^* < \infty$$

then

$$E|v_2^*(t, x) - v_1^*(t, x)|^p \leq c(p) a^{p/2} e \left( \frac{1}{2m} \right) c_1^* \int_0^t e^{-2m(t-s)} ds$$

finally leading to

$$E|v_n^*(t, x) - v_{n-1}^*(t, x)|^p \leq c(p) (n-1)^{p/2} \left( \frac{ac_2}{2m} \right) \cdot c_1^*, \quad t \geq 0, \ x \in \mathbb{R}^d.$$
Hence,
\[
\sup_{t \geq 0} E\left\| v_n^* (t, \cdot ) - v_{n-1}^* (t, \cdot ) \right\|_{p, \varrho}^p \leq c(p)^{p(n-1)} \left( \frac{ac_\sigma^2}{2m} \right)^{(n-1)p/2} c_1^* \int_{\mathbb{R}^d} (1 + |x|^2)^{-\varrho/2} \, dx,
\]
and, because the assumption of the lemma yields \( c(p)^2 ac_\sigma^2 < 2m \) in the nuclear case, \( v_n^* \) converges to the unique solution \( v^* \) of (Eq[m]) if \( n \to \infty \). Moreover

\[
\sup_{t \geq 0} \mathbb{E}\left[ \left\| v_n^* (t, \cdot ) - v_{n-1}^* (t, \cdot ) \right\|_{p, \varrho}^p \right]^{1/p} = \sup_{t \geq 0} \left( \mathbb{E} \left[ \sum_{n=1}^{\infty} \left( v_n^* (t, \cdot ) - v_{n-1}^* (t, \cdot ) \right) \right]_{p, \varrho}^p \right)^{1/p}
\]

\[
\leq \sup_{t \geq 0} \sum_{n=1}^{\infty} \mathbb{E}\left[ \left\| v_n^* (t, \cdot ) - v_{n-1}^* (t, \cdot ) \right\|_{p, \varrho}^p \right]^{1/p}
\]

\[
\leq \sum_{n=1}^{\infty} \left[ c(p)^{p(n-1)} \left( \frac{ac_\sigma^2}{2m} \right)^{(n-1)p/2} c_1^* \right]^{1/p} \times \left( \int_{\mathbb{R}^d} (1 + |x|^2)^{-\varrho/2} \, dx \right)^{1/p} < \infty
\]

proving the lemma in the nuclear case if (2) is true.

Before we show (2), let us discuss the cylindrical case in a similar way. Here we get

\[
\mathbb{E}\left| v_2^* (t, x) - v_1^* (t, x) \right|^p \leq c(p)^p \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} e^{-2m(t-s)} G^2 (t-s, x-y) \left[ \sigma(v_1^*(s,y)) - \sigma(0) \right]^2 \, dy \, ds \right]^{p/2}
\]

\[
\leq c(p)^p c_\sigma^p \left( \int_0^t \int_{\mathbb{R}} e^{-2ms} G^2 (s,y) \, dy \, ds \right)^{p/2} c_1^*,
\]

where

\[
\int_0^t \int_{\mathbb{R}} e^{-2ms} G^2 (s,y) \, dy \, ds = (8\pi \lambda)^{-1/2} \int_0^t s^{-1/2} e^{-2ms} \, ds
\]

\[
< (16\pi \lambda m)^{-1/2} \Gamma \left( \frac{1}{2} \right) = \frac{1}{\sqrt{16\lambda m}},
\]

and, because here \( c(p)^4 c_\sigma^4 < 16\lambda m \) follows from the assumption of the lemma, we can proceed as in the above proof for the nuclear case.
It remains to show that (2) is really true. From the definition of \( v_1 \) follows

\[
E|v_1^*(s, y)|^p \leq 3^{p-1} \left| \int_{\mathbb{R}^d} e^{-ms} G(s, y - z) |\theta(z)| \, dz \right|^p
\]

\[
+ 3^{p-1} E \left| \int_0^s \int_{\mathbb{R}^d} e^{-m(s-r)} G(s-r, y-z) |\sigma(0)| \, dW_{rz} \right|^p
\]

\[
+ 3^{p-1} E \left| \int_0^s \int_{\mathbb{R}^d} e^{-m(s-r)} G(s-r, y-z) h^*(v^*(r,z)) \, dz \, dr \right|^p
\]

\[
= 3^{p-1} (I_1 + I_2 + I_3),
\]

where

\[
I_1 \leq \sup_{z \in \mathbb{R}^d} |\theta_0(z)|^p e^{-pms} \leq \sup_{z \in \mathbb{R}^d} |\theta_0(z)|^p, \quad s \geq 0, \ y \in \mathbb{R}^d.
\]

