Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 332 (2007) 123–136

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Four positive periodic solutions for the first order differential system ☆

Zhengqiu Zhang *, Hengsheng Tang

Department of Applied Mathematics, Hunan University, Changsha 410082, PR China

Received 17 January 2006

Available online 7 November 2006

Submitted by J. Mawhin

Abstract

In this paper, we establish the existence of four positive periodic solutions for the first order differential system by using the continuation theorem of coincidence degree theory. When our result is applied to a competition Lotka–Volterra population model, we obtain the existence of four positive periodic solutions for this model.

Keywords: Four positive periodic solutions; Differential system; Competition Lotka–Volterra model; The continuation theorem of coincidence degree theory

1. Introduction

In this paper, we consider the following first order differential system:

$$\begin{cases} x'(t) = x(t)F_1\big(t, x(t), y(t)\big) - h_1(t), \\ y'(t) = y(t)F_2\big(t, x(t), y(t)\big) - h_2(t), \end{cases} \tag{1.1}$$

where $h_i : R \to R^+$ are continuous ω-periodic functions, $i = 1, 2$, with $\omega > 0$, $F_i : R \times R \times R \to R$ are continuous and ω-periodic with respect to its first variable, $F_i(t, x, y)$ is monotonously decreasing in x for fixed t, y and in y for fixed x, t, $\forall x, t, y \in R$, $i = 1, 2$.

☆ The project supported by NNSF of China (10271044).

* Corresponding author.
E-mail address: z_q_zhang@sina.com.cn (Z. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.09.071

When $F_1(t,x(t),y(t)) = a_1(t) - b_1(t)x(t) - c_1(t)y(t)$, $F_2(t,x(t),y(t)) = a_2(t) - b_2(t)x(t) - c_2(t)y(t)$, system (1.1) reduces to the following well-known competition Lotka–Volterra population model with stocking:

$$\begin{cases} x(t) = x(t)\big(a_1(t) - b_1(t)x(t) - c_1(t)y(t)\big) - h_1(t), \\ y(t) = y(t)\big(a_2(t) - b_2(t)x(t) - c_2(t)y(t)\big) - h_2(t), \end{cases} \tag{1.2}$$

where $a_i(t)$, $b_i(t)$, $c_i(t)$ and $h_i(t)$ are all positive continuous ω-periodic functions. On the existence of positive periodic solutions to system (1.2), few results are found in the literature. This motivates us investigate the existence of a positive periodic solution or multiple positive periodic solutions for system (1.1) and (1.2). Recently, the powerful and effective method of coincidence degree has been applied to study the existence of periodic solutions in periodic equations or systems and a number of good results have been obtained, for example, see Zhang and Wang [1–4] and Li [5]. However, the existence results of multiple periodic solutions established by using coincidence degree theory for periodic systems or equations are very scarce. So, in this paper, our purpose is to study the existence of multiple periodic solutions to system (1.1) by applying the method of coincidence degree theory. The paper is organized as follows. In Section 2, by employing the continuation theorem of coincidence degree theory, we establish a sufficient condition for the existence of four positive periodic solutions of system (1.1). In Section 3, we illustrate our result with a competition Lotka–Volterra population model.

2. Existence of four positive periodic solutions

For the readers' convenience, we first summarize a few concepts from the book by Gaines and Mawhin.

Let X and Z be normed vector spaces. Let $L : \mathrm{Dom} \subset X \to Z$ be a linear mapping and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}\, L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by Kp. If Ω is a bounded open subset of X, the mapping N is called L-compact on Ω, if $QN(\overline{\Omega})$ is bounded and $Kp(I - Q)N : \overline{\Omega} \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof of our existence result, we need the following continuation theorem.

Theorem 2.1 *(Continuation Theorem, Gaines and Mawhin [6]). Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg\{JQNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution lying in $\mathrm{Dom}\, L \cap \overline{\Omega}$.

For the sake of convenience, we introduce some notations as follows:

$$a_1^M = \max_{t\in[0,\omega]} F_1(t,0,0), \qquad a_2^M = \max_{t\in[0,\omega]} F_2(t,0,0), \qquad \overline{f} = \frac{1}{\omega}\int_0^{\omega} f(t)\,dt,$$

$$f^l = \min_{t\in[0,\omega]} f(t), \qquad f^M = \max_{t\in[0,\omega]} f(t),$$

here f is a continuous ω-periodic function.

Theorem 2.2. *Assume that the following conditions hold*:

(i) *there exists a constant A such that when* $x \geqslant A$,

$$F_1\big(t, e^x, 0\big) \leqslant 0, \quad \forall t \in R;$$

(ii) *there exists a constant* C *such that when* $x \geqslant C$,

$$F_2\big(t, 0, e^x\big) \leqslant 0, \quad \forall t \in R;$$

(iii) *there exist two positive constants* $l_- < l_+$ *with* $l_- > h_1^l/a_1^M$, $A > \ln l_+$, *such that for* $\forall t \in R$ *and* $\ln l_- \leqslant x \leqslant \ln l_+$,

$$e^x F_1\big(t, e^x, e^C\big) \geqslant h_1(t).$$

(iv) *there exist two positive constants* $u_- < u_+$ *with* $\ln u_+ < C$, $u_- > h_2^l/a_2^M$, *such that for* $\forall t \in R$ *and* $\ln u_- \leqslant x \leqslant \ln u_+$,

$$e^x F_2\big(t, e^A, e^x\big) \geqslant h_2(t).$$

Then system (1.1) *has at least four different positive* ω*-periodic solutions.*

Proof. Since we are concerned with positive periodic solutions of system (1.1), we make the change of variables

$$x(t) = \exp\big(u_1(t)\big), \qquad y(t) = \exp\big(u_2(t)\big). \tag{2.1}$$

Then system (1.1) is rewritten as

$$\begin{cases} u_1'(t) = F_1\big(t, e^{u_1(t)}, e^{u_2(t)}\big) - h_1(t)e^{-u_1(t)}, \\ u_2'(t) = F_2\big(t, e^{u_1(t)}, e^{u_2(t)}\big) - h_2(t)e^{-u_2(t)}. \end{cases} \tag{2.2}$$

