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On the regularity of local cohomology of bigraded algebras

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Abstract

The Hilbert functions and the regularity of the graded components of local cohomology of a bigraded algebra are considered. Explicit bounds for these invariants are obtained for bigraded hypersurface rings.

Introduction

In this paper we study algebraic properties of the graded components of local cohomology of a bigraded K-algebra. Let P_0 be a Noetherian ring, $P = P_0[y_1, \dots, y_n]$ be the polynomial ring over P_0 with the standard grading and $P_+ = (y_1, \dots, y_n)$ the irrelevant graded ideal of P. Then for any finitely generated graded P-module M, the local cohomology modules $H^i_{P_+}(M)$ are naturally graded P-modules and each graded component $H^i_{P_+}(M)_j$ is a finitely generated P_0-module. In case $P_0 = K[x_1, \dots, x_m]$ is a polynomial ring, the K-algebra P is naturally bigraded with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. In this situation, if M is a finitely generated bigraded P-module, then each of the modules $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module.

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We are interested in the Hilbert functions and the Castelnuovo–Mumford regularity of these modules.

In Section 1 we introduce the basic facts concerning graded and bigraded local cohomology and give a description of the local cohomology of a graded (bigraded) P-module from its graded (bigraded) P-resolution.

In Section 2 we use a result of Gruson, Lazarsfeld and Peskine on the regularity of reduced curves, in order to show that the regularity of $H^i_{P_+}(M)_j$ as a function in j is bounded provided that $\dim_{P_0} M/P_+M \leqslant 1$.

The rest of the paper is devoted to study of the local cohomology of a hypersurface ring $R = P/fP$ where $f \in P$ is a bihomogeneous polynomial.

In Section 3 we prove that the Hilbert function of the top local cohomology $H^n_{P_+}(R)_j$ is a nonincreasing function in j. If moreover, the ideal $I(f)$ generated by all coefficients of f is $\mathfrak{m}$-primary where $\mathfrak{m}$ is the graded maximal ideal of P_0, then by a result of Katzman and Sharp the P_0-module $H^i_{P_+}(R)_j$ is of finite length. In particular, in this case the regularity of $H^i_{P_+}(R)_j$ is also a nonincreasing function in j.

In the following section we compute the regularity of $H^i_{P_+}(R)_j$ for a special class of hypersurfaces. For the computation we use in an essential way a result of Stanley and Watanabe. They showed that a monomial complete intersection has the strong Lefschetz property. Stanley used the hard Lefschetz theorem, while Watanabe representation theory of Lie algebras to prove this result. Using these facts the regularity and the Hilbert function of $H^i_{P_+}(P/f^r_\lambda P)_j$ can be computed explicitly. Here $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$. As a consequence we are able to show that $H^{n-1}_{P_+}(P/f^r P)_j$ has a linear resolution and its Betti numbers can be computed. We use these results in the last section to show that for any bigraded hypersurface ring $R = P/fP$ for which $I(f)$ is $\mathfrak{m}$-primary, the regularity of $H^i_{P_+}(R)_j$ is linearly bounded in j.

1. Basic facts about graded and bigraded local cohomology

Let P_0 be a Noetherian ring, and let $P = P_0[y_1, \dots, y_n]$ be the polynomial ring over P_0 in the variables $y_1, \dots, y_n$. We let $P_j = \bigoplus_{|b|=j} P_0 y^b$ where $y^b = y_1^{b_1} \dots y_n^{b_n}$ for $b = (b_1, \dots, b_n)$, and where $|b| = \sum_i b_i$. Then P is a standard graded P_0-algebra and P_j is a free P_0-module of rank $\binom{n+j-1}{n-1}$.

In most cases we assume that P_0 is either a local ring with residue class field K, or $P_0 = K[x_1, \dots, x_m]$ is the polynomial ring over the field K in the variables $x_1, \dots, x_m$.