For \( I_2 \) we obtain

\[
I_2 \leq c(p) \left( \frac{a\sigma(0)^2}{2m} \right)^{p/2}, \quad s \geq 0, \ y \in \mathbb{R}^d
\]

and

\[
I_2 \leq c(p) \left( \frac{\sigma(0)^2}{\sqrt{16\lambda m}} \right)^{p/2}, \quad s \geq 0, \ y \in \mathbb{R}
\]

in the nuclear case and the cylindrical case, respectively, simply copying the way we estimated \( E|v_2^*(t, x) - v_1^*(t, x)|^p \). So, (2) is shown if we can estimate \( I_3 \) uniformly in \( s \geq 0 \) and \( y \in \mathbb{R}^d \).

In a first step, the definition of \( h^* \) gives

\[
I_3 \leq \left( \frac{1}{m} \right)^{p-1} \int_0^s \int_{\mathbb{R}^d} e^{-m(s-r)} G(s-r, y-z) E|h^*(v^*(r,z))|^p \, dz \, dr
\]

\[
\leq \left( \frac{2}{m} \right)^{p-1} \int_0^s \int_{\mathbb{R}^d} e^{-m(s-r)} G(s-r, y-z)
\times (c_{f,k}^p + E|h^*(v^*(r,z))|^p 1_{v^*(r,z) \leq 0}) \, dz \, dr
\]

\[
\leq \left( \frac{2c_{f,k}}{m} \right)^p + \left( \frac{2}{m} \right)^{p-1} \int_0^s \int_{\mathbb{R}^d} e^{-m(s-r)} G(s-r, y-z)
\times E|h^*(v^*(r,z))|^p 1_{v^*(r,z) \leq 0} \, dz \, dr
\]

for all \( s \geq 0 \) and \( y \in \mathbb{R}^d \). Because of

\[
h^*(u) \geq - \tilde{m}u, \quad u \in \mathbb{R},
\]

we have

\[
v^*(r,z) \geq \tilde{v}(r,z)
\]
for all \( z \in \mathbb{R}^d \), \( r \geq 0 \), by the comparison theorem (Manthey and Zausinger, 1999, Theorem 3.3.1), where \( \tilde{v} \) denotes the unique solution of Eq. (Eq[m]) with \( F = -\tilde{m}Id \). As a consequence,
\[
E|\tilde{h}^*(v^*(r,z))|^p 1_{\{v^*(r,z) \leq 0\}} \leq E|h^*(\tilde{v}(r,z))|^p 1_{\{\tilde{v}(r,z) \leq 0\}}
\]
for all \( z \in \mathbb{R}^d \) and \( r \geq 0 \) since \( h^* \) is monotonously decreasing and \( h^* 1_{(-\infty,0]} \geq 0 \). But we know that \( f \) is of at most polynomial growth leading to
\[
h^*(u) \leq -\tilde{m}u + \tilde{c}_v(1 + |u|^\nu) - f(0) + c_{f,\kappa}
\]
for all \( u \leq 0 \), and thus
\[
E|h^*(\tilde{v}(r,z))|^p 1_{\{\tilde{v}(r,z) \geq 0\}} \leq 3^{p-1}\tilde{m}^p E\tilde{v}(r,z)^p + 3^{p-1}(\tilde{c}_v + c_{f,\kappa} - f(0))^p + 3^{p-1}\tilde{c}_v^p E\tilde{v}(r,z)^p
\]
for all \( z \in \mathbb{R}^d \) and \( r \geq 0 \). Hence, \( I_3 \) can uniformly be estimated in \( s \geq 0 \) and \( y \in \mathbb{R}^d \) if
\[
\sup_{t \geq 0, x \in \mathbb{R}^d} E\tilde{v}(t,x)^p < \infty.
\]
As in the first part of the proof with respect to \( v^* \) we now approximate \( \tilde{v} \) by a sequence \( (\tilde{v}_n)_{n=0}^\infty \) defined by
\[
\tilde{v}_0 \equiv 0
\]
as well as
\[
\tilde{v}_n(t,x) := \int_{\mathbb{R}^d} e^{-(m+\tilde{m})y} G(t,x-y)\theta_0(y) \text{d}y
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} e^{-(m+\tilde{m})(t-s)} G(t-s,x-y)\sigma(\tilde{v}_{n-1}(s,y)) \text{d}W_{s,y}.
\]
It is again sufficient for (5) to show that
\[
\sup_{t \geq 0, x \in \mathbb{R}^d} E\tilde{v}_1(t,x)^p < \infty
\]
because
\[
c(p
\]
\[
\frac{a\sigma_0^2}{2(m+\tilde{m})} (p/2)^\nu
\]
\[
\nu
\]
\[
c(p
\]
\[
\frac{\sigma_0^2}{\sqrt{16\lambda(m+\tilde{m})}} (p/2)^\nu
\]
for sufficiently large \( \tilde{m} \).

