JOURNAL OF ALGEBRA **81**, 420–433 (1983)

A Presentation of Topoi as Algebraic Relative to Categories or Graphs

EDUARDO J. DUBUC*

Departamento de Matematicas, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina

AND

G. M. KELLY*

Department of Pure Mathematics, University of Sydney, N.S.W. 2006, Australia

Communicated by Saunders MacLane

Received March 26, 1982

1. INTRODUCTION

A functor is *conservative* if it reflects isomorphisms, and is *finitary* if it preserves filtered colimits; a monad is *finitary* if its functor-part is so. Whenever $U: \mathscr{L} \to \mathscr{K}$ is conservative, it makes good sense to think of an $\mathscr{L}$-object as *a* $\mathscr{K}$*-object with extra structure*; and to regard the extra structure as *algebraic* when U is monadic, and as *finitary* when U is finitary. In the monadic case, U is of course finitary exactly when the corresponding monad T on $\mathscr{K}$ is so.

When $\mathscr{K}$ is locally finitely presentable, and everything is enriched over a suitable closed category $\mathscr{V}$, one can make a precise analysis of the nature of a finitary monad T on $\mathscr{K}$; this will be done in this generality in the proposed paper of Kelly [6]. Such a T is the left Kan extension of its restriction to the finitely-presentable objects of $\mathscr{K}$; and Tc, for a finitely-presentable $c \in \mathscr{K}$, may be thought of as *the* $\mathscr{K}$*-object of operations of arity* c. When $\mathscr{K} = \mathscr{V} = \mathbf{Set}$, the finitely-presentable sets are just the finite cardinals $n \in \mathbf{N}$, and Tn is the set of n-ary operations in the corresponding Lawvere theory $\mathscr{T}$, which is given by $\mathscr{T}(n, m) = (Tn)^m$.

As in that case $\mathscr{K} = \mathscr{V} = \mathbf{Set}$ of classical finitary one-sorted universal algebra, T may be given not directly but by a *presentation*: we have for each

* The authors gratefully acknowledge the assistance of the Australian Research Grants Committee.

0021-8693/83 $3.00

finitely-presentable c a $\mathscr{K}$-object Sc of *basic operations of arity* c, from which we build up a $\mathscr{K}$-object Rc of *derived operations of arity* c; and then we have for each c a $\mathscr{K}$-object Ec of *equations* between derived operations of arity c. Often, in practice, Sc and Ec are "empty" except for a few values of c, and the presentation is "finite." Then a very direct way of proving $U: \mathscr{L} \to \mathscr{K}$ monadic is by exhibiting the $\mathscr{L}$-objects as the algebras for such a presentation, and verifying that the maps in $\mathscr{L}$ are the maps in $\mathscr{K}$ preserving the basic operations.

This notion of "a presentation of an algebraic theory on $\mathscr{K}$" becomes less abstract when $\mathscr{K}$ is, say, the category **Cat** of small categories or the category **Gph** of small graphs. The case $\mathscr{K} = \mathbf{Cat}$, of categories with algebraic extra structure, was sketched in Kelly's 1979 Oberwolfach lecture [3], both in the classical context $\mathscr{V} = \mathbf{Set}$ and in the enriched contexts $\mathscr{V} = \mathbf{Cat}$ and $\mathscr{V} = \mathbf{Gpd}$ (the category of small groupoids).

In the meantime, A. Burroni has independently given the analogous analysis of finitary monads in the case $\mathscr{K} = \mathbf{Gph}$, $\mathscr{V} = \mathbf{Set}$; and has made the surprising observation that many of the categories-with-extra-structure that occur in practice are monadic not only over **Cat** but over **Gph**: they may be presented as graphs with algebraic extra structure, given by operations whose arities are finite graphs, and equations between derived operations. Burroni's detailed results are to appear in his forthcoming thesis; they have been communicated in lectures at the *Troisième Colloque sur les Catégories, Amiens, 7–12 July 1980*; at the *Journées Faisceaux et Logique, Université Paris Nord, 23–24 May 1981*; and at the *Cambridge Summer Meeting in Category Theory, 19–25 July 1981*. Some of these results are also sketched in Burroni [1], where explicit presentations are given for a number of extra structures on a category.