Let

$$X = Z = \big\{u = (u_1, u_2)^T \in C\big(R, R^2\big)\colon\ u(t+\omega) = u(t)\big\}$$

and define

$$\|u\| = \sum_{i=1}^{2} \max_{t\in[0,\omega]} \big|u_i(t)\big|, \quad u \in X \text{ or } Z.$$

Equipped with the above norm $\|.\|$, X and Z are Banach spaces.

Let

$$Nu = \begin{bmatrix} F_1(t, e^{u_1(t)}, e^{u_2(t)}) - h_1(t)e^{-u_1(t)} \\ F_2(t, e^{u_1(t)}, e^{u_2(t)}) - h_2(t)e^{-u_2(t)} \end{bmatrix}, \quad u \in X,$$

$Lu = u' = \frac{du(t)}{dt}$, $Pu = \frac{1}{\omega}\int_0^\omega u(t)\,dt$, $u \in X$, $Qz = \frac{1}{\omega}\int_0^\omega z(t)\,dt$, $z \in Z$. Thus it follows that $\operatorname{Ker} L = R^2$, $\operatorname{Im} L = \{z \in Z\colon \int_0^\omega z(t)\,dt = 0\}$ is closed in Z, $\dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_p(z) = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt.$$

Then

$$QNu = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega G_1(s)\,ds \\ \frac{1}{\omega}\int_0^\omega G_2(s)\,ds \end{bmatrix}$$

and

$$K_p(I-Q)Nu = \begin{bmatrix} \int_0^t G_1(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t G_1(s)\,ds\,dt + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega G_1(s)\,ds \\ \int_0^t G_2(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t G_2(s)\,ds\,dt + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega G_2(s)\,ds \end{bmatrix},$$

where

$$G_1(s) = F_1\big(s, e^{u_1(s)}, e^{u_2(s)}\big) - h_1(s)e^{-u_1(s)},$$
$$G_2(s) = F_2\big(s, e^{u_1(s)}, e^{u_2(s)}\big) - h_2(s)e^{-u_2(s)}.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. It is not difficult to show that $K_p(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela–Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to use Theorem 2.1, we have to find at least four appropriate open bounded subsets in X. Considering the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{cases} u_1'(t) = \lambda\big(F_1\big(t, e^{u_1(t)}, e^{u_2(t)}\big) - h_1(t)e^{-u_1(t)}\big), \\ u_2'(t) = \lambda\big(F_2\big(t, e^{u_1(t)}, e^{u_2(t)}\big) - h_2(t)e^{-u_2(t)}\big). \end{cases} \tag{2.3}$$

Suppose that $u \in X$ is a solution of system (2.3) for some $\lambda \in (0,1)$. Then there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i = 1,2.$$

It is clear that $u_i'(\xi_i) = 0$, $u_i'(\eta_i) = 0$, $i = 1,2$. From this and (2.3), we obtain

$$\begin{cases} F_1\big(\xi_1, e^{u_1(\xi_1)}, e^{u_2(\xi_1)}\big) = h_1(\xi_1)e^{-u_1(\xi_1)}, & (2.4) \\ F_2\big(\xi_2, e^{u_1(\xi_2)}, e^{u_2(\xi_2)}\big) = h_2(\xi_2)e^{-u_2(\xi_2)} & (2.5) \end{cases}$$

and

$$\begin{cases} F_1\big(\eta_1, e^{u_1(\eta_1)}, e^{u_2(\eta_1)}\big) = h_1(\eta_1)e^{-u_1(\eta_1)}, & (2.6) \\ F_2\big(\eta_2, e^{u_1(\eta_2)}, e^{u_2(\eta_2)}\big) = h_2(\eta_2)e^{-u_2(\eta_2)}. & (2.7) \end{cases}$$

(2.4) gives

$$F_1\big(\xi_1, e^{u_1(\xi_1)}, 0\big) > 0,$$

which, together with condition (i) in Theorem 2.2, implies that

$$u_1(\xi_1) < A. \tag{2.8}$$

(2.6) gives

$$F_1(\eta_1, 0, 0) > h_1(\eta_1)e^{-u_1(\eta_1)},$$

that is,

$$u_1(\eta_1) > \ln \frac{h_1^l}{F_1(\eta_1, 0, 0)} \geqslant \ln \frac{h_1^l}{a_1^M}. \tag{2.9}$$

(2.5) implies that

$$F_2\big(\xi_2, 0, e^{u_2(\xi_2)}\big) > 0,$$

from which, together with condition (ii), we have

$$u_2(\xi_2) < C. \tag{2.10}$$

(2.7) gives

$$F_2(\eta_2, 0, 0) > h_2(\eta_2)e^{-u_2(\eta_2)},$$

that is,

$$u_2(\eta_2) > \ln \frac{h_2^l}{F_2(\eta_2, 0, 0)} \geqslant \ln \frac{h_2^l}{a_2^M}. \tag{2.11}$$