We always assume that all P-modules considered here are finitely generated and graded. In case that P_0 is a polynomial ring, then P itself is bigraded, if we assign to each x_i the bidegree $(1, 0)$ and to each y_j the bidegree $(0, 1)$. In this case we assume that all P-modules are even bigraded. Observe that if M is bigraded, and if we set

$$M_j = \bigoplus_i M_{(i,j)}.$$

Then $M = \bigoplus_j M_j$ is a graded P-module and each graded component M_j is a finitely generated graded P_0-module, with grading $(M_j)_i = M_{(i,j)}$ for all i and j.

Now let $S = K[y_1, \dots, y_n]$. Then $P = P_0 \otimes_K K[y_1, \dots, y_n] = P_0 \otimes_K S$. Let $P_+ := \bigoplus_{j>0} P_j$ be the irrelevant graded ideal of the P_0-algebra P.

Next we want to compute the graded P-modules $H^i_{P_+}(P)$. Observe that there are isomorphisms of graded R-modules

$$\begin{aligned} H^i_{P_+}(P) &\cong \varinjlim_{k\geqslant 0} \operatorname{Ext}^i_P\big(P/(P_+)^k, P\big) \\ &\cong \varinjlim_{k\geqslant 0} \operatorname{Ext}^i_{P_0\otimes_K S}\big(P_0 \otimes_K S/(y)^k, P_0 \otimes_K S\big) \\ &\cong P_0 \otimes_K \varinjlim_{k\geqslant 0} \operatorname{Ext}^i_P\big(S/(y)^k, S\big) \\ &\cong P_0 \otimes_K H^i_{(y)}(S). \end{aligned}$$

Since $H^i_{S_+}(S) = 0$ for $i \neq n$, we get

$$H^i_{P_+}(P) = \begin{cases} P_0 \otimes_k H^n_{(y)}(S) & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Let M be a graded S-module. We write $M^\vee = \operatorname{Hom}_K(M, K)$ and consider $M^\vee$ a graded S-module as follows: for $\varphi \in M^\vee$ and $f \in S$ we let $f\varphi$ be the element in $M^\vee$ with

$$f\varphi(m) = \varphi(fm) \quad \text{for all } m \in M,$$

and define the grading by setting $(M^\vee)_j := \operatorname{Hom}_K(M_{-j}, K)$ for all $j \in \mathbb{Z}$.

Let ω_S be the canonical module of S. Note that $\omega_S = S(-n)$, since S is a polynomial ring in n indeterminates. By the graded version of the local duality theorem, see [1, Example 13.4.6] we have $H^n_{S_+}(S)^\vee = S(-n)$ and $H^i_{S_+}(S) = 0$ for $i \neq n$. Applying again the functor $(\_)^\vee$ we obtain

$$H^n_{S_+}(S) = \operatorname{Hom}_K\big(S(-n), K\big) = \operatorname{Hom}_K(S, K)(n).$$

We can thus conclude that

$$H^n_{S_+}(S)_j = \operatorname{Hom}_k(S, K)_{n+j} = \operatorname{Hom}_K(S_{-n-j}, K) \quad \text{for all } j \in \mathbb{Z}.$$

Let $S_l = \bigoplus_{|a|=l} Ky^a$. Then

$$\operatorname{Hom}_K(S_{-n-j}, K) = \bigoplus_{|a|=-n-j} Kz^a,$$

where $z \in \operatorname{Hom}_K(S_{-n-j}, K)$ is the K-linear map with

$$z^a\big(y^b\big) = \begin{cases} z^{a-b}, & \text{if } b \leqslant a, \\ 0, & \text{if } b \nleqslant a. \end{cases}$$

Here we write $b \leqslant a$ if $b_i \leqslant a_i$ for $i = 1, \ldots, n$. Therefore $H^n_{S_+}(S)_j = \bigoplus_{|a|=-n-j} Kz^a$, and this implies that

$$H^n_{P_+}(P)_j = P_0 \otimes_K H^n_{(y)}(S)_j = \bigoplus_{|a|=-n-j} P_0 z^a. \tag{1}$$

Hence we see that $H^n_{P_+}(P)_j$ is free P_0-module of rank $\binom{-j-1}{n-1}$. Moreover, if P_0 is graded

$$H^n_{P_+}(P)_{(i,j)} = \bigoplus_{|b|=-n-j} (P_0)_i z^b = \bigoplus_{\substack{|a|=i \\ |b|=-n-j}} K x^a z^b.$$

The next theorem describes how the local cohomology of a graded P-module can be computed from its graded free P-resolution.

