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Note

Laplacian spectra and invariants of graphs

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Abstract

For a connected graph G of order $n \geqslant 2$ with positive Laplacian eigenvalues $\lambda_2, \ldots, \lambda_n$, let

$$b(G) = \frac{n-1}{1/\lambda_2 + \cdots + 1/\lambda_n}.$$

In this note we derive bounds on some graph invariants (edge-density in cuts, isoperimetric number, mean distance, edge-forwarding index, edge connectivity, etc) in terms of $b(G)$.

Keywords: Graph spectrum; Graph invariants; Graph Laplacian

1. Introduction

For any graph G with no loops, and no multiple edges, let $V = V(G)$ and $E = E(G)$ be the vertex set and the edge set of G, respectively. Let $A(G)$ be the adjacency matrix of G, and $D(G) = \mathrm{diag}(\deg(v))_{v \in V(G)}$ the degree matrix of G. Then the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G. The *Laplacian eigenvalues* of a graph are defined to be the eigenvalues of its Laplacian matrix. In recent years, the relationships between a graph and its Laplacian eigenvalues have been investigated fruitfully. The Laplacian matrix $L(G)$ is positive semidefinite.

Throughout this note, G denotes a non-trivial graph. For a connected graph G of order n, let

$$0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$$

be its Laplacian eigenvalues of G (repeated according to their multiplicities).

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The second smallest eigenvalue $a(G)=\lambda_2$ is called the *algebraic connectivity* of G. It is well-known that $a(G)$ is related to several important graph invariants. For a connected graph of order n, we define

$$b(G)=\frac{n-1}{1/\lambda_2+\cdots+1/\lambda_n}$$

In other words, $b(G)$ is the harmonic mean of the $n-1$ positive Laplacian eigenvalues.

In the present note, we shall show that $b(G)$ is also related to some important graph invariants (edge density in cuts, isoperimetric number, edge-forwarding index, etc).

2. The fundamental inequality

We shall use the symbol $(T_1,\ldots,T_k)$ to denote the spanning forest of a graph G with connected components $T_1,\ldots,T_k$, and the set of spanning forests of G with k connected components will be denoted by $\mathrm{For}_G(k)$. For a partition $(X_1,\ldots,X_k)$ of $V(G)$, we denote by $\delta(X_1,\ldots,X_k)$ the set of edges whose end vertices are in distinct partition classes. In this section we begin with the following:

Lemma 2.1. *Let G be a connected graph of order n. then*

$$\sum|\delta(V(T_1),\ldots,V(T_{k+1}))|=(n-k)|\mathrm{For}_G(k)|,$$

where the sum is taken over all spanning forests $(T_1,\ldots,T_{k+1})$ with $k+1$ connected components.

Proof. Take a forest $(T_1,\ldots,T_{k+1})\in\mathrm{For}_G(k+1)$, and add an edge in $\delta(V(T_1),\ldots,V(T_{k+1}))$ to the forest $(T_1,\ldots,T_{k+1})$. Then we obtain a forest in $\mathrm{For}_G(k)$. Conversely from a forest $S=(S_1,\ldots,S_k)$ in $\mathrm{For}_G(k)$ remove one edge in S. Then we obtain a forest in $\mathrm{For}_G(k+1)$. Since S contains $n-k$ edges, $\sum|\delta(V(T_1),\ldots,V(T_{k+1}))|$ is equal to $(n-k)|\mathrm{For}_G(k)|$. $\square$

Lemma 2.2 (Biggs [3, Theorem 7.5]). *Let G be a connected graph of order n with positive Laplacian eigenvalues $\lambda_2,\ldots,\lambda_n$. Then*

$$\sum|V(T_1)|\cdots|V(T_k)|=e_{n-k}(\lambda_2,\ldots,\lambda_n),\quad k=1,2,\ldots,n-1,$$

where the sum is taken over all spanning forests with k connected components, and $e_{n-k}(\lambda_2,\ldots,\lambda_n)$ denotes the elementary symmetric polynomial of degree $n-k$ in $\lambda_2,\ldots,\lambda_n$.