From (2.4), we have

$$e^{u_1(\xi_1)} F_1\big(\xi_1, e^{u_1(\xi_1)}, e^{u_2(\xi_1)}\big) = h_1(\xi_1),$$

which, together with (2.10), implies that

$$e^{u_1(\xi_1)} F_1\big(\xi_1, e^{u_1(\xi_1)}, e^{C}\big) < h_1(\xi_1).$$

From this and condition (iii), we have

$$u_1(\xi_1) < \ln l_- \quad \text{or} \quad u_1(\xi_1) > \ln l_+. \tag{2.12}$$

Similarly, from (2.6), we have

$$u_1(\eta_1) > \ln l_+ \quad \text{or} \quad u_1(\eta_1) < \ln l_-. \tag{2.13}$$

From (2.5), we obtain

$$e^{u_2(\xi_2)} F_2\big(\xi_2, e^{u_1(\xi_2)}, e^{u_2(\xi_2)}\big) = h_2(\xi_2),$$

which, together with (2.8), implies that

$$e^{u_2(\xi_2)} F_2\big(\xi_2, e^{A}, e^{u_2(\xi_2)}\big) < h_2(\xi_2).$$

From this and condition (iv), it follows that

$$u_2(\xi_2) > \ln u_+ \quad \text{or} \quad u_2(\xi_2) < \ln u_-. \tag{2.14}$$

Similarly, from (2.7), we obtain

$$u_2(\eta_2) > \ln u_+ \quad \text{or} \quad u_2(\eta_2) < \ln u_-. \tag{2.15}$$

From (2.8), (2.9), (2.12) and (2.13), we obtain for $\forall t \in [0, \omega]$,

$$\ln \frac{h_1^l}{a_1^M} < u_1(t) < \ln l_- \quad \text{or} \quad \ln l_+ < u_1(t) < A. \tag{2.16}$$

From (2.10), (2.11), (2.14) and (2.15), we obtain for $\forall t \in [0, \omega]$,

$$\ln \frac{h_2^l}{a_2^M} < u_2(t) < \ln u_- \quad \text{or} \quad \ln u_+ < u_2(t) < C. \tag{2.17}$$

Obviously, $\ln l_\pm$, $\ln u_\pm$, $\ln \frac{h_1^l}{a_1^M}$, A and C are independent of λ. Now let

$$\Omega_1 = \left\{ u = (u_1, u_2)^T \in X\colon\ u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right),\ u_2(t) \in \left(\ln \frac{h_2^l}{a_2^M}, \ln u_- \right) \right\},$$

$$\Omega_2 = \left\{ u = (u_1, u_2)^T \in X\colon\ u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right),\ u_2(t) \in (\ln u_+, C) \right\},$$

$$\Omega_3 = \left\{ u = (u_1, u_2)^T \in X\colon\ u_1(t) \in (\ln l_+, A),\ u_2(t) \in \left(\ln \frac{h_2^l}{a_2^M}, \ln u_- \right) \right\},$$

$$\Omega_4 = \left\{ u = (u_1, u_2)^T \in X\colon\ u_1(t) \in (\ln l_+, A),\ u_2(t) \in (\ln u_+, C) \right\}.$$

Then Ω_i $(i = 1, 2, 3, 4)$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3, 4$. Thus Ω_i $(i = 1, 2, 3, 4)$ satisfies the requirement (a) in Theorem 2.1.

Now, we prove that (b) of Theorem 2.1 holds, i.e., we prove that when $u \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap R^2$, $QNu \neq (0, 0)^T$ $(i = 1, 2, 3, 4)$. If it is not true, then for some $u \in \partial\Omega_i \cap R^2$, we have

$$\int_0^\omega F_1\big(t, e^{u_1}, u^{u_2}\big)\, dt = \int_0^\omega h_1(t)\, dt e^{-u_1},$$

$$\int_0^\omega F_2\big(t, e^{u_1}, u^{u_2}\big)\, dt = \int_0^\omega h_2(t)\, dt e^{-u_2}.$$

Thus there exist two points t_i $(i = 1, 2)$ such that

$$F_1\big(t_1, e^{u_1}, e^{u_2}\big) = h_1(t_1) e^{-u_1}, \tag{2.18}$$

$$F_2\big(t_2, e^{u_1}, e^{u_2}\big) = h_2(t_2) e^{-u_2}. \tag{2.19}$$

Following the arguments of (2.8)–(2.15), we obtain

$$\ln \frac{h_1^l}{a_1^M} < u_1 < A, \quad u_1 > \ln l_+ \quad \text{or} \quad u_1 < \ln l_-,$$

$$\ln \frac{h_2^l}{a_2^M} < u_2 < C, \quad u_2 > \ln u_+ \quad \text{or} \quad u_2 < \ln u_-.$$

Then $\ln \frac{h_1^l}{a_1^M} < u_1 < \ln l_-$, $\ln l_+ < u_1 < A$, $\ln \frac{h_2^l}{a_2^M} < u_2 < \ln u_-$, $\ln u_+ < u_2 < C$. Hence $u \in \Omega_1 \cap R^2$ or $u \in \Omega_2 \cap R^2$ or $u \in \Omega_3 \cap R^2$ or $u \in \Omega_4 \cap R^2$. This proves that (b) in Theorem 2.1 holds.