But the wanted estimation for \( \tilde{v}_1 \) holds true since for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \) resp. \( x \in \mathbb{R} \) we have
\[
E\tilde{v}_1(t,x)^p \leq 2^{p-1} \sup_{y \in \mathbb{R}^d} |\theta_0(y)|^p + 2^{p-1} c(p
\]
\[
\frac{a\sigma(0)^2}{2(m+\tilde{m})} (p/2)^\nu
\]
resp.
\[
E\tilde{v}_1(t,x)^{pv} \leq 2^{pv-1}\sup_{y \in \mathbb{R}^d} |\theta_0(y)|^{pv} + 2^{pv-1}c(pv)^{pv}\left(\frac{\sigma(0)^2}{\sqrt{16\lambda(m + \tilde{m})}}\right)^{(p/2)v}
\]
finishing the proof of the lemma. \(\square\)

**Lemma 2.** Define
\[
h_\bullet(u) = \begin{cases} 
-\tilde{m}u + \inf_{u \geq 0} (f(v) - f(0) - c_{f,\kappa}), & u \geq 0, \\
-c_{f,\kappa}, & u \leq 0,
\end{cases}
\]
where \(\tilde{m}\) is a sufficiently large number, and denote by \(v_\bullet = (v_\bullet(t,x))_{t \geq 0, x \in \mathbb{R}^d}\) the unique solution of (Eq[m]) with \(F = h_\bullet\). If
\[
m > c(p)^4 \frac{c_\sigma^4}{16\lambda}\text{ resp. } m > c(p)^2 \frac{ac_\sigma^2}{2}
\]
in the cylindrical resp. nuclear case, then
\[
\sup_{t \geq 0} E\|v_\bullet(t, \cdot)\|_{p,\nu}^p < \infty.
\]

**Proof.** We copy the proof of Lemma 1 only modifying the estimation of \(I_3\) as follows:
Instead of (3) we get
\[
h_\bullet(u) \leq -\tilde{m}u, \quad u \in \mathbb{R},
\]
thus the comparison theorem gives
\[
v_\bullet(r,z) \leq \tilde{v}(r,z)
\]
for Lebesgue-a.e. \(z \in \mathbb{R}^d\) a.s., \(r \geq 0\). However, the wanted analogue to (4), i.e.
\[
E|h_\bullet(v_\bullet(r,z))|^p1_{\{v_\bullet(r,z) \geq 0\}} \leq E|h_\bullet(\tilde{v}(r,z))|^p1_{\{\tilde{v}_\bullet(r,z) \geq 0\}}, \quad z \in \mathbb{R}^d, \quad r \geq 0,
\]
holds true since \(h_\bullet\) is monotonously decreasing and \(h_\bullet 1_{[0,\infty)} \leq 0\). \(\square\)

