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Impulsive control for stability of n-species Lotka–Volterra cooperation models with finite delays

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ABSTRACT

In this paper, the asymptotic behavior of some n-species Lotka–Volterra cooperation systems with finite delays and impulsive perturbations at fixed moments of time is studied. By using the Lyapunov–Razumikhin method sufficient conditions for uniform asymptotic stability of the solutions are obtained. We shall show, also, that the role of impulses in controlling the behavior of solutions of impulsive differential equations is very important.

1. Introduction

Owing to their theoretical and practical significance, the Lotka–Volterra systems with time delays have been studied extensively. See, for example, [1–5] and the references cited therein. In addition to these, the books of Gopalsamy [6] and Kuang [7] are good sources for these topics of Lotka–Volterra type systems.

Wei and Wang [5] are investigated the asymptotic behavior of the periodic solutions of the following Lotka–Volterra cooperation system with finite delays

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \frac{x_i(t-\tau_{ii}(t))}{a_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{n} b_j(t)x_j(t-\tau_{ij}(t))} - c_i(t)x_i(t)\right], \tag{1.1}$$

where $i, j = 1, \ldots, n$; $t \geq 0$; $x_i(t)$ denotes the density of species i at the moment t; $r_i(t),\ a_i(t),\ b_i(t),\ c_i(t)\ (i = 1, 2, \ldots, n)$ are the system parameters; $0 \leq \tau_{ij} \leq \tau$, $\tau = const$.

However, in the study of the dynamic relationship between species, the effect of some impulsive factors has been ignored, which exists widely in the real world. For example, the birth of many species is an annual birth pulse or harvesting. Moreover, the human beings have been harvesting or stocking species at some time, then the species is affected by another impulsive type. Such factors have a great impact on the population growth. If we incorporate these impulsive factors into the model

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of population interaction, the model must be governed by impulsive functional differential system

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left[r_i(t) - \dfrac{x_i(t-\tau_{ii}(t))}{a_i(t) + \sum\limits_{\substack{j=1\\ j\neq i}}^{n} b_j(t)x_j(t-\tau_{ij}(t))} - c_i(t)x_i(t)\right], \quad t \neq t_k, \\ x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)), \quad i = 1, \dots, n, \ k = 1, 2, \dots, \end{cases} \tag{1.2}$$

where $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. The numbers $x_i(t_k)$ and $x_i(t_k^+)$ are, respectively, the population densities of species i before and after impulse perturbation at the moment t_k; and I_{ik} are functions which characterize the magnitude of the impulse effect on the species i at the moments t_k.

The main purpose of this paper is to investigate the role of impulses in control of asymptotic behavior of system (1.2). By means of piecewise continuous Lyapunov functions [8] and Razumikhin technique [9,10] sufficient conditions for uniform asymptotic stability of a nonzero solution are obtained. An example is considered to illustrate our results. The example also shows that by means of appropriate impulsive perturbations we can control the system's population dynamics [11,1].

2. Preliminaries

Let $R_+ = [0, \infty)$, R^n be the n-dimensional Euclidean space and $\|x\| = |x_1| + \cdots + |x_2|$ denote the norm of $x \in R^n$. Let $J \subset R$ be an interval. Define the following class of functions:

$$CB[J, R] = \{\sigma \in C[J, R] : \sigma(t) \text{ is bounded on } J\}.$$

Let $\varphi \in CB[[-\tau, 0], R^n]$, $\varphi = col(\varphi_1, \ \varphi_2, \ \dots, \ \varphi_n)$. We denote by $x(t) = x(t; 0, \varphi) = col(x_1(t; 0, \varphi), x_2(t; 0, \varphi), \dots, x_n(t; 0, \varphi))$ the solution of system (1.2), satisfying the initial conditions

$$\begin{cases} x_i(s; 0, \varphi) = \varphi_i(s), \quad s \in [-\tau, 0], \\ x_i(0^+; 0, \varphi) = \varphi_i(0), \quad i = 1, \dots, n, \end{cases} \tag{2.1}$$

and by $J^+(0, \varphi)$ — the maximal interval of type $[0, \beta)$ in which the solution $x(t; 0, \varphi)$ is defined.

Let $\|\varphi\|_\tau = \max_{s\in[-\tau,0]} \|\varphi(s)\|$ be the norm of the function $\varphi \in CB[[-\tau, 0], R^n]$.