Theorem 2.3. *Let G be a connected graph of order n. Then we have*

$$\min\frac{|\delta(X_1,\ldots,X_{k+1})|}{|X_1|\cdots|X_{k+1}|}\leqslant\frac{(n-k)|\mathrm{For}(k)|}{e_{n-1-k}(\lambda_2,\ldots,\lambda_n)}\leqslant\max\frac{|\delta(X_1,\ldots,X_{k+1})|}{|X_1|\cdots|X_{k+1}|},$$

where the minimum and the maximum are taken over all vertex partitions $(X_1, \ldots, X_{k+1})$ *with* $k+1$ *classes.*

Proof. Let

$$c_k = \min \frac{|\delta(X_1, \ldots, X_{k+1})|}{|X_1| \cdots |X_{k+1}|}.$$

Then we have

$$c_k \sum |V(T_1)| \cdots |V(T_{k+1})| \leqslant \sum |\delta(T_1, \ldots, T_{k+1})|,$$

where the sum is taken over all spanning forests with $k+1$ connected components. From Lemmas 2.1 and 2.2, we then obtain the result. By similar arguments as above, we obtain the second inequality. □

For a subset $X \subset V(G)$, let X^c denote the complement of X in $V(G)$. For a proper subset X of $V(G)$, the quantity

$$\frac{|\delta(X, X^c)|}{|X||X^c|}$$

is called the *edge-density* of the edge cut $\delta(X, X^c)$.

It is known [2,6] that the edge-density of any edge-cut is between λ_2/n and λ_n/n. The following result gives an upper bound on the minimal edge-density in terms of $b(G)$:

Proposition 2.4. *Let G be a connected graph of order n. Then*

$$\min \left\{ \frac{|\delta(X, X^c)|}{|X||X^c|} \right\} \leqslant \frac{b(G)}{n},$$

where the minimum is taken over all proper subsets of $V(G)$.

Proof. By the matrix-tree theorem [3, Corollary 6.5],

$$|\mathrm{For}_G(1)| = \frac{\lambda_2 \cdots \lambda_n}{n}$$

and hence the result follows from Theorem 2.3. □

Although simple to prove, Proposition 2.4 is fundamental in this note for obtaining bounds on graph invariants.

Proposition 2.5. *Let G be a connected graph of order n with edge connectivity* $\eta(G)$. *Then*

$$\eta(G) \leqslant \frac{nb(G)}{4}.$$

Proof. From the proof of Theorem 2.3, we see that there exists a spanning forest (T_1, T_2) such that

$$\frac{|\delta(V(T_1), V(T_2))|}{|V(T_1)||V(G_2)|} \leqslant \frac{b(G)}{n}.$$

Since $\eta(G) \leqslant |\delta(V(T_1), V(T_2))|$ and $|V(T_1)||V(T_2)| \leqslant n^2/4$, the result follows. $\square$

3. Graph invariants and $b(G)$

The *isoperimetric number* $i(G)$ of a graph G of order n is defined as

$$i(G) = \min\left\{\frac{|\delta(X, X^c)|}{|X|};\ X \subset V(G),\ 0 < |X| \leqslant \frac{n}{2}\right\}.$$

Discrete versions of the Cheeger inequality are known [1,6]. As a straightforward application of Proposition 2.4 we obtain the following upper bound on $i(G)$.

Theorem 3.1. *Let G be a connected graph of order n. Then*

$$i(G) \leqslant \frac{n-1}{n} b(G).$$

Let u and v be vertices of a connected graph G. The *distance* $d_G(u, v)$ between u and v is the length of the shortest path between u and v in G.

The *mean distance* $\rho(G)$ of G is equal to the average of all distances between distinct vertices of G.

$$\rho(G) := \frac{\sum_{(u,v) \in \binom{V(G)}{2}} d(u, v)}{\binom{|V(G)|}{2}}.$$

If G is a tree, by a theorem of B.D. McKay, the mean distance $\rho(G)$ is equal to $2/b(G)$ (see [7, Theorem 4.3] for a proof). In [7], some bounds on $\rho(G)$ are derived.

Lemma 3.2. *Let G be a graph of order n. Then*

$$\sum \rho(T) = \frac{\sum |\delta(V(T_1), V(T_2))||V(T_1)||V(T_2)|}{\binom{n}{2}},$$

where the first sum is taken over all spanning trees T of G, while the second sum is taken over all spanning forests (T_1, T_2) of G with two connected components.

Proof. For two vertices u and v of G, take a spanning tree T, and delete one edge from the u–v path in T. Then we get a spanning forest (T_1, T_2) of G, and u and v are contained in distinct partition classes of the vertex partition $(V(T_1), V(T_2))$. For given vertices u and v, and a spanning tree T, there are $d_T(u,v)$ such edges. Conversely, for a spanning forest (T_1,T_2) of G, and vertices u and v contained in distinct partition classes of $(V(T_1),V(T_2))$, add one edge in $\delta(V(T_1),V(T_2))$. Then we obtain a spanning tree of G. For a given spanning forest (T_1,T_2) we can choose $|V(T_1)||V(T_2)|$ such vertices u and v. By the double counting argument, we then obtain the result. □

The following theorem gives an upper bound for the mean distance of a connected graph in terms of $b(G)$.