Among these structures is that of an elementary topos. Unfortunately the presentation given by Burroni in this case is incomplete, in the crucial matter of the subobject-classifier Ω. He has an operation, of arity "arrow," which to every $f: A \to B$ associates a $\chi f: B \to \Omega$, to be interpreted as the characteristic map of the monomorphic part i of f; and he has an equation asserting that the pullback of "true" along χf is isomorphic to i. However, there is nothing to ensure that, for an arbitrary $g: B \to \Omega$, the characteristic map of the pullback of "true" along g is again g; and in fact a model of his presentation is given by **Set** when we take for Ω any set with at least two elements. This cannot be put right just by introducing the missing equation, to hold for each $g: B \to \Omega$; for the terms so equated are not operations in the allowable sense. An operation of arity "arrow" must assign something to *every* arrow $B \to C$, and not just to every arrow with the fixed codomain Ω; see Section 2 below.

Our present purpose is firstly to complete Burroni's presentation of topoi, and thus to confirm that they are indeed monadic over graphs. We recover the missing equation above by introducing two extra axioms; but even this is

481 81 2-10

not in fact enough. More operations and more axioms are needed to ensure that two *isomorphic* monomorphisms $A \to B$ have the same characteristic map $B \to \Omega$. Our second purpose is to give similar algebraic presentations of such related structures as quasi-topoi and categories-with-a-natural-numbers-object. In some cases we prove monadicity only over **Cat**, and not over **Gph**. In any case, monadicity over **Cat** is important because, **Cat** unlike **Gph** being a **Gpd**-category, we can consider the **Gpd**-enrichability of a monad T on **Cat**, which allows the consideration of morphisms of algebras preserving the structure only to within isomorphism: see Section 8 below.

Accordingly, we give in Section 8 a precise statement of the results of the analysis in [6], as they apply to the cases $\mathscr{K} = \mathbf{Cat}$ or **Gph**, with $\mathscr{V}' = \mathbf{Set}$. However, we give the explicit presentation of the various structures before this abstract theory, preceding them only by some further informal elaboration, in Section 2, of the ideas sketched above. This should make the article more easily readable: we all learnt presentations of the theories of groups and of rings before we learnt abstract universal algebra. It does no harm to the logic: for the explicit presentations, along with the results of Section 8, constitute a formal proof of monadicity.

2. Presentations of a Finitary Monad on Cat

Categories with some extra structure are the algebras for a finitary monad on **Cat** precisely when the extra structure can be presented by *operations*, whose *arities* are finitely-presentable categories, subjected to *equational axioms* between derived operations. If c is a finitely-presentable category, an operation of arity c is realized in a model $\mathscr{A}$ by a rule which, to every diagram $\phi: c \to \mathscr{A}$ of type c in $\mathscr{A}$, assigns either an object or a morphism of $\mathscr{A}$. Among the processes which produce *derived* operations from the *basic* ones, we may count composition; but it is more elegant to regard composites of basic operations as new basic ones, so that the basic operations of arity c in fact form a *category*: this is the position we take in Section 8 below. In this view derived operations are formed much as in classical universal algebra; from a d-ary operation and a "d-ad" of c-ary operations we get a new c-ary operation. In particular we get a c-ary operation from a d-ary operation and a functor $d \to c$. The axioms are to be equations between pairs of derived operations of the same arity. The morphisms of models are just those functors that preserve the operations. We say that the operations and the equations form a *presentation* of the extra structure as *algebraic*, relative to the structure of a category.

If we have such a presentation in which the arities of the basic operations and of the equations are free categories on finite graphs, we have a stronger conclusion: not only is the forgetful functor to categories finitarily monadic.

but even the forgetful functor to graphs is so. In this case the models can be seen as *graphs* with algebraic extra structure. Such a presentation over graphs coincides with what Burroni [1] calls "une théorie équationelle graphique."