**Proof of Theorem 3.** Choose \(\kappa\) and \(p\) as in the theorem. Obviously, the unique solution \(u\) of Eq. (Eq) also solves (Eq[\(\kappa\)]) with \(F = f + \kappa Id\).
Now, for \(m = \kappa\), we introduce \(h_\bullet, v_\bullet\) resp. \(h_\bullet, v_\bullet\) as in Lemma 1 resp. Lemma 2 and remark that (f3), or better its equivalent version given in Remark 2(c), implies
\[
h_\bullet(u) \leq f(u) + \kappa u \leq h_\bullet(u), \quad u \in \mathbb{R}.
\]
Hence, from the comparison theorem follows that
\[
v_\bullet(t,x) \leq u(t,x) \leq v_\bullet(t,x)
\]
for all \(x \in \mathbb{R}^d\) and \(t \geq 0\) thus
\[
\sup_{t \geq 0} E\|u(t, \cdot)\|_{p,\nu}^p < \infty
\]
is an easy consequence of the corresponding property of \(v^*\) resp. \(v^*\) stated in Lemma 2 resp. Lemma 1. □

**Proof of Corollary 1.** Starting with \(v_0 \equiv 0\), we set

\[
v_n(t,x) := \int_{\mathbb{R}^d} G(t,x - y) \theta_0(y) \, dy + \int_0^t \int_{\mathbb{R}^d} G(t - s,x - y) \sigma(v_{n-1}(s,y)) \, dW_{sy}
\]

and if we can show

\[
|v_2(t,x) - v_1(t,x)|^2 \leq c_2^2 \left[ a + \sum_{k=1}^{\infty} a_k (\pi \lambda)^{-d/2} 2^{-(3/2)d-1} (d-2) \right] \cdot c_1, \quad t \geq 0, \ x \in \mathbb{R}^d
\]

as well as

\[
c_1 = \sup_{s \geq 0, y \in \mathbb{R}^d} \mathbb{E}|v_1(s,y)|^2 < \infty,
\]

then the assertion of the corollary follows doing similar steps to those made in the proof of Lemma 1. But,

\[
\mathbb{E}|v_2(t,x) - v_1(t,x)|^2
\]

\[
= \mathbb{E} \int_0^t \sum_{k=1}^{\infty} a_k \left( \int_{\mathbb{R}^d} G(t-s,x-y) [\sigma(v_1(s,y)) - \sigma(0)] e_k(y) \, dy \right)^2 \, ds
\]

\[
\leq c_2^2 \left[ a \mathbb{E} \int_{t-1}^t \sum_{k=1}^{\infty} a_k \left( \int_{\mathbb{R}^d} G(t-s,x-y) |v_1(s,y)| |e_k(y)| \, dy \right)^2 \, ds
\]

\[
+ \mathbb{E} \int_0^t \sum_{k=1}^{\infty} a_k \left( \int_{\mathbb{R}^d} G(t-s,x-y) |v_1(s,y)| |e_k(y)| \, dy \right)^2 \, ds
\]

\[
\leq c_2^2 \left[ a c_1 + \mathbb{E} \int_0^{t-1} \sum_{k=1}^{\infty} a_k \left( \int_{\mathbb{R}^d} G(t-s,x-y) |v_1(s,y)| |e_k(y)| \, dy \right)^2 \, ds \right]
\]

\[
\leq c_2^2 \left[ a c_1 + \sum_{k=1}^{\infty} a_k \int_0^t \int_{\mathbb{R}^d} G^2(t-s,x-y) \, dy \int_{\mathbb{R}^d} \mathbb{E}|v_1(s,y)|^2 e_k(y)^2 \, dy \, ds \right]
\]

\[
\leq c_2^2 \left[ a c_1 + \sum_{k=1}^{\infty} a_k \int_0^t \int_{\mathbb{R}^d} G^2(t,x) \, dx \, dt \cdot c_1 \right],
\]
where
\[
\int_1^\infty \int_{\mathbb{R}^d} G^2(t,x) \, dx \, dt = (\pi \lambda)^{-d/2} 2^{-(3/2)d-1} (d - 2)
\]
is only finite if \( d \geq 3 \).