Theorem 1.1. *Let M be a finitely generated graded P-module. Let $\mathbb{F}$ be a graded free P-resolution of M. Then we have graded isomorphisms*

$$H^{n-i}_{P_+}(M) \cong H_i\big(H^n_{P_+}(\mathbb{F})\big).$$

Proof. Let

$$\mathbb{F}: \cdots \to F_2 \to F_1 \to F_0 \to 0.$$

Applying the functor $H^n_{P_+}$ to $\mathbb{F}$, we obtain the complex

$$H^n_{P_+}(\mathbb{F}): \cdots \to H^n_{P_+}(F_2) \to H^n_{P_+}(F_1) \to H^n_{P_+}(F_0) \to 0.$$

We see that

$$H^n_{P_+}(M) = \operatorname{Coker}\big(H^n_{P_+}(F_1) \to H^n_{P_+}(F_0)\big) = H_0\big(H^n_{P_+}(\mathbb{F})\big),$$

since $H^i_{P_+}(N) = 0$ for each $i > n$ and all finitely generated P-modules N.

We define the functors:

$$\mathcal{F}(M) := H^n_{P_+}(M) \quad \text{and} \quad \mathcal{F}_i(M) := H^{n-i}_{P_+}(M).$$

The functors $\mathcal{F}_i$ are additive, covariant and strongly connected, i.e., for each short exact sequence $0 \to U \to V \to W \to 0$ one has the long exact sequence

$$0 \cdots \to \mathcal{F}_i(U) \to \mathcal{F}_i(V) \to \mathcal{F}_i(W) \to \mathcal{F}_{i-1}(U) \to \cdots \to \mathcal{F}_0(V) \to \mathcal{F}_0(W) \to 0.$$

Moreover, $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i(F) = H^{n-i}_{P_+}(F) = 0$ for all $i > 0$ and all free P-modules F. Therefore, the theorem follows from the dual version of [1, Theorem 1.3.5]. $\square$

Note that if M is a finitely generated bigraded P-module. Then $H^n_{P_+}(M)$ with natural grading is also a finitely generated bigraded P-module, and hence in Theorem 1.1 we have bigraded isomorphisms

$$H^{n-i}_{P_+}(M) \cong H_i\big(H^n_{P_+}(\mathbb{F})\big).$$

2. Regularity of the graded components of local cohomology for modules of small dimension

Let $P_0 = K[x_1, \dots, x_m]$, and M be a finitely generated graded P_0-module. By Hilbert's syzygy theorem, M has a graded free resolution over P_0 of the form

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \bigoplus_{j=1}^{t_i} P_0(-a_{ij})$ for some integers a_{ij}. Then the Castelnuovo–Mumford regularity $\operatorname{reg}(M)$ of M is the nonnegative integer

$$\operatorname{reg} M \leqslant \max_{i,j}\{a_{ij} - i\}$$

with equality holding if the resolution is minimal. If M is an Artinian graded P_0-module, then

$$\operatorname{reg}(M) = \max\{j\colon\ M_j \neq 0\}.$$

We also use the following characterization of regularity

$$\operatorname{reg}(M) = \min\{\mu\colon\ M_{\geqslant \mu} \text{ has a linear resolution}\}.$$