Finally, we show that (c) in Theorem 2.1 holds. We classify our proof into two steps.

Step 1. We show that for $i = 1, 2, 3, 4$,

$$\begin{aligned}
&\deg\{JQNu, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\} \\
&\quad = \deg\{\big(F_1(t_1, e^{u_1}, e^{u_2}) - h_1(t_1)e^{-u_1}, F_2(t_2, e^{u_1}, e^{u_2}) - h_2(t_2)e^{-u_2}\big)^T, \\
&\qquad \Omega_i \cap \operatorname{Ker} L, (0,0)^T\} \\
&\quad = \deg\{\big(a_1 - b_1 e^{u_1} - c_1 e^{u_2} - h_1 e^{-u_1}, a_2 - b_2 e^{u_1} - c_2 e^{u_2} - h_2 e^{-u_2}\big)^T, \\
&\qquad \Omega_i \cap \operatorname{Ker} L, (0,0)^T\},
\end{aligned}$$

where a_i, b_i, c_i, h_i are constants defined below. To this end, we define $\phi_1 : \operatorname{Ker} L \times [0,1] \to X$ by

$$\begin{aligned}
\phi_1(u_1, u_2, \mu_1) &= \mu_1 \begin{pmatrix} a_1 - b_1 e^{u_1} - c_1 e^{u_2} - h_1 e^{-u_1} \\ a_2 - b_2 e^{u_1} - c_2 e^{u_2} - h_2 e^{-u_2} \end{pmatrix} \\
&\quad + (1 - \mu_1) \begin{pmatrix} F_1(t_1, e^{u_1}, e^{u_2}) - h_1(t_1)e^{-u_1} \\ F_2(t_2, e^{u_1}, e^{u_2}) - h_2(t_2)e^{-u_2} \end{pmatrix},
\end{aligned}$$

where $\mu_1 \in [0,1]$ is a parameter, a_i, b_i, c_i, h_i $(i = 1, 2)$ are some chosen positive constants such that

$$\begin{aligned}
&A > \ln \frac{a_1}{b_1}, \qquad C > \ln \frac{a_2}{c_2}, \qquad \frac{h_1^l}{a_1^M} < \frac{h_1}{a_1}, \\
&l_+ < \frac{a_1 - c_1 e^C + \sqrt{(a_1 - c_1 e^C)^2 - 4b_1 h_1}}{2b_1}, \\
&l_- > \frac{a_1 - c_1 e^C - \sqrt{(a_1 - c_1 e^C)^2 - 4b_1 h_1}}{2b_1}, \\
&\frac{h_2^l}{a_2^M} < \frac{h_2}{a_2}, \\
&u_+ < \frac{a_2 - b_2 e^A + \sqrt{(a_2 - b_2 e^A)^2 - 4b_2 c_2}}{2c_2}, \\
&u_- > \frac{(a_2 - b_2 e^A)^2 - \sqrt{(a_2 - b_2 e^A)^2 - 4b_2 c_2}}{2c_2}.
\end{aligned}$$

When $(u_1, u_2)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap R^2$, $i = 1, 2, 3, 4$, u is a constant vector in R^2 with $u \in \partial\Omega_i$. We will show that when $u \in \partial\Omega_i \cap R^2$, $\phi_1(u_1, u_2, \mu_1) \neq (0,0)^T$. If the conclusion is not true, i.e., some constant vector u in R^2 with $u \in \partial\Omega_i$ satisfies $\phi_1(u_1, u_2, \mu_1) = (0,0)^T$, then we have

$$\begin{cases}
\mu_1\big(a_1 - b_1 e^{u_1} - c_1 e^{u_2} - h_1 e^{-u_1}\big) \\
\quad + (1 - \mu_1)\big(F_1(t_1, e^{u_1}, e^{u_2}) - h_1(t_1)e^{-u_1}\big) = 0, & (2.20) \\
\mu_1\big(a_2 - b_2 e^{u_1} - c_2 e^{u_2} - h_2 e^{-u_2}\big) \\
\quad + (1 - \mu_1)\big(F_2(t_2, e^{u_1}, e^{u_2}) - h_2(t_2)e^{-u_2}\big) = 0. & (2.21)
\end{cases}$$

Claim 1. $u_1 < A$.

Otherwise, $u_1 \geqslant A$, from (i) of Theorem 2.2, we have

$$\begin{aligned}
&(1-\mu_1)\big(F_1\big(t_1, e^{u_1}, e^{u_2}\big) - h_1(t_1)e^{-u_1}\big) + \mu_1\big(a_1 - b_1 e^{u_1} - c_1 e^{u_2} - h_1 e^{-u_1}\big)\\
&\quad < (1-\mu_1)\big(F_1\big(t_1, e^{u_1}, 0\big) - h_1(t_1)e^{-u_1}\big) + \mu_1\big(a_1 - b_1 e^{u_1} - h_1 e^{-u_1}\big)\\
&\quad < (1-\mu_1)\big(-h_1 e^{-u_1}\big) + \mu_1\big(a_1 - b_1 e^{u_1}\big)\\
&\quad < 0.
\end{aligned}$$

Hence Claim 1 holds.

Claim 2. $u_1 > \ln \frac{h_1^l}{a_1^M}$.