In denoting arities, we write 0, 1, 2, 3,... for the discrete category with the indicated number of objects, and **0**, **1**, **2**, **3**,... for the corresponding ordinal, so that **3** denotes the category $(0 \to 1 \to 2)$. We also write $\mathbf{m} \vee \mathbf{n}$ for the coproduct of the finite ordinals **m** and **n** with their terminal object identified; so that $\mathbf{2} \vee \mathbf{3}$ is the category $(\cdot \to \cdot \leftarrow \cdot \leftarrow \cdot)$.

Some writers tend to use "equational" for what we call "algebraic" or "monadic." There is a danger in this usage. Merely having all the axioms equational does not imply monadicity; the operations must themselves be of the right kind. The essence of an operation of arity c is that it assigns a value to *every* diagram $\phi\colon c \to \mathscr{A}$, and not just to special such diagrams. Thus, in the theory of non-monoidal closed categories, there is an operation $j_A\colon I \to [A, A]$ of arity 1, from which we get a derived operation of arity **2** sending $f\colon A \to B$ to $f^* = [1, f]\, j_A\colon I \to [A, B]$. The desired axiom that $f \mapsto f^*$ gives a bijection $\mathscr{A}(A, B) \to \mathscr{A}(I, [A, B])$ can be expressed equationally by introducing an operation $(\ )^\dagger$ sending each $g\colon I \to [A, B]$ to some $g^\dagger\colon A \to B$, and setting $f^{*\dagger} = f$ and $g^{\dagger *} = g$. But $(\ )^\dagger$ is not an operation of arity **2**, or of any arity, in the present sense; and in fact it is shown in [4] that the forgetful functor from non-monoidal closed categories to categories is not monadic.

With these remarks, we pass directly to the concrete examples, referring the reader to Section 8 for a more precise account of presentations.

3. Finite Limits

To establish our notation for the later sections, we recall from [1], with inessential changes, Burroni's presentation of categories with finite limits. The arities are all free on finite graphs, so that we have monadicity over graphs.

To give the terminal object, we give an operation of arity 0 whose value is an object 1, and one of arity 1 that assigns, to each object A, a morphism $A \to 1$; and we impose a single equation of arity **2**:

$$\begin{tikzcd} A \arrow[d, "x"'] \arrow[dr] & \\ B \arrow[r] & 1. \end{tikzcd}$$

Having the terminal object, we can construct all finite limits if we can construct fibred products. We present these in an unsymmetric way, namely, as the operation of pulling back a "variable" $x\colon X \to B$ along a "fixed" $f\colon A \to B$, providing a right adjoint to composition with f.

Suppressing the fixed f from parts of the notation, we begin with three operations of arity $\mathbf{2} \vee \mathbf{2}$, giving the pullback-object and its two projections, and one equation of arity $\mathbf{2} \vee \mathbf{2}$ expressing the commutativity of

$$\begin{tikzcd} P(f,x) \arrow[r, "\varepsilon x"] \arrow[d, "f^*x"'] & X \arrow[d, "x"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Thus ε is the counit of the adjunction. To define pulling-back along f on morphisms in $\mathscr{A}/B$, and to express the naturality of ε, we introduce an operation $P(f, x, h)$ of arity $\mathbf{2} \vee \mathbf{3}$ and two equations of this arity, namely, the commutativity of the remaining regions in

$$\begin{tikzcd} P(f,xh) \arrow[r, "\varepsilon(xh)"] \arrow[dr, "P(f,x,h)"] \arrow[ddr, "f^*(xh)"'] & Z \arrow[dr, "h"] & \\ & P(f,x) \arrow[r, "\varepsilon x"] \arrow[d, "f^*x"] & X \arrow[d, "x"] \\ & A \arrow[r, "f"'] & B. \end{tikzcd}$$

To ensure that $P(f, x, h)$ is in fact functorial in h we need the axiom $P(f, x, 1_X) = 1_{P(f,x)}$ of arity $\mathbf{2} \vee \mathbf{2}$, and an evident axiom of arity $\mathbf{2} \vee \mathbf{4}$ that we leave the reader to write out.