Introduce the following conditions:

H2.1. The functions $r_i(t)$, $a_i(t)$, $b_i(t)$ and $c_i(t)$ are continuous, positive and bounded on R_+.
H2.2. $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.
H2.3. $I_{ik} \in C[R_+, R]$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$.
H2.4. $x_i + I_{ik}(x_i) \geq 0$ for $x_i \in R_+$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$.

Given a continuous function $g(t)$ which is defined on J, $J \subseteq R$, we set

$$g^L = \inf_{t\in J} g(t), \qquad g^M = \sup_{t\in J} g(t).$$

Introduce the following notations:

$$G_k = (t_{k-1}, t_k) \times R_+^n, k = 1, 2, \dots; \quad G = \cup_{k=1}^{\infty} G_k;$$

$V_0 = \{V : [0, \infty) \times R_+^n \to R_+ : V \in C[G, R_+],\ t \in [0, \infty), V$ is locally Lipschitzian in $x \in R_+^n$ on each of the sets G_k, $V(t_k^-, x) = V(t_k, x)$ and $V(t_k^+, x) = \lim_{\substack{t\to t_k\\ t>t_k}} V(t, x)$ exists$\}$.

For $V \in V_0$ and for any $(t, x) \in [t_{k-1}, t_k) \times R_+^n$, the right-hand derivative of the function $V \in V_0$ with respect to system (1.2) is defined by

$$D^+_{(1.2)} V(t, x(t)) = \lim_{h\to 0^+} \sup \frac{1}{h}\Big[V(t+h, x(t+h)) - V(t, x(t))\Big].$$

In the proofs of the main theorem, we shall use the following lemmas.

Lemma 2.1. *Let the conditions* H2.1–H2.3 *hold.*
Then $J^+(0, \varphi) = [0, \infty)$.

Proof. Since the condition H2.1 holds, then from the existence theorem for the corresponding system without impulses [6,7,5], it follows that the solution $x(t) = x(t; 0, \varphi)$ of problem (1.2), (2.1) is defined on $[0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \dots$. From conditions H2.2 and H2.3, we conclude that it is continuable for $t \geq 0$. □

Lemma 2.2. *Assume that:*

1. *Conditions* H2.1–H2.4 *hold.*
2. $x(t) = x(t;0,\varphi) = col(x_1(t;0,\varphi), x_2(t;0,\varphi), \ldots, x_n(t;0,\varphi))$ *is a solution of* (1.2), (2.1) *such that*

$$x_i(s) = \varphi_i(s) \geq 0, \qquad \sup \varphi_i(s) < \infty, \quad \varphi_i(0) > 0,$$

$1 \leq i \leq n$.

Then $x_i(t) > 0$, $1 \leq i \leq n$, $t \in [0, \infty)$.

Lemma 2.3. *Assume that:*

1. *Conditions of* Lemma 2.2 *hold.*
2. *The functions* I_{ik} *are such that*

$$-x_i \leq I_{ik}(x_i) \leq 0 \quad \text{for } x_i \in R_+,\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots.$$

Then there exist positive constants m *and* $M < \infty$ *such that*

$$m \leq x_i(t) \leq M, \quad t \in [0, \infty). \tag{2.2}$$

The proofs of Lemmas 2.2 and 2.3 are similar to the proofs of Assertion 1 and Assertion 2 of Lemma 3.1 in [1] and we will omit them here.

3. Uniform asymptotic stability

Let $\phi \in CB[[-\tau, 0], R^n_+]$, $\phi = col(\phi_1, \phi_2, \ldots, \phi_n)$ and $x^*(t) = x^*(t;0,\phi) = col(x^*_1(t;0,\phi), x^*_2(t;0,\phi), \ldots, x^*_n(t;0,\phi))$ be a solution of system (1.2), satisfying the initial conditions

$$\begin{cases} x^*_i(s;0,\phi) = \phi_i(s), & s \in [-\tau, 0], \\ x^*_i(0^+;0,\phi) = \phi_i(0), & i = 1, 2, \ldots, n. \end{cases}$$

In the next, we shall suppose that

$$\begin{aligned} &\varphi_i(s) \geq 0, \quad \sup \varphi_i(s) < \infty, \qquad \varphi_i(0) > 0, \\ &\phi_i(s) \geq 0, \quad \sup \phi_i(s) < \infty, \qquad \phi_i(0) > 0,\ i = 1, 2, \ldots, n. \end{aligned}$$