Finally, having these estimates in mind, the finiteness of \( c_1 \) follows as the finiteness of \( I_1, I_2 \) in the proof of Lemma 1. \( \square \)

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Appendix.

Proposition A.1. Fix \( p \geq 1 \) as well as \( t > 0 \) and choose \( \bar{q} \geq 0 \) such that \( q - d > \bar{q} \). Then
\[
S(t) : L^p_{\bar{q}}(\mathbb{R}^d) \to L^p_{\bar{q}}(\mathbb{R}^d)
\]
is a compact operator.

Proof (Sketch). If \( \{u_n\} \) is a bounded sequence in \( L^p_{\bar{q}}(\mathbb{R}^d) \) then we have to show that \( \{S(t)u_n\} \) contains a convergent subsequence in \( L^p_{\bar{q}}(\mathbb{R}^d) \).

For \( R > 0 \), let \( K_R \) denote the compact set \( \{x \in \mathbb{R}^d : |x| \leq R\} \) and consider the restrictions \( v_n|_{K_R} \) of \( v_n = S(t)u_n \), \( n = 1, 2, \ldots \). Of course, the sequence \( \{v_n|_{K_R}\} \) presents a bounded subset of the space of continuous functions on \( K_R \) which is even equicontinuous; therefore it contains a convergent subsequence by Arzelà–Ascoli. As a consequence, if \( \mathbf{1}_{K_R} \) denotes the indicator function of the subset \( K_R \) then easy calculations show that the sequence \( \{\mathbf{1}_{K_R}v_n\} \) contains a convergent subsequence in \( L^p_{\bar{q}}(\mathbb{R}^d) \).

We now construct a subsequence \( \{v_{m_R}\}_{R=1}^{\infty} \) of \( \{v_n\} \) which converges in \( L^p_{\bar{q}}(\mathbb{R}^d) \). Let \( \{\mathbf{1}_{K_i}v_n\} \) be a subsequence of \( \{\mathbf{1}_{K_i}v_n\} \) which converges in \( L^p_{\bar{q}}(\mathbb{R}^d) \) to \( \tilde{v}_1 \). Then there is a subsequence of \( \{\mathbf{1}_{K_2}v_{n_k}\} \) given by \( (n_k^2) \) which converges in \( L^p_{\bar{q}}(\mathbb{R}^d) \) to \( \tilde{v}_2 \) satisfying \( \mathbf{1}_{K_i}\tilde{v}_2 = \tilde{v}_1 \) and, successively, for each \( R = 1, 2, \ldots \), we find a subsequence \( \{\mathbf{1}_{K_R}v_{n_k}\}_{k=1}^{\infty} \) which converges in \( L^p_{\bar{q}}(\mathbb{R}^d) \) to \( \tilde{v}_R \) satisfying \( \mathbf{1}_{K_R}\tilde{v}_{R+1} = \tilde{v}_R \). So, for each \( R = 1, 2, \ldots \),
\[
\exists k_R \forall k > k_R : \|\mathbf{1}_{K_R}v_{n_k^R} - \tilde{v}_R\|_{p,\bar{q}} < \frac{1}{R}
\]
and we set
\[
m_R = n_k^R, \quad R = 1, 2, \ldots .
\]

Because of
\[
v_n = \mathbf{1}_{K_R}v_n + \mathbf{1}_{K_R^c}v_n, \quad n = 1, 2, \ldots ,
\]
\( \{v_{m_R}\}_{R=1}^{\infty} \) converges to \( \lim_{R \to \infty} \tilde{v}_R \) in \( L^p_0(\mathbb{R}^d) \) since for all \( \varepsilon > 0 \) there exists an \( R_0 > 0 \) such that
\[
\forall R \geq R_0 \quad \forall n: \|1_{K^c_R}v_n\|_{p,q} < \varepsilon.
\]