The unit η of the adjunction is introduced by an operation of arity $\mathbf{3}$ and an equation of that arity:

$$\begin{tikzcd} Y \arrow[r, "\eta y"] \arrow[dr, "y"'] & P(f,fy) \arrow[d, "f^*fy"] \\ & A; \end{tikzcd}$$

while its naturality is asserted by an equation of arity $\mathbf{4}$:

$$\begin{tikzcd} Z \arrow[r, "\eta(yh)"] \arrow[d, "h"'] & P(f,fyh) \arrow[d, "P(f,fy,h)"] \\ Y \arrow[r, "\eta y"'] & P(f,fy). \end{tikzcd}$$

Finally, the triangular equations for the adjunction are given by the equation $\varepsilon(fy) \cdot \eta y = 1_Y$ of arity **3** and the equation $P(f, x, \varepsilon x) \cdot \eta(f^*x) = 1_{P(f,x)}$ of arity $\mathbf{2} \vee \mathbf{2}$.

Of course categories with finite *colimits* are algebraic over graphs in the same way, with arities suitably modified.

4. Cartesian-Closed and Locally-Cartesian-Closed Categories

A presentation of locally-cartesian-closed categories as algebraic over graphs is given by Burroni in a similar way: namely, by writing down, in terms of a unit and a counit and triangular equations, the existence of a right adjoint Π_f to f^*. We have nothing to add to this, but one word of warning to insert.

From the fact that locally-cartesian-closed categories are algebraic over graphs, it does not follow automatically that cartesian closed categories are so. The point is that an operation of arity **2**, say, must, as we said in Section 2, assign some value to *every* morphism; and not just to a morphism with some restricted codomain like 1.

This means that all the arities in the former presentation must be reassessed, if we seek to present merely cartesian closed categories. When this is done, it does turn out to be the case that we have a presentation of cartesian closed categories as algebraic over graphs.

5. Factorizations of Morphisms

Suppose we have in our theory a derived operation of arity c whose value is a morphism $x: A \to B$. We can easily modify the theory so as to force x to be an isomorphism: we merely introduce a new operation $\tilde{x}: B \to A$ of arity c and impose the two new equations $x\tilde{x} = 1_B$ and $\tilde{x}x = 1_A$ of arity c.

If our theory is already rich enough to include finite limits, we can force x to be a monomorphism just by imposing the equation $x^*x = \varepsilon x: P(x, x) \to A$, asserting that the two projections of its pullback by itself are equal.

Again, if our models already have both finite limits and finite colimits, we can assert that x is a regular epimorphism by forcing the derived operation $\bar{x}$ to be an isomorphism, where

$$\begin{tikzcd} P(x,x) \arrow[r, shift left, "\varepsilon x"] \arrow[r, shift right, "x^*x"'] & A \arrow[r, "qx"] \arrow[dr, "x"'] & Q(x) \arrow[d, "\bar{x}"] \\ & & B \end{tikzcd}$$

with qx being the coequalizer.

In the presence of finite limits and colimits, therefore, we can assert that every morphism factorizes as a regular epimorphism followed by a monomorphism, just by forcing $\bar{x}$ above to be a monomorphism. It is easy to make the further demand that such factorizations be stable under pullback; thus exhibiting regular categories with finite colimits as algebraic over graphs.

Similarly, in the presence of finite limits and colimits, we can assert that every morphism factorizes as a regular epimorphism followed by a regular monomorphism: either by forcing $\bar{x}$ above to be a regular monomorphism, or directly as in Example 7 of [1].

6. A Regular-Subobject Classifer; Quasi-Topoi and Topoi

Suppose we already have finite limits and finite colimits. We here show that we can add, algebraically over graphs, a regular-subobject classifier. Thus, by adding local cartesian closedness, we shall have a presentation of Penon's quasi-topoi [9] as algebraic over graphs; while by further adding the requirement that every morphism factorize as a regular epimorphism followed by a regular monomorphism, we shall have a presentation of elementary topoi as algebraic over graphs.