Let M be a finitely generated bigraded P-module, thus $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module. Let $f_{i,M}$ be the numerical function given by

$$f_{i,M}(j) = \operatorname{reg} H^i_{P_+}(M)_j$$

for all j. In this section we show that $f_{i,M}$ is bounded provided that M/P_+M has Krull dimension $\leqslant 1$. There are some explicit examples which show that the condition $\dim_{P_0} M/P_+M \leqslant 1$ is indispensable. We postpone the example to Section 4. First one has the following

Lemma 2.1. *Let M be a finitely generated graded P-module. Then*

$$\dim_{P_0} M_i \leqslant \dim_{P_0} M/P_+M \quad \textit{for all } i.$$

Proof. Let $r = \min\{j\colon\ M_j \neq 0\}$. We prove the lemma by induction on $i \geqslant r$. Let $i = r$. Note that

$$M/P_+M = M_r \oplus M_{r+1}/P_1M_r \oplus \cdots.$$

It follows that M_r is a direct summand of the P_0-module M/P_+M, so that $\dim_{P_0} M_r \leqslant \dim_{P_0} M/P_+M$. We now assume that $i > r$ and $\dim_{P_0} M_j \leqslant \dim_{P_0} M/P_+M$, for $j = r, \ldots, i-1$. We will show that $\dim_{P_0} M_i \leqslant \dim_{P_0} M/P_+M$. We consider the exact sequence of P_0-modules

$$0 \to P_1M_{i-1} + \cdots + P_{i-r}M_r \to M_i \xrightarrow{\varphi} (M/P_+M)_i \to 0.$$

By the induction hypothesis, one easily deduces that

$$\dim_{P_0} \sum_{j=1}^{i-r} P_j M_{i-j} \leqslant \dim_{P_0} M/P_+M,$$

and since $(M/P_+M)_i$ is a direct summand of M/P_+M it also has dimension $\leqslant \dim_{P_0} M/P_+M$. Therefore, by the above exact sequence, $\dim M_i \leqslant \dim_{P_0} M/P_+M$, too. □

The following lemma is needed for the proof of the next proposition.

Lemma 2.2. *Let M be a finitely generated graded P-module. Then there exists an integer i_0 such that*

$$\operatorname{Ann}_{P_0} M_i = \operatorname{Ann}_{P_0} M_{i+1} \quad \textit{for all } i \geqslant i_0.$$

Proof. Since $P_1M_i \subseteq M_{i+1}$ for all i and M is a finitely generated P-module, there exists an integer t such that $P_1M_i = M_{i+1}$ for all $i \geqslant t$. This implies that $\operatorname{Ann}_{P_0} M_t \subseteq \operatorname{Ann}_{P_0} M_{t+1} \subseteq \cdots$. Since P_0 is Noetherian, there exists an integer k such that $\operatorname{Ann}_{P_0} M_{t+k} = \operatorname{Ann}_{P_0} M_i$ for all $i \geqslant t+k = i_0$. □

Proposition 2.3. *Let M be a finitely generated graded P-module. Then*

$$\dim_{P_0} H^i_{P_+}(M)_j \leqslant \dim_{P_0} M_j \quad \textit{for all } i \textit{ and } j \gg 0.$$

Proof. Let $P_+ = (y_1, \ldots, y_n)$. Then by [1, Theorem 5.1.19] we have

$$H^i_{P_+}(M) \cong H^i\big(C(M)^\bullet\big) \quad \text{for all } i \geqslant 0,$$

where $C(M)^\bullet$ denote the (extended) Čech complex of M with respect to $y_1, \ldots, y_n$ defined as follows:

$$C(M)^\bullet : 0 \to C(M)^0 \to C(M)^1 \to \cdots \to C(M)^n \to 0$$

with

$$C(M)^t = \bigoplus_{1\leqslant i_1<\cdots<i_t\leqslant n} M_{y_{i_1}\ldots y_{i_t}},$$