Otherwise, $u_1 \leqslant \ln \frac{h_1^l}{a_1^M}$, then

$$\begin{aligned}
&(1-\mu_1)\big(F_1\big(t_1, e^{u_1}, e^{u_2}\big) - h_1(t_1)e^{-u_1}\big) + \mu_1\big(a_1 - b_1 e^{u_1} - c_1 e^{u_2} - h_1 e^{-u_1}\big)\\
&\quad < (1-\mu_1)\big(F_1(t_1, 0, 0) - h_1^l e^{-u_1}\big) + \mu_1\big(a_1 - h_1 e^{-u_1}\big)\\
&\quad < (1-\mu_1)\big(a_1^M - h_1^l e^{-u_1}\big) + \mu_1\big(a_1 - h_1 e^{-u_1}\big)\\
&\quad < 0.
\end{aligned}$$

Hence Claim 2 holds.

Claim 3. $u_2 < C$.

Otherwise, $u_2 \geqslant C$, from (ii) in Theorem 2.2, we have

$$\begin{aligned}
&(1-\mu_1)\big(F_2\big(t_2, e^{u_1}, e^{u_2}\big) - h_2(t_2)e^{-u_2}\big) + \mu_1\big(a_2 - b_2 e^{u_1} - c_2 e^{u_2} - h_2 e^{-u_2}\big)\\
&\quad < (1-\mu_1)\big(F_2\big(t_2, 0, e^{u_2}\big) - h_2(t_2)e^{-u_2}\big) + \mu_1\big(a_2 - c_2 e^{u_2}\big)\\
&\quad < 0.
\end{aligned}$$

Hence Claim 3 holds.

Claim 4. $u_2 > \ln \frac{h_2^l}{a_2^M}$.

Otherwise, $u_2 \leqslant \ln \frac{h_2^l}{a_2^M}$. Then

$$\begin{aligned}
&(1-\mu_1)\big(F_2\big(t_2, e^{u_1}, e^{u_2}\big) - h_2(t_2)e^{-u_2}\big) + \mu_1\big(a_2 - b_2 e^{u_1} - c_2 e^{u_2} - h_2 e^{-u_2}\big)\\
&\quad < (1-\mu_1)\big(F_1(t_2, 0, 0) - h_2^l e^{-u_2}\big) + \mu_1\big(a_2 - h_2 e^{-u_2}\big)\\
&\quad < (1-\mu_1)\big(a_2^M - h_2^l e^{-u_2}\big) + \mu_1\big(a_2 - h_2 e^{-u_2}\big)\\
&\quad < 0.
\end{aligned}$$

Hence Claim 4 holds.

Claim 5. $u_1 > \ln l_+$ *or* $u_1 < \ln l_-$.

Otherwise, $\ln l_- \leqslant u_1 \leqslant \ln l_+$, from (iii) in Theorem 2.2, we obtain

$$\begin{aligned}
&(1-\mu_1)\big(F_1\big(t_1, e^{u_1}, e^{u_2}\big) - h_1(t_1)e^{-u_1}\big) + \mu_1\big(a_1 - b_1e^{u_1} - c_1e^{u_2} - h_1e^{-u_1}\big)\\
&\quad \geqslant (1-\mu_1)\big(F_1\big(t_1, e^{u_1}, e^{C}\big) - h_1^l e^{-u_1}\big) + \mu_1\big(a_1 - b_1e^{u_1} - c_1e^{C} - h_1e^{-u_1}\big)\\
&\quad > (1-\mu_1)\big(F_1\big(t_1, e^{u_1}, e^{C}\big) - h_1(t_1)e^{-u_1}\big) - \mu_1e^{-u_1}\big(b_1e^{2u_1} + h_1 + c_1e^{C}e^{u_1} - a_1e^{u_1}\big)\\
&\quad > 0.
\end{aligned}$$

Hence Claim 5 holds.

Claim 6. $u_2 > \ln u_+$ *or* $u_2 < \ln u_-$.

Otherwise, $\ln u_- \leqslant u_2 \leqslant \ln u_+$. Then from (iv) in Theorem 2.2, we have

$$\begin{aligned}
&(1-\mu_1)\big(F_2\big(t_2, e^{u_1}, e^{u_2}\big) - h_2(t_2)e^{-u_2}\big) + \mu_1\big(a_2 - b_2e^{u_1} - c_2e^{u_2} - h_2e^{-u_2}\big)\\
&\quad > (1-\mu_1)\big(F_2\big(t_2, e^{A}, e^{u_2}\big) - h_2(t_2)e^{-u_2}\big) + \mu_1\big(a_2 - b_2e^{A} - c_2e^{u_2} - h_2e^{-u_2}\big)\\
&\quad > -\mu_1e^{-u_2}\big(c_2e^{2u_2} - a_2e^{u_2} + h_2 + b_2e^{A}e^{u_2}\big)\\
&\quad > 0.
\end{aligned}$$

Hence Claim 6 holds.