We begin with two operations of arity 0, giving an object Ω of $\mathscr{A}$ and a morphism $t\colon 1 \to \Omega$. Then we give an operation of arity **2** sending any morphism $x\colon X \to B$ to a morphism $\chi x\colon B \to \Omega$, plus an equation of this arity asserting the commutativity of

$$\begin{tikzcd} X \arrow[r] \arrow[d, "x"'] & 1 \arrow[d, "t"] \\ B \arrow[r, "\chi x"'] & \Omega. \end{tikzcd}$$

We now have the derived operation $\hat{x}$ of arity **2** given by the morphism so denoted in

$$\begin{tikzcd} X \arrow[dr, "\hat{x}"] \arrow[drr, bend left] \arrow[ddr, bend right, "x"'] & & \\ & P(\chi x, t) \arrow[r] \arrow[d, "(\chi x)^* t"] & 1 \arrow[d, "t"] \\ & B \arrow[r, "\chi x"'] & \Omega; \end{tikzcd}$$

and we force $\hat{x}$ to be epimorphic, as in Section 5, by an equation of arity **2**. Since t is a coretraction with left inverse $\Omega \to 1$, and is hence a regular monomorphism, its pullback $(\chi x)^* t$ is also a regular monomorphism. If, as in the case of topoi, one wanted to impose factorizations into a regular epimorphism followed by a regular monomorphism, one could just force $\hat{x}$ here to be instead a regular epimorphism.

This is where Burroni [1] stops in his attempt to present topoi as algebraic over graphs; but it is not enough. If we restrict the function χ for the moment to those $x: X \to B$ which are regular monomorphisms, we have the composite

$$\text{Reg Mono } B \xrightarrow[\chi]{} \mathscr{A}(B, \Omega) \xrightarrow[(\ )^* t]{} \text{Reg Mono } B; \tag{1}$$

since the epimorphism $\hat{x}$ is an isomorphism when x is a regular monomorphism, the axioms above ensure that (1) sends each x to an isomorph of x; but they fail to ensure that χ is surjective. To see this, take $\mathscr{A} = \mathbf{Set}$ and let Ω be any set with at least two elements t and f; the axioms to date are satisfied if we define χx by $(\chi x) b = t$ when $b \in \operatorname{im} x$ and $(\chi x) b = f$ otherwise.

Accordingly, we introduce two more axioms. The first, of arity 0, asserts that $\chi t = 1: \Omega \to \Omega$. The second, of arity $\mathbf{2} \vee \mathbf{2}$, contemplates maps $x: X \to B$ and $y: Y \to B$, and asserts the naturality property of χ given by $\chi(y^* x) = \chi x \cdot y$. Replacing B, X, x, Y, y in this second axiom by Ω, 1, t, b, g and using the first axiom now gives $g = \chi(g^* t)$ for each $g: B \to \Omega$.

So now χ is surjective; and in fact it is clear that (1) now gives a bijection between $\mathscr{A}(B, \Omega)$ and those "special" regular monomorphisms into B of the form $g^* t$ for some $g: B \to \Omega$. Yet this is still less than what we want. We want the χ of (1) to factorize through the canonical quotient map $\text{Reg Mono } B \to \text{Reg Sub } B$ from the monomorphisms onto the subobjects they represent, which is equivalently to require that each regular subobject contain only one "special" regular monomorphism $g^* t$. We can force this by adding another equation, whose arity is the category

$$\begin{tikzcd} X \arrow[dd, shift right, "u"'] \arrow[dr, "x"] & \\ & B \\ Z \arrow[uu, shift right, "v"'] \arrow[ur, "z"'] & \end{tikzcd} \tag{2}$$

where $uv = 1$, $vu = 1$, $zu = x$, and $xv = z$, asserting that $\chi x = \chi z$. This is enough to show algebraicity over categories; but not to show algebraicity over graphs, since the arity (2) is not the free category on a finite graph.

To overcome this we introduce a new operation $m\colon \Omega \times \Omega \to \Omega$ of arity 0, which will interpret as the usual order-relation on Ω, although we ask of it here only that it be reflexive and anti-symmetric. We express the reflexivity by the axiom

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "m"'] & \Omega \end{tikzcd}$$

of arity 0, where Δ is the diagonal. To express the anti-symmetry, we write m^{op} for the composite of m with the twist map $\Omega \times \Omega \to \Omega \times \Omega$. Then, since we clearly also have $m^{op}\Delta = t$, there are derived operations rendering commutative

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d] \arrow[dr, "\Delta"] & P(m^{op}, t) \arrow[d, "(m^{op})^*t"] \\ P(m, t) \arrow[r, "m^*t"'] & \Omega \times \Omega, \end{tikzcd}$$

and hence a derived operation s rendering commutative

$$\begin{tikzcd} \Omega \arrow[r, "s"] \arrow[dr] & P(m^*t, (m^{op})^*t) = P \arrow[d] \\ & \Omega \times \Omega. \end{tikzcd}$$

We now assert the anti-symmetry of m by forcing s (which is already a monomorphism since Δ is) to be an isomorphism, by introducing a new operation $r\colon P \to \Omega$ of arity 0 and a new equation $sr = 1$ of that arity.