and where the differentiation $C(M)^t \to C(M)^{t+1}$ is given on the component

$$M_{y_{i_1}\ldots y_{i_t}} \to M_{y_{j_1}\ldots y_{j_{t+1}}}$$

to be the homomorphism

$$(-1)^{s-1} nat : M_{y_{i_1}\ldots y_{i_t}} \to (M_{y_{i_1}\ldots y_{i_t}})_{y_{j_s}},$$

if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \hat{j}_s, \ldots, j_{t+1}\}$ and 0 otherwise. We set $\mathcal{I} = \{i_1, \ldots, i_t\}$ and $y_{\mathcal{I}} = y_{i_1} \ldots y_{i_t}$. For $m/y_{\mathcal{I}}^k \in M_{y_{\mathcal{I}}}$, m homogeneous, we set $\deg m/y_{\mathcal{I}}^k = \deg m - \deg y_{\mathcal{I}}^k$. Then we can define a grading on $M_{y_{\mathcal{I}}}$ by setting

$$(M_{y_{\mathcal{I}}})_j = \{m/y_{\mathcal{I}}^k \in M_{y_{\mathcal{I}}} \colon \deg m/y_{\mathcal{I}}^k = j\} \quad \text{for all } j.$$

In view of Lemma 2.2 there exists an ideal $I \subseteq P_0$ and an integer j_0 such that $\operatorname{Ann}_{P_0} M_j = I$ for all $j \geqslant j_0$. We now claim that $I \subseteq \operatorname{Ann}_{P_0}(M_{y_{\mathcal{I}}})_j$ for all $j \geqslant j_0$. Let $a \in I$ and $m/y_{\mathcal{I}}^k \in (M_{y_{\mathcal{I}}})_j$ for some integer k. We may choose an integer l such that

$$\deg m + \deg y_{\mathcal{I}}^l = \deg m y_{\mathcal{I}}^l = t \geqslant j_0.$$

Thus $am/y_{\mathcal{I}}^k = amy_{\mathcal{I}}^l/y_{\mathcal{I}}^{k+l} = 0$, because $my_{\mathcal{I}}^l \in M_t$. Thus we have

$$\dim_{P_0}(M_{y_{\mathcal{I}}})_j = \dim_{P_0} P_0/\operatorname{Ann}(M_{y_{\mathcal{I}}})_j \leqslant \dim_{P_0} P_0/I = \dim_{P_0} M_j.$$

Since $H^i_{P_+}(M)_j$ is a subquotient of the jth graded component of $C(M)^i$, the desired result follows. □

Now we can state the main result of this section as follows.

Theorem 2.4. *Let M be a finitely generated bigraded P-module such that*

$$\dim_{P_0} M/P_+M \leqslant 1.$$

Then for all i the functions $f_{i,M}(j) = \operatorname{reg} H^i_{P_+}(M)_j$ are bounded.

In a first step we prove the following

Proposition 2.5. *Let M be a finitely generated bigraded P-module with*

$$\dim_{P_0} M/P_+M \leqslant 1.$$

Then the function $f_{n,M}(j) = \operatorname{reg} H^n_{P_+}(M)_j$ is bounded above.

Proof. By the bigraded version of Hilbert's syzygy theorem, M has a bigraded free resolution of the form

$$\mathbb{F}: 0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \bigoplus_{k=1}^{t_i} P(-a_{ik}, -b_{ik})$. Applying the functor $H^n_{P_+}(-)_j$ to this resolution yields a graded complex of free P_0-modules

$$H^n_{P_+}(\mathbb{F})_j : 0 \to H^n_{P_+}(F_k)_j \to \cdots \to H^n_{P_+}(F_1)_j \to H^n_{P_+}(F_0)_j \to H^n_{P_+}(M)_j \to 0.$$

Theorem 1.1, together with Proposition 2.3, Lemma 2.1 and our assumption imply that for $j \gg 0$ we have

$$\dim_{P_0} H_i\big(H^n_{P_+}(\mathbb{F})_j\big) = \dim_{P_0} H^{n-i}_{P_+}(M)_j \leqslant \dim_{P_0} M/P_+M \leqslant 1 \leqslant i \quad \text{for all } i \geqslant 1.$$