In fact,
\[
\|1_{K^c_R}v_n\|_{p,q}^p = \int_{\{|x| > R\}} \int_{\mathbb{R}^d} G(t,x-y)(1 + |y|^2)^{\tilde{\alpha}/2} p u_n(y)(1 + |y|^2)^{-\tilde{\alpha}/2} p \, dy \right)^p \\
\times (1 + |x|^2)^{-\tilde{\alpha}/2} \, dx
\]
\[
\leq \|u_n\|_{p,\tilde{\alpha}}^p \int_{\{|x| > R\}} \int_{\mathbb{R}^d} G(t,x-y)^{p/(p-1)} \, dy \right)^{p-1} \\
\leq \|u_n\|_{p,\tilde{\alpha}}^p \int_{\mathbb{R}^d} G(t,z)^{p/(p-1)} 2(1 + |z|^2)^{\tilde{\alpha}/2(p-1)} \, dz \right)^{p-1} \\
\times \int_{\{|x| > R\}} (1 + |x|^2)^{(\tilde{\alpha} - q)/2} \, dx
\]
for all \( n = 1, 2, \ldots \) and the assertion immediately follows from the assumptions because the above integral of which integrand includes \( G \) as a factor is finite.

**Proof of Theorem 2.** Fix \( \tilde{\alpha}, \bar{q} \in \mathbb{R} \) such that
\[
\tilde{\alpha} > d \quad \text{and} \quad \bar{q} > \tilde{\alpha} + d
\]
and remark that, if \( q > 1 \) then
\[
H_\varphi \Phi = \int_0^1 (1-t)^{q-1} S(1-t) \Phi(t, \cdot) \, dt, \quad \Phi \in L^q([0,1]; L^p_\tilde{\alpha}(\mathbb{R}^d))
\]
defines a compact operator \( H_\varphi : L^q([0,1]; L^p_\tilde{\alpha}(\mathbb{R}^d)) \to L^p_0(\mathbb{R}^d) \) for every \( \varphi \in (1/q, 1] \). Indeed, applying that \( S(t) : L^p_\varphi(\mathbb{R}^d) \to L^p_\varphi(\mathbb{R}^d), \ t > 0 \) are compact operators (see Proposition 1, Appendix), the last result follows as in the proof of Lemma 6.1.4 in Da Prato and Zabczyk (1996).

As a consequence, for every \( r > 0 \) and \( \bar{q} \in (1/q, 1] \), the set
\[
K(r) = \{ S(1)\tilde{\alpha} + H_1 \Phi + H_\varphi \Psi : \|\tilde{\alpha}\|_{p,\tilde{\alpha}} < r; \|\Phi\|_{L^q([0,1]; L^p_\tilde{\alpha}(\mathbb{R}^d))} < r, \|\Psi\|_{L^q([0,1]; L^p_\bar{q}(\mathbb{R}^d))} < r \}
\]
is relatively compact in \( L^p_\bar{q}(\mathbb{R}^d) \).

In a first step we show that if \( q \geq p \) is especially chosen to be greater than 4 and greater than 2 in the cylindrical and the nuclear case, respectively, then there exists a
A uniform constant $c > 0$ such that

$$\mathbb{P}\left( \{ u(\tilde{\theta}, 1, \cdot) \in K(r) \} \right) \geq 1 - cr^{-q}(1 + \| \tilde{\theta} \|_{p,v,\tilde{\theta}}^{q} + \| \tilde{\theta} \|_{p,v,\tilde{\theta}}^{q})$$

(A.1)

holds true for all $r > 0$ and all $\tilde{\theta}$ satisfying $\| \tilde{\theta} \|_{p,v,\tilde{\theta}} < r \cdot (\int_{\mathbb{R}^d} (1 + |x|^2)^{\tilde{\theta}/2} \, dx)^{p-p/v}$.