From above claims, we have

$$\begin{aligned}
&\ln\frac{h_1^l}{a_1^M} < u_1 < A, \quad u_1 > \ln l_+ \quad \text{or} \quad u_1 < \ln l_-,\\
&\ln\frac{h_2^l}{a_2^M} < u_2 < C, \quad u_2 > \ln u_+ \quad \text{or} \quad u_2 < \ln u_-.
\end{aligned}$$

Therefore

$$u \in \Omega_1 \cap R^2 \quad \text{or} \quad u \in \Omega_2 \cap R^2 \quad \text{or} \quad u \in \Omega_3 \cap R^2 \quad \text{or} \quad u \in \Omega_4 \cap R^2.$$

This contradicts $u \in \partial\Omega_i \cap R^2$, $i = 1, 2, 3, 4$. By topological degree theory and taking $J = I$ since $\operatorname{Ker} L = \operatorname{Im} Q$, we have for $i = 1, 2, 3, 4$,

$$\begin{aligned}
&\deg\big\{JQNu, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}\\
&\quad = \deg\big\{\phi_1(u_1, u_2, 0), \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}\\
&\quad = \deg\big\{\phi_1(u_1, u_2, 1), \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}\\
&\quad = \deg\big\{\big(a_1 - b_1e^{u_1} - c_1e^{u_2} - h_1e^{-u_1}, a_2 - b_2e^{u_1} - c_2e^{u_2} - h_2e^{-u_2}\big),\\
&\qquad \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}.
\end{aligned} \tag{2.22}$$

Step 2. We show that for $i = 1, 2, 3, 4$,

$$\begin{aligned}
&\deg\big\{\big(a_1 - b_1e^{u_1} - c_1e^{u_2} - h_1e^{-u_1}, a_2 - b_2e^{u_1} - c_2e^{u_2} - h_2e^{-u_2}\big)^T, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}\\
&\quad = \deg\big\{\big(a_1 - b_1e^{u_1} - h_1e^{-u_1}, a_2 - c_2e^{u_2} - h_2e^{-u_2}\big)^T, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}.
\end{aligned}$$

To this end, we define a mapping $\phi_2 : \operatorname{Ker} L \times [0,1] \to X$ by

$$\phi_2(u_1,u_2,\mu_2)=\begin{pmatrix} a_1-b_1e^{u_1}-\mu_2c_1e^{u_2}-h_1e^{-u_1}\\ a_2-\mu_2b_2e^{u_1}-c_2e^{u_2}-h_1e^{-u_2}\end{pmatrix},$$

where $\mu_2\in[0,1]$ is a parameter. When $u\in\partial\Omega_i\cap\operatorname{Ker}L=\partial\Omega_i\cap R^2$, u is a constant vector in R^2 with $u\in\partial\Omega_i$, $i=1,2,3,4$. We prove that when $u\in\partial\Omega_i\cap\operatorname{Ker}L$, $\phi_2(u_1,u_2,\mu_2)\neq(0,0)^T$. If it is not true, then some constant vector u with $u\in\partial\Omega_i$ satisfies

$$a_1-b_1e^{u_1}-c_1\mu_2e^{u_2}-h_1e^{-u_1}=0, \tag{2.23}$$

$$a_2-\mu_2b_2e^{u_1}-c_2e^{u_2}-h_2e^{-u_2}=0. \tag{2.24}$$

(2.23) implies

$$b_1e^{u_1}<a_1,\qquad h_1e^{-u_1}<a_1,$$

that is,

$$u_1<\ln\frac{a_1}{b_1}<A,\qquad u_1>\ln\frac{h_1}{a_1}>\ln\frac{h_1^l}{a_1^M}. \tag{2.25}$$

(2.24) implies that

$$c_2e^{u_2}<a_2,\qquad h_2e^{-u_2}<a_2,$$

that is,

$$u_2<\ln\frac{a_2}{c_2}<C,\qquad u_2>\ln\frac{h_2}{a_2}>\ln\frac{h_2^l}{a_2^M}. \tag{2.26}$$

(2.23) gives

$$0=b_1e^{2u_1}-\left(a_1-\frac{c_1a_2}{c_2}\right)e^{u_1}+h_1,$$

that is,

$$u_1>\ln\frac{a_1-\frac{c_1a_2}{c_2}+\sqrt{(a_1-\frac{c_1a_2}{c_2})^2-4b_1h_1}}{2b_1}>\ln l_+ \tag{2.27}$$

or

$$u_1<\ln\frac{a_1-\frac{c_1a_2}{c_2}-\sqrt{(a_1-\frac{c_1a_2}{c_2})^2-4b_1h_1}}{2b_1}<\ln l_-. \tag{2.28}$$

(2.24) gives

$$0=c_2e^{2u_2}+h_2-a_2e^{u_2}+\mu_2b_2e^{u_1+u_2}<c_2e^{2u_2}+h_2-\left(a_2-\frac{b_2a_1}{b_1}\right)e^{u_2},$$

that is,

$$u_2>\ln\frac{a_2-\frac{b_2a_1}{b_1}+\sqrt{(a_2-\frac{b_2a_1}{b_1})^2-4c_2h_2}}{2c_2}>\ln u_+ \tag{2.29}$$

or

$$u_2<\ln\frac{a_2-\frac{b_2a_1}{b_1}-\sqrt{(a_2-\frac{b_2a_1}{b_1})^2-4c_2h_2}}{2c_2}<\ln u_-. \tag{2.30}$$

From (2.25)–(2.30), we obtain

$$\ln\frac{h_1}{a_1} < u_1 < \ln l_- \quad \text{or} \quad \ln l_+ < u_1 < \ln\frac{a_1}{b_1},$$
$$\ln\frac{h_2}{a_2} < u_2 < \ln u_- \quad \text{or} \quad \ln u_+ < u_2 < \ln\frac{a_2}{c_2}.$$