Theorem 3.3. *Let G be a connected graph of order n with m edges. Then*

$$\rho(G) \leqslant \frac{2(m-n+2)}{b(G)},$$

with equality if and only if G is a tree.

Proof. For a spanning forest (T_1,T_2) of G, since each T_i, $i=1,2$, contains $|V(T_i)|-1$ edges, G has at least $|V(T_1)|+|V(T_2)|-2+|\delta(V(T_1),V(T_2))|$ edges, and so we have

$$|\delta(V(T_1),V(T_2))| \leqslant m-n+2.$$

Then by Lemma 2.2,

$$\sum |\delta(V(T_1),V(T_2))||V(T_1)||V(T_2)| \leqslant (m-n+2)e_{n-2}(\lambda_2,\dots,\lambda_n),$$

where the sum is taken over all spanning forests (T_1,T_2) with two connected components.

On other hand, since $\rho(T) \geqslant \rho(G)$, we have

$$\sum_{T\in \mathrm{For}_G(1)} \rho(T) \geqslant \rho(G)|\mathrm{For}_G(1)|.$$

Therefore, the inequality follows from Lemma 3.2 and the matrix-tree theorem. If G is a tree, then $m-n+2=1$, and the equality holds by the theorem of McKay, mentioned above. Conversely if equality holds, then from the proof, we see that $\rho(G)=\rho(T)$ for each spanning tree T. This implies that G itself is a tree. □

In a connected graph G, we denote by $D_G(u,v)$ the length of the longest path between u and v. Let

$$\Xi(G) := \frac{\sum_{(u,v)\in\binom{V(G)}{2}} D_G(u,v)}{\binom{|V(G)|}{2}}.$$

Proposition 3.4. *Let G be a connected graph with edge connectivity $\eta(G)$. Then*

$$\Xi(G) \geqslant \frac{2\eta(G)}{b(G)}.$$

Proof. For each spanning forest (T_1,T_2) of G, $\eta(G)\leqslant|\delta(V(T_1),V(T_2))|$. Then by similar arguments as in the proof of Theorem 3.3, the result follows. □

From Proposition 3.4, we obtain information about Laplacian spectra of graphs without long paths.

Corollary 1. *Let G be a connected graph without a path of length $k(k\geqslant 1)$. Then*

$$b(G)\geqslant\frac{2\eta(G)}{k}.$$

Proof. This is a direct consequence of Proposition 3.4. □

The edge-forwarding index of a graph is a useful notion used in the theory of communication networks. A *routing* R is a collection of paths $R(u,v)$, specified for each ordered pair (u,v) of distinct vertices of G. For each edge e, let $\pi(G,R,e)$ be the number of paths in R going through the edge e. Let

$$\pi(G,R)=\max\{\pi(G,R,e)\},$$

where the maximum is taken over all edges e of G. Then the *edge-forwarding index* $\pi(G)$ of G is defined as the minimum of $\pi(G,R)$ taken over all routings R of G.

Proposition 3.5. *Let G be a connected graph of order n. Then*

$$\pi(G)\geqslant\frac{2n}{b(G)}.$$

Proof. For every proper subset S of $V(G)$, it is known [8, Proposition 3.7] that

$$\pi(G)\geqslant\frac{2|S||S^c|}{|\delta(S,S^c)|}.$$

Then the result follows from Proposition 2.4.

Definition 3.6. A connected graph G is said to be *orbital regular* if, for some subgroup H of the automorphism group $\mathrm{Aut}(G)$, H satisfies the following two conditions:

(1) H acts regularly on each of its orbits in $\{(u,v);\ u,v\in V(G),\ u\neq v\}$.
(2) $E(G)=\{\{u,v\}\in E(G)\colon\ (u,v)\in S\}$ for an H-*orbit* S.

Proposition 3.7. *Let G be an orbital regular graph of order n and degree k. Then*

$$\rho(G)\geqslant\frac{nk}{(n-1)b(G)}.$$

Proof. In [9] it is shown that, if G is an orbital regular graph of order n with m edges,

$$\pi(G) = \frac{1}{m}\binom{n}{2}\rho(G).$$

From Proposition 3.5, we then obtain the result. □

Proposition 3.8. *Let G be a Cayley graph of order n. Then*

$$\rho(G) \geqslant \frac{n}{(n-1)b(G)}.$$

Proof. By [4, Corollary 110],

$$\pi(G) \leqslant (n-1)\rho(G).$$

Then the result follows from Proposition 3.5. □

Proposition 3.9. *Let G be a connected vertex-transitive graph of order n with diameter D. Then*

$$D \geqslant \frac{n}{2(n-1)b(G)}.$$

Proof. It is known [5, p. 83] that $i(G) \geqslant 1/2D$. Therefore the result follows from Theorem 3.1. □

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