Applying the factorization formula

$$u(\tilde{\theta}, 1, \cdot) = S(1) \tilde{\theta} + H_1 f(u(\tilde{\theta}, \cdot, \cdot)) + \sin \frac{\pi}{p} H_x Y_x(\tilde{\theta}, \cdot, \cdot)$$

for some $\alpha \in (1/q, 1/4)$ resp. $\alpha \in (1/q, 1/2)$ in the cylindrical resp. nuclear case, where

$$Y_x(\tilde{\theta}, t, \cdot) = \int_0^t (t-s)^{-\alpha} S(t-s) \sigma(u(\tilde{\theta}, s, \cdot)) \, dW(s), \quad t \geq 0,$$

we only have to verify that

$$\mathbb{E} \int_0^1 \| f(u(\tilde{\theta}, t, \cdot)) \|_{p,v,\tilde{\theta}}^q \, dt \leq \tilde{c}(1 + \| \tilde{\theta} \|_{p,v,\tilde{\theta}}^{q})$$

(A.2)

as well as

$$\mathbb{E} \int_0^1 \| Y_x(\tilde{\theta}, t, \cdot) \|_{p,v,\tilde{\theta}}^q \, dt \leq \tilde{c}(1 + \| \tilde{\theta} \|_{p,v,\tilde{\theta}}^{q})$$

(A.3)

for some constant $\tilde{c} > 0$, and (A.1) can be obtained as in the proof of Lemma 6.1.5 in Da Prato and Zabczyk (1996).

In what follows, we use to denote various constants by the same symbol $\tilde{c}$ or, if necessary, by $\tilde{c}_1, \tilde{c}_2$ and so on. In every case, these constants do not depend on the arguments $\tilde{\theta}, t, x$ of $u$, but, they might depend on the fixed parameters $p, q, v, q, x$ or on the covariance of the driving Wiener process $W(t)$ in a way which becomes clear from the calculations. At first, we may estimate

$$\mathbb{E} \int_0^1 \| f(u(\tilde{\theta}, t, \cdot)) \|_{p,v,\tilde{\theta}}^q \, dt$$

$$\leq \tilde{c}_0 \int_0^1 \, dt \mathbb{E} \left( \int_{\mathbb{R}^d} (1 + |u(\tilde{\theta}, t, x)|^v (1 + |x|^2)^{-\tilde{\theta}/2} \, dx)^q/p \right)^q/p$$

(by Remark 1c) and, because of $\tilde{q} > d$, we only need to proceed estimating

$$\mathbb{E} \left( \int_{\mathbb{R}^d} |u(\tilde{\theta}, t, x)|^v (1 + |x|^2)^{-\tilde{\theta}/2} \, dx \right)^q/p = \mathbb{E} \| u(\tilde{\theta}, t, \cdot) \|_{p,v,\tilde{\theta}}^{q}$$

uniformly in $t \in [0, 1]$. Because of $q \geq p$, the right-hand side can be dominated by $\tilde{c}(1 + \| \tilde{\theta} \|_{p,v,\tilde{\theta}}^{q})$ for some constant $\tilde{c}$ just applying Theorem 1 with respect to the time horizon $T = 1$. Thus, (A.2) holds true for every $\tilde{\theta} \in L_{\tilde{\theta}}(\mathbb{R}^d)$.
We now show (A.3). Since $L^p_{\tilde{E}}(\mathbb{R}^d)$ is an M-type 2 Banach space, from the generalized Burkholder–Davis–Gundy inequality (cf. Dettweiler, 1991) follows that

$$
E \int_0^1 \| Y_x(\tilde{\theta}, t, \cdot) \|^q_{p, \tilde{E}} \, dt
$$

$$
\leq \tilde{c}_1 \int_0^1 dt \left( \int_0^t \mathbb{E} \left[ \left( \sum_k (t-s)^{-2x} (S(t-s)[\sigma(u(\tilde{\theta}, s, \cdot)) \sqrt{\alpha_k e_k}]^2 \right] \right)^{1/2} \| \|_{p, \tilde{E}} \, ds \right)^{q/2}
$$

$$
\leq \tilde{c}_2 E \int_0^t dt \left( \int_0^t (t-s)^{-2x} \| S(t-s) \| \sigma(u(\tilde{\theta}, s, \cdot)) \|_{p, \tilde{E}} \, ds \right)^{q/2}
$$