Hence $u \in \Omega_1 \cap R^2$ or $u \in \Omega_2 \cap R^2$ or $u \in \Omega_3 \cap R^2$ or $u \in \Omega_4 \cap R^2$. This contradicts the fact that $u \in \partial\Omega_i \cap R^2$, $i = 1, 2, 3, 4$. According to topological degree theory, we have for $i = 1, 2, 3, 4$,

$$\begin{aligned}
&\deg\big\{\big(a_1 - b_1e^{u_1} - c_1e^{u_2} - h_1e^{-u_1}, a_2 - b_2e^{u_1} - c_2e^{u_2} - h_2e^{-u_2}\big)^T, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\} \\
&\quad = \deg\big\{\phi_2(u_1, u_2, 1), \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\} \\
&\quad = \deg\big\{\phi_2(u_1, u_2, 0), \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\} \\
&\quad = \deg\big\{\big(a_1 - b_1e^{u_1} - h_1e^{-u_1}, a_2 - c_2e^{u_2} - h_2e^{-u_2}\big)^T, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}.
\end{aligned} \tag{2.31}$$

From (2.22) and (2.31), we have

$$\begin{aligned}
&\deg\big\{JQNu, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\} \\
&\quad = \deg\big\{\big(a_1 - b_1e^{u_1} - h_1e^{-u_1}, a_2 - c_2e^{u_2} - h_2e^{-u_2}\big)^T, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\big\}.
\end{aligned}$$

Note that the system of algebraic equations:

$$a_1 - b_1e^x - h_1e^{-x} = 0,$$
$$a_2 - c_2e^y - h_2e^{-y} = 0$$

has four distinct solutions

$$\big(x_1^*, y_1^*\big) = \left(\ln\frac{a_1 + \sqrt{a_1^2 - 4b_1h_1}}{2b_1}, \ln\frac{a_2 + \sqrt{a_2^2 - 4c_2h_2}}{2c_2}\right),$$
$$\big(x_2^*, y_2^*\big) = \left(\ln\frac{a_1 + \sqrt{a_1^2 - 4b_1h_1}}{2b_1}, \ln\frac{a_2 - \sqrt{a_2^2 - 4c_2h_2}}{2c_2}\right),$$
$$\big(x_3^*, y_3^*\big) = \left(\ln\frac{a_1 - \sqrt{a_1^2 - 4b_1h_1}}{2b_1}, \ln\frac{a_2 + \sqrt{a_2^2 - 4c_2h_2}}{2c_2}\right),$$
$$\big(x_4^*, y_4^*\big) = \left(\ln\frac{a_1 - \sqrt{a_1^2 - 4b_1h_1}}{2b_1}, \ln\frac{a_2 - \sqrt{a_2^2 - 4c_2h_2}}{2c_2}\right).$$

It is easy to verify that

$$\ln l_+ < \ln\frac{a_1 + \sqrt{a_1^2 - 4b_1h_1}}{2b_1} < \ln\frac{a_1}{b_1} < A,$$
$$\ln\frac{h_1^l}{a_1^M} < \ln\frac{a_1 - \sqrt{a_1^2 - 4b_1h_1}}{2b_1} < \ln l_-,$$
$$\ln u_+ < \ln\frac{a_2 + \sqrt{a_2^2 - 4c_2h_2}}{2c_2} < \ln\frac{a_2}{c_2} < C,$$

$$\ln \frac{h_2^l}{a_2^M} < \ln \frac{a_2 - \sqrt{a_2^2 - 4c_2h_2}}{2c_2} < \ln u_-.$$

Therefore

$$\left(x_1^*, y_1^*\right) \in \Omega_4, \qquad \left(x_2^*, y_2^*\right) \in \Omega_3, \qquad \left(x_3^*, y_3^*\right) \in \Omega_2, \qquad \left(x_4^*, y_4^*\right) \in \Omega_1.$$

A direct computation gives for $i = 1, 2, 3, 4$,

$$\begin{aligned} &\deg\left\{JQNu, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\begin{vmatrix} -b_1x^* + \frac{h_1}{x*} & 0 \\ 0 & -c_2y^* + \frac{h_2}{y^*} \end{vmatrix} \\ &\quad = \operatorname{sign}\left\{\frac{1}{x^*y^*}\left[b_1c_2\left(x^*\right)^2\left(y^*\right)^2 - b_1h_2\left(x^*\right)^2 - h_1c_2\left(y^*\right)^2 + h_1h_2\right]\right\} \end{aligned}$$

where $(x^*, y^*) = (e^{x_i^*}, e^{y_i^*})$. Since

$$b_1\left(x^*\right)^2 = a_1x^* - h_1, \qquad c_2\left(y^*\right)^2 = a_2y^* - h_2,$$

then

$$\begin{aligned} &\deg\left\{JQNu, \Omega_i \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\left[\left(a_1x^* - h_1\right)\left(a_2y^* - h_2\right) - h_2\left(a_1x^* - h_1\right) - h_1\left(a_2y^* - h_2\right) + h_1h_2\right] \\ &\quad = \operatorname{sign}\left[\left(a_1x^* - 2h_1\right)\left(a_2y^* - 2h_2\right)\right], \end{aligned}$$

where $(x^*, y^*) = (e^{x_i^*}, e^{y_i^*})$.