To obtain the effect of the rejected axiom in (2), it clearly suffices to add one new axiom of arity **3**, which for $x\colon X \to B$ and $v\colon Z \to X$ asserts the commutativity of

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\binom{\chi(xv)}{\chi x}"'] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "m"'] & \Omega. \end{tikzcd}$$

7. A Natural-Numbers Object

The question of the algebraicity over graphs of a topos with a natural-numbers object, which Burroni left open in [1], is in fact easily answered positively by using Freyd's criterion [2, Theorem 5.43], as Burroni observed in his 1981 Paris lecture. We have only to give operations $o\colon 1 \to N$ and $s\colon N \to N$ of arity 0, and the equations of arity 0 which assert

$$1 \xrightarrow[o]{} N \xleftarrow[s]{} N$$

to be a coproduct diagram, and

$$N \underset{1}{\overset{s}{\rightrightarrows}} N \longrightarrow 1$$

to be a coequalizer diagram. (For the first, we force the map (o, s): $1 + N \to N$ to be an isomorphism, and similarly for the second.)

More generally, we can consider a natural-numbers object (N, o, s) in any category with finite limits. What we mean by such an object, in this generality, is what was called in Section 23 of [5] the *algebraically free monoid* on the object 1. By Theorem 23.1 of [5], it consists of an object N and morphisms $o\colon 1 \to N$ and $s\colon N \to N$, with the following universal property: given $x\colon Y \to X$ and $f\colon X \to X$, there is a unique $S(x,f)$ rendering commutative the diagram

$$\begin{tikzcd}
 & N \times Y \arrow[r, "s \times Y"] \arrow[dd, "S(x,f)"] & N \times Y \arrow[dd, "S(x,f)"] \\
Y \arrow[ur, "\binom{o}{1_Y}"] \arrow[dr, "x"'] & & \\
 & X \arrow[r, "f"'] & X.
\end{tikzcd} \tag{3}$$

The special case $Y = 1$ of (3) is the property originally used by Lawvere to characterize a natural-numbers object; and this special case clearly implies the general case when the model $\mathscr{A}$ is cartesian closed.

To present a category with finite products and a natural numbers object as algebraic over categories, we take the operations N, o, s, of arity 0 and the operation $S(x,f)$ whose arity is the free category on the graph $\cdot \to \cdot\circlearrowleft$. We impose as axioms, of this latter arity, the commutativity of the two regions in (3).

We need more axioms to ensure that $S(x,f)$ is the *only* morphism rendering (3) commutative. We first impose the axiom asserting that, given $y: Y \to Z$ and a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & X, \end{array}$$

we have $hS(y, g) = S(hy, f)$. (The arity of this axiom, because of the commutativity condition $fh = hg$, is not the free category on a finite graph; so that our argument does not give algebraicity over graphs.) We finally impose the axiom of arity 1 asserting that, when $X = N \times Y$ in (3), and

$$x = \begin{pmatrix} o \\ 1_Y \end{pmatrix}$$

and $f = s \times Y$, we have

$$S\left(\begin{pmatrix} o \\ 1_Y \end{pmatrix}, s \times Y\right) = 1_{N \times Y}.$$

These last two axioms together clearly imply that any $h: N \times Y \to X$ rendering (3) commutative is in fact $S(x, f)$.

8. A Precise Description of Algebraic Presentations

The analysis of [6] shows that categories with some extra structure are the algebras for a finitary monad on **Cat** precisely when they are the models for a theory (S, E) as described below, given by *basic operations* S and *equations* E; we then say that (S, E) is a *presentation* of the extra structure as algebraic, relative to the structure of a category.