Theorem 3.1. *Assume that:*

1. *Conditions of* Lemma 2.3 *hold.*
2. $m \leq x_i + I_{ik}(x_i) \leq M$ *for* $m \leq x_i \leq M$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$.
3. *There exists a nonnegative constant* μ *such that*

$$m \min_{1 \leq i \leq n} c^L_i \geq \mu + M^2 \sum_{i=1}^{n} \max_{j \neq i} \frac{b^M_j}{\left(a^L_i + m \sum\limits_{\substack{s=1 \\ s \neq i}}^{n} b^L_s\right)^2} > 0.$$

Then the solution $x^*(t)$ *of system* (1.2) *is uniformly asymptotically stable.*

Proof. Consider the Lyapunov function

$$V(t, x(t)) = \sum_{i=1}^{n} \left| \ln \frac{x_i(t)}{x^*_i(t)} \right|.$$

By Mean Value Theorem and by (2.2), it follows that for any closed interval contained in $[0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$ and for all $i = 1, 2, \ldots$

$$\frac{1}{M}|x_i(t) - x^*_i(t)| \leq |\ln x_i(t) - \ln x^*_i(t)| \leq \frac{1}{m}|x_i(t) - x^*_i(t)|. \tag{3.1}$$

From the inequalities (3.1), we obtain

$$\begin{aligned} V(0^+, x(0^+)) &= \sum_{i=1}^{n} |\ln x_i(0^+) - \ln x^*_i(0^+)| \\ &\leq \frac{1}{m}|\varphi_i(0) - \phi_i(0)| \leq \frac{1}{m}\|\varphi - \phi\|_\tau. \end{aligned} \tag{3.2}$$

For $t > 0$ and $t = t_k, k = 1, 2, \ldots$, we have

$$\begin{aligned} V(t_k^+, x(t_k^+)) &= \sum_{i=1}^n \left| \ln \frac{x_i(t_k^+)}{x_i^*(t_k^+)} \right| \\ &= \sum_{i=1}^n \left| \ln \frac{x_i(t_k) + I_{ik}(x_i(t_k))}{x_i^*(t_k) + I_{ik}(x_i^*(t_k))} \right| \le \sum_{i=1}^n \left| \ln \frac{M}{m} \right| = \sum_{i=1}^n \left| \ln \frac{m}{M} \right| \\ &\le \sum_{i=1}^n \left| \ln \frac{x_i(t_k)}{x_i^*(t_k)} \right| = V(t_k, x(t_k)). \end{aligned} \tag{3.3}$$

For $t \ge 0$ and $t \ne t_k, k = 1, 2, \ldots$, we have

$$\begin{aligned} D^+_{(1.2)} V(t, x(t)) &= \sum_{i=1}^n \left(\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_i^*(t)}{x_i^*(t)} \right) \operatorname{sgn}\left(x_i(t) - x_i^*(t)\right) \\ &\le \sum_{i=1}^n \left\{ - c_i(t)|x_i(t) - x_i^*(t)| - \frac{1}{a_i(t) + \sum\limits_{\substack{j=1 \\ j\ne i}}^n b_j(t) x_j^*(t - \tau_{ij}(t))} |x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t))| \right. \\ &\quad \left. + \sum_{\substack{j=1 \\ j\ne i}}^n \frac{b_j(t) x_i(t - \tau_{ii}(t)) |x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t))|}{\left(a_i(t) + \sum\limits_{\substack{s=1 \\ s\ne i}}^n b_s(t) x_s(t - \tau_{is}(t))\right)\left(a_i(t) + \sum\limits_{\substack{s=1 \\ s\ne i}}^n b_s(t) x_s^*(t - \tau_{is}(t))\right)} \right\} \\ &\le \sum_{i=1}^n \left\{ -c_i^L |x_i(t) - x_i^*(t)| + \sum_{\substack{j=1 \\ j\ne i}}^n \frac{M b_j^M}{\left(a_i^L + m \sum\limits_{\substack{s=1 \\ s\ne i}}^n b_s^L\right)^2} |x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t))| \right\} \\ &\le -\min_{1\le i\le n} c_i^L \sum_{i=1}^n |x_i(t) - x_i^*(t)| + \left(\sum_{i=1}^n \max_{j\ne i} \frac{M b_j^M}{\left(a_i^L + m \sum\limits_{\substack{s=1 \\ s\ne i}}^n b_s^L\right)^2} \right) \sum_{i=1}^n \sup_{s\in[t-\tau, t]} |x_i(s) - x_i^*(s)|. \end{aligned}$$