Thus

$$\begin{aligned} &\deg\left\{JQNu, \Omega_4 \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\left\{\left[\frac{a_1(a_1 + \sqrt{a_1^2 - 4b_1h_1})}{2b_1} - 2h_1\right]\left[\frac{a_2(a_2 + \sqrt{a_2^2 - 4c_2h_2})}{2c_2} - 2h_2\right]\right\} \\ &\quad = 1, \\ &\deg\left\{JQNu, \Omega_3 \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\left\{\left[\frac{a_1(a_1 + \sqrt{a_1^2 - 4b_1h_1})}{2b_1} - 2h_1\right]\left[\frac{a_2(a_2 - \sqrt{a_2^2 - 4c_2h_2})}{2c_2} - 2h_2\right]\right\} \\ &\quad = -1, \\ &\deg\left\{JQNu, \Omega_2 \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\left\{\left[\frac{a_1(a_1 - \sqrt{a_1^2 - 4b_1h_1})}{2b_1} - 2h_1\right]\left[\frac{a_2(a_2 + \sqrt{a_2^2 - 4c_2h_2})}{2c_2} - 2h_2\right]\right\} \\ &\quad = -1, \\ &\deg\left\{JQNu, \Omega_1 \cap \operatorname{Ker} L, (0,0)^T\right\} \\ &\quad = \operatorname{sign}\left\{\left[\frac{a_1(a_1 - \sqrt{a_1^2 - 4b_1h_1})}{2b_1} - 2h_1\right]\left[\frac{a_2(a_2 - \sqrt{a_2^2 - 4c_2h_2})}{2c_2} - 2h_2\right]\right\} \\ &\quad = 1. \end{aligned}$$

So far, we have proved that Ω_i $(i = 1, 2, 3, 4)$ satisfies all assumptions in Theorem 2.1. Hence, system (2.2) has at least four ω-periodic solutions. Thus by (2.1) system (1.1) has at least four different positive ω-periodic solutions. This completes the proof of Theorem 2.2. □

3. An example

Consider the following competition Lotka–Volterra population model with stocking:

$$\begin{cases} x'(t) = x(t)\big(a_1(t) - b_1(t)x(t) - c_1(t)y(t)\big) - h_1(t), \\ y'(t) = y(t)\big(a_2(t) - b_2(t)x(t) - c_2(t)y(t)\big) - h_2(t), \end{cases} \tag{3.1}$$

where $x(t)$ and $y(t)$ denote the densities of two competition species, $a_i(t)$, $b_i(t)$, $c_i(t)$, $h_i(t)$ $(i = 1, 2)$ are all positive continuous ω-periodic functions.

In Theorem 2.2, $F_1(t, e^x, e^y) = a_1(t) - b_1(t)e^x - c_1(t)e^y$, $F_2(t, e^x, e^y) = a_2(t) - b_2(t)e^x - c_2(t)e^y$. If $x \geqslant \ln \frac{a_1^M}{b_1^l} \stackrel{\text{def}}{=} A$, then $F_1(t, e^x, 0) \leqslant 0$, $\forall t \in R$; if $x \geqslant \ln \frac{a_2^M}{c_2^l} \stackrel{\text{def}}{=} C$, then $F_2(t, 0, e^y) \leqslant 0$.

Let

$$l_\pm = \frac{a_1^l - c_1^M e^C \pm \sqrt{(a_1^l - c_1^M e^C)^2 - 4b_1^M h_1^M}}{2b_1^M},$$

$$u_\pm = \frac{a_2^l - b_2^M e^A \pm \sqrt{(a_2^l - b_2^M e^A)^2 - 4c_2^M h_2^M}}{2c_2^M}.$$

If

$$a_1^l > c_1^M e^C + 2\sqrt{b_1^M h_1^M} \quad \text{and} \quad a_2^l > b_2^M e^A + 2\sqrt{c_2^M h_2^M},$$

then when $\ln l_- \leqslant x \leqslant \ln l_+$,

$$e^x F_1\big(t, e^x, e^C\big) \geqslant h_1(t),$$

and when $\ln u_- \leqslant x \leqslant \ln u_+$,

$$e^x F_2\big(t, e^A, e^x\big) \geqslant h_2(t).$$

Therefore the conditions in Theorem 2.2 are satisfied. Thus we get the following theorem.

Theorem 3.1. *If*

$$a_1^l > c_1^M e^C + 2\sqrt{b_1^M h_1^M} \quad \textit{and} \quad a_2^l > b_2^M e^A + 2\sqrt{c_2^M h_2^M},$$

then system (3.1) *has at least four different positive ω-periodic solutions.*

References

[1] Zhengqiu Zhang, Zhicheng Wang, The existence of a periodic solution for a generalized prey–predator system with delay, Math. Proc. Cambridge Philos. Soc. 137 (2004) 475.

[2] Zhengqiu Zhang, Zhicheng Wang, Periodic solution of a two-species ratio-dependent predator–prey system with time delay in a two-patch environment, ANZIAM J. 45 (2003) 233–244.

[3] Zhengqiu Zhang, Xianwu Zeng, On a periodic stage-structure model, Appl. Math. Lett. 16 (7) (2003) 1053–1061.
[4] Zhengqiu Zhang, Jun Wu, Zhicheng Wang, Periodic solution of nonautonomous stage-structured cooperative system, Comput. Math. Appl. 47 (2004) 699–706.
[5] Y. Li, Periodic solutions of a periodic delay predator–prey system, Proc. Amer. Math. Soc. 127 (1999) 1331–1335.
[6] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Non-Linear Differential Equations, Springer, Berlin, 1979.