S is a function which to each finitely-presentable category c assigns a category Sc of *basic operations of arity* c; in practice Sc will often be empty except for a few values of c. A *model* of S (we have no equations, as yet) is a category $\mathscr{A}$ together with a function which assigns to every functor $\phi: c \to \mathscr{A}$ a functor $\phi_{\mathscr{A}}: Sc \to \mathscr{A}$. In other words, to each operation ω of arity c and to each diagram ϕ in $\mathscr{A}$ of type c is assigned, as the *value* of ω at ϕ, an object or a morphism $\phi_{\mathscr{A}}(\omega)$ of $\mathscr{A}$, according as ω is an object or a morphism of Sc; and this in a functorial way.

We can now build up the category Rc of *derived* c-ary operations. Setting $R_0 c = c$, we inductively define $R_n c$ for $n \in \mathbf{N}$ by

$$R_{n+1}c = c + \sum_d \mathbf{Cat}(d, R_n c) \times Sd, \tag{4}$$

wherein the set of functors $\mathbf{Cat}(d, R_n c)$ is viewed as a discrete category, and the coproduct $\sum_d$ is taken over all finitely presentable categories d. (It is of course intended that this set of arities d contain only one representative of each isomorphism class.) The evident canonical maps $R_n c \to R_{n+1} c$ turn out to be inclusions; and we set Rc equal to the colimit—in fact the union—of the sequence $R_0 c \to R_1 c \to R_2 c \to \cdots$. The *value* $\phi_{\mathscr{A}}(\tau)$, in a model $\mathscr{A}$ of S, of a derived c-ary operation τ at $\phi: c \to \mathscr{A}$ is defined inductively. When τ belongs to the summand c of (4), we have $\phi_{\mathscr{A}}(\tau) = \phi(\tau)$; these are the "identity operations." When $\tau = (\psi, \omega)$ where $\psi: d \to R_n c$ and ω is an object or a morphism in Sd, we have $\phi_{\mathscr{A}}(\tau) = (\phi_{\mathscr{A}}\psi)_{\mathscr{A}}(\omega)$, where $\phi_{\mathscr{A}}\psi$ here is the composite of $\psi: d \to R_n c$ and $\phi_{\mathscr{A}}: R_n c \to \mathscr{A}$. The reader will have no trouble in thinking out what this means in elementary terms, and in checking that what we asserted above to be derived operations really are such: the values of n which occur in the examples are very small.

The *equations* E of a theory (S, E) consist of a function which assigns, to each arity c, a set Ec of pairs (τ, τ') of morphisms in Rc; in practice Ec will often be empty except for a few values of c. A *model* of (S, E) is a model $\mathscr{A}$ of S such that, for every c and every $(\tau, \tau') \in Ec$, we have $\phi_{\mathscr{A}}(\tau) = \phi_{\mathscr{A}}(\tau')$ for every $\phi: c \to \mathscr{A}$.

These models are the objects of a category (S, E)-Alg. A morphism $\mathscr{A} \to \mathscr{B}$ in this category is a functor $F: \mathscr{A} \to \mathscr{B}$ such that $F\phi_{\mathscr{A}} = (F\phi)_{\mathscr{B}}$: $Sc \to \mathscr{B}$ for each c and each $\phi: c \to \mathscr{A}$. In other words, F preserves the basic operations on the nose.

The conclusion of the analysis in [6] is that (S, E)-Alg $= T$-Alg for a certain finitary monad $T = (T, \eta, \mu)$ on $\mathbf{Cat}$; and that every finitary monad on $\mathbf{Cat}$ admits such a presentation (S, E).

The explicit presentations we have given above are not exactly in the above form, as they stand; but they can easily be brought into this form. It suffices to illustrate by the presentation in Section 3 of a category with a terminal object.