From (3.1) for any solution $x(t)$ of (1.2) such that

$$V(s, x(s)) \le V(t, x(t)), \quad t - \tau \le s \le t,\ t \ne t_k,\ k = 1, 2, \ldots,$$

we have

$$\sum_{i=1}^n |x_i(s) - x_i^*(s)| \le \frac{M}{m} \sum_{i=1}^n |x_i(t) - x_i^*(t)|.$$

Then

$$D^+ V_{(1.2)}(t, x(t)) \le -\frac{\mu}{m} \sum_{i=1}^n |x_i(t) - x_i^*(t)| \le -\mu V(t, x(t)),$$

$t \ge 0$ and $t \ne t_k, k = 1, 2, \ldots$.

From the last estimate, (3.2) and (3.3), we get

$$V(t, x(t)) \le V(0^+, x(0^+)) e^{-\mu t}, \quad t \in [0, \infty).$$

So,

$$\begin{aligned}\|x(t)-x^*(t)\| &= \sum_{i=1}^{n}\left|x_i(t)-x_i^*(t)\right| \le MV(t,x(t))\\ &\le MV(0^+,x(0^+))e^{-\mu t} \le \frac{M}{m}\|\varphi-\phi\|_\tau e^{-\mu t}, \quad t\in[0,\infty),\end{aligned}$$

and this completes the proof of the theorem. □

4. An example

The system

$$\begin{cases}\dot{x}(t) = x(t)\left[\dfrac{907}{224} - \dfrac{x(t-\tau_{11})}{1+4y(t-\tau_{12})} - 16x(t)\right],\\ \dot{y}(t) = y(t)\left[15 - \dfrac{y(t-\tau_{22})}{1+2x(t-\tau_{21})} - 14y(t)\right],\end{cases} \tag{4.1}$$

with parameters $r_1 = \frac{907}{224}$, $r_2 = 15$, $a_1 = a_2 = 1$, $b_1 = 2$, $b_2 = 4$, $c_1 = 16$ and $c_2 = 14$ has a uniformly asymptotically stable [5] equilibrium point $(x^*, y^*) = (0, 1)$ which implies the first species will go extinct.

However, for the impulsive Lotka–Volterra system

$$\begin{cases}\dot{x}(t) = x(t)\left[\dfrac{907}{224} - \dfrac{x(t-\tau_{11})}{1+4y(t-\tau_{12})} - 16x(t)\right], \quad t\ne t_k,\\ \dot{y}(t) = y(t)\left[15 - \dfrac{y(t-\tau_{22})}{1+2x(t-\tau_{21})} - 14y(t)\right], \quad t\ne t_k,\\ \Delta x(t_k) = -\dfrac{1}{4}\left(x(t_k) - \dfrac{1}{4}\right), \quad k=1,2,\ldots,\\ \Delta y(t_k) = -\dfrac{11}{15}\left(y(t_k) - \dfrac{45}{44}\right), \quad k=1,2,\ldots,\end{cases}$$

where $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$, the point $(x^*, y^*) = (\frac{1}{4}, \frac{45}{44})$ is an equilibrium which is uniformly asymptotically stable. In fact, all conditions of Theorem 3.1 are satisfied for $\mu = 1.524$, $m = \frac{1}{4}$ and $M = \frac{45}{44}$ and

$$\begin{aligned}\frac{1}{4} &\le x(t_k) + I_{1k}(x(t_k)) = \frac{12x(t_k)+1}{16} \le \frac{45}{44},\\ \frac{1}{4} &\le y(t_k) + I_{2k}(y(t_k)) = \frac{4y(t_k)}{15} + \frac{3}{4} \le \frac{45}{44}\end{aligned}$$

for $\frac{1}{4} \le x(t_k) \le \frac{45}{44}$, $\frac{1}{4} \le y(t_k) \le \frac{45}{44}$, $k = 1, 2, \ldots$.

This example shows that the impulsive perturbations can prevent the population from going extinct. In short, by impulsive controls of the population numbers of the first and the second species at fixed moments, such as stocking and harvesting, we can control the system's population dynamics.

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