(A.4)

in the nuclear case (cf. the proof of Lemma 1 above) which yields

$$
\leq \tilde{c}_3 E \int_0^t dt \left( \int_0^t (t-s)^{-2x} \| S(t-s) \| \sigma(u(\tilde{\theta}, s, \cdot)) \|_{p, \tilde{E}} \, ds \right)^{q/2}
$$

because $(S(t))_{t \geq 0}$ is a strongly continuous semigroup on $L^p_{\tilde{E}}(\mathbb{R}^d)$. Hence, by Hausdorff–Young’s inequality, we can continue estimating

$$
\leq \tilde{c}_3 \left( \int_0^1 t^{-2x} \, dt \right)^{q/2} \int_0^1 E \| \sigma(u(\tilde{\theta}, t, \cdot)) \|^q_{p, \tilde{E}} \, dt.
$$

Of course, $\int_0^1 t^{-2x} \, dt$ is finite since $x < \frac{1}{2}$, and the linear growth of the lipschitz continuous coefficient $\sigma$ implies

$$
E \| \sigma(u(\tilde{\theta}, t, \cdot)) \|^q_{p, \tilde{E}} \leq \tilde{c}_4 (1 + \| \tilde{\theta} \|^q_{p, \tilde{E}}), \quad t \in [0, 1],
$$

applying Theorem 1 once more; finally leading to (A.3) in the nuclear case because $p \nu \geq p$.

In the cylindrical case, instead or (A.4), we get

$$
\leq \tilde{c}_2 E \int_0^1 dt \left( \int_0^t (t-s)^{-2x} \| \int_\mathbb{R} G^2(t-s, x-y) \sigma^2(u(\tilde{\theta}, s, y)) \, dy \|_{p/2, \tilde{E}} \, ds \right)^{q/2}
$$

$$
= \tilde{c}_2 E \int_0^1 dt \left( \int_0^t (8\pi\lambda)^{-1/2} (t-s)^{-2x-1/2} \| \tilde{S}(t-s) \| \sigma^2(u(\tilde{\theta}, s, \cdot)) \|_{p/2, \tilde{E}} \, ds \right)^{q/2},
$$

where

$$
[\tilde{S}(t)u](x) := \int_\mathbb{R} (2\pi\lambda t)^{-1/2} \exp \left\{ -\frac{|x-y|^2}{2\lambda t} \right\} u(y) \, dy, \quad t \geq 0
$$

also defines a strongly continuous semigroup on $L^p_{\tilde{E}}(\mathbb{R})$. Thus, again by Hausdorff–Young’s inequality,

$$
E \int_0^1 \| Y_x(\tilde{\theta}, t, \cdot) \|^q_{p, \tilde{E}} \, dt \leq \tilde{c}_3 \left( \int_0^1 t^{-2x-1/2} \, dt \right)^{q/2} \int_0^1 E \| \sigma^2(u(\tilde{\theta}, t, \cdot)) \|^q_{p/2, \tilde{E}} \, dt.
$$
Because of
\[ \| \sigma^2(u(\tilde{\theta}, t, \cdot)) \|_{p, \bar{\delta}}^{q/2} = \| \sigma(u(\tilde{\theta}, t, \cdot)) \|_{p, \bar{\delta}}^{q}, \quad t \in [0, 1], \]
(A.3) follows as in the nuclear case since \( \alpha < \frac{1}{4} \).
Altogether, (A.1) is shown and we proceed copying the proof of Theorem 6.1.2 in Da Prato and Zabczyk (1996). Namely, we similarly arrive at the inequality
\[ P(\{ (u(\theta_0, t, \cdot) \in K(r) \} \geq (1 - cr^{-q}(1 + r_1^{q_v} + r_1^{q}))P(\{ \| u(\theta_0, t - 1, \cdot) \|_{p, \bar{\delta}} < r_1 \}) \]
for all \( t > 1 \) and all \( r > r_1 \cdot (\int_{\mathbb{R}^d} (1 + |x|^2)^{q_v/2} \, dx) (r^{p_v - p})^{1/p_v}; \) thus, first taking \( r_1 \) and then \( r \) sufficiently large, we can also finish similarly, because \( (u(\theta_0, t, \cdot))_{t \geq 0} \) is bounded in probability in \( L_{p, \bar{\delta}}^{r_1}(\mathbb{R}^d) \).

References