The basic operations of arity 0 form the one-object category 1, realized in a model $\mathscr{A}$ by the object 1. The basic operations of arity 1 form the category $t: P \to Q$ isomorphic to $\mathbf{2}$, and are realized in a model $\mathscr{A}$ by maps $tA: PA \to QA$ for each $A \in \mathscr{A}$. The basic operations of arity $\mathbf{2}$ (corresponding to our "equation") form the category

$$\begin{tikzcd} L \arrow[d, "a"'] \arrow[dr, "c"] & \\ M \arrow[r, "b"'] & N \end{tikzcd}$$

isomorphic to **3**, and are realized in a model $\mathscr{A}$ by a commutative diagram

$$\begin{tikzcd} Lx \arrow[d, "ax"'] \arrow[dr, "cx"] & \\ Mx \arrow[r, "bx"'] & Nx \end{tikzcd}$$

for each map $x: A \to B$ in $\mathscr{A}$. There are no basic operations of other arities. The equations, in the formal sense above, between derived operations are: $PA = A$, $QA = 1$, $ax = x$, $bx = t(\operatorname{cod} x)$, $cx = t(\operatorname{dom} x)$—which further imply $Lx = \operatorname{dom} x$, $Mx = \operatorname{cod} x$, $Nx = 1$. Equations between maps in fact suffice; we can replace $PA = A$ by $1_{PA} = 1_A$.

It may seem unnatural to consider morphisms of algebras which preserve the structure on the nose. Thus, in the case of topoi seen as (S, E)-algebras, we have *chosen* limits, *chosen* colimits, a *chosen* internal-hom, and a *chosen* subobject classifier; and the morphisms are to preserve these strictly. Yet in real life a logical morphism preserves such things only to within isomorphism.

This apparent unnaturality is in fact of no importance. Every monad on **Cat** that we have considered above in fact admits enrichment to a $\mathscr{V}$-monad, where $\mathscr{V}$ is the cartesian closed category **Gpd** of small groupoids: see [6]. For such an enriched monad we have the notion, as in [8], of a morphism "preserving the structure to within coherent isomorphisms"; which gives exactly the morphisms that occur in nature. Moreover, general results about categories of algebras with these "natural" morphisms will be proved in [7].

The whole analysis above can be repeated with **Cat** replaced by **Gph**, and with finitely-presentable categories replaced by finite graphs as the arities; except that the set Ec of equations in Rc must now explicitly allow pairs (σ, σ') of *objects* in Rc as well as pairs (τ, τ') of arrows. Of course we cannot now speak of enrichability over **Gpd**, since **Gph** is not a **Gpd**-category. For this reason we have put the emphasis in the examples on the case of **Cat**, although establishing wherever possible the stronger monadicity over graphs.

For the purpose of the latter, we recall that Burroni [1] gives a presentation of categories as algebraic relative to graphs, in which composition of morphisms and identity morphisms now appear as basic operations. It easily follows that any presentation (S, E) of an extra structure as algebraic relative to categories gives rise to a presentation of it as algebraic relative to graphs, *provided* that Sc and Ec are empty except when c is the free category Fg on a finite graph g. We have only to present Sc by generators and relations (which is how it is usually given, in any case), and observe that, since the coproduct in (4) is now in effect taken over finite graphs, the notion of derived operation is not essentially changed in passing to the presentation over graphs.

REFERENCES

1. A. BURRONI, Algèbres graphiques, *Cahiers Topologie Géom. Différentielle* **22** (1981), 249–265.
2. P. J. FREYD, Aspects of topoi, *Bull. Austral. Math. Soc.* **7** (1972), 1–76, 467–480.
3. G. M. KELLY, Syntax and semantics for categories with structure, *Math. Forschungsinstitut Oberwolfach*, Tagungsbericht 29/1979.
4. G. M. KELLY, Examples of non-monadic structures on categories, *J. Pure Appl. Algebra* **18** (1980), 59–66.
5. G. M. KELLY, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bull. Austral. Math. Soc.* **22** (1980), 1–83.
6. G. M. KELLY, Essentially algebraic structures in the enriched context, II, in preparation.
7. G. M. KELLY, Categories with structure—Biadjoints for algebraic functors, in preparation.
8. G. M. KELLY AND R. STREET, Review of the elements of 2-categories, *in* "Category Seminar," pp. 75–103, Lecture Notes in Mathematics No. 420, Springer-Verlag, New York/Berlin, 1974.
9. J. PENON, Sur les quasi-topos, *Cahiers Topologie Géom. Différentielle* **18** (1977), 181–218.