Moreover we know that

$$H^n_{P_+}(M) = H_0\big(H^n_{P_+}(\mathbb{F})\big).$$

Then by a theorem of Lazarsfeld [6, Lemma 1.6], see also [4, Theorem 12.1], one has

$$\operatorname{reg} H^n_{P_+}(M)_j = \operatorname{reg} H_0\big(H^n_{P_+}(\mathbb{F})\big)_j \leqslant \max\big\{b_i\big(H^n_{P_+}(\mathbb{F})_j\big) - i \text{ for all } i \geqslant 0\big\},$$

where $b_i(H^n_{P_+}(\mathbb{F})_j)$ is the maximal degree of the generators of $H^n_{P_+}(F_i)_j$. Note that

$$H^n_{P_+}(F_i)_j = \bigoplus_{k=1}^{t_i} \bigoplus_{|a|=-n-j+b_{ik}} P_0(-a_{ik})z^a.$$

Thus we conclude that

$$\operatorname{reg} H^n_{P_+}(M)_j \leqslant \max_{i,k}\{a_{ik} - i\} = c \quad \text{for } j \gg 0,$$

as desired. □

Next we want to give a lower bound for the functions $f_{i,M}$. We first prove

Proposition 2.6. *Let*

$$\mathbb{G}: 0 \to G_p \xrightarrow{d_p} G_{p-1} \to \cdots \to G_1 \xrightarrow{d_1} G_0 \to 0,$$

be a complex of free P_0-modules, where $G_i = \bigoplus_j P_0(-a_{ij})$ for all $i \geqslant 0$. Let $m_i = \min_j\{a_{ij}\}$. Then

$$\operatorname{reg} H_i(\mathbb{G}) \geqslant m_i.$$

Proof. Since $H_i(\mathbb{G}) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$ and $\operatorname{Ker} d_i \subseteq G_i$ for all $i \geqslant 0$, it follows that

$$\begin{aligned}\operatorname{reg} H_i(\mathbb{G}) &\geqslant \text{largest degree of generators of } H_i(\mathbb{G})\\ &\geqslant \text{lowest degree of generators of } H_i(\mathbb{G})\\ &\geqslant \text{lowest degree of generators of } \operatorname{Ker} d_i\\ &\geqslant \text{lowest degree of generators of } G_i\\ &= m_i,\end{aligned}$$

as desired. □

Corollary 2.7. *Let M be a finitely generated bigraded P-module. Then for each i, the function $f_{i,M}$ is bounded below.*

Proof. Let $\mathbb{G}$ be the complex $H^n_{P_+}(\mathbb{F})_j$ in the proof of Proposition 2.5, then the assertion follows from Proposition 2.6. □

Proof of Theorem 2.4. Because of Corollary 2.7 it suffices to show that for each i, $f_{i,M}$ is bounded above.

There exists an exact sequence $0 \to U \to F \xrightarrow{\varphi} M \to 0$ of finitely generated bigraded P-modules where F is free. This exact sequence yields the exact sequence of P_0-modules

$$0 \to H^{n-1}_{P_+}(M)_j \to H^n_{P_+}(U)_j \to H^n_{P_+}(F)_j \xrightarrow{\varphi} H^n_{P_+}(M)_j \to 0.$$

Let $K_j := \operatorname{Ker} \varphi$. We consider the exact sequences

$$0 \to K_j \to H^n_{P_+}(F)_j \to H^n_{P_+}(M)_j \to 0,$$
$$0 \to H^{n-1}_{P_+}(M)_j \to H^n_{P_+}(U)_j \to K_j \to 0.$$

Thus we have

$$\operatorname{reg} K_j \leqslant \max\bigl\{\operatorname{reg} H^n_{P_+}(F)_j, \operatorname{reg} H^n_{P_+}(M)_j + 1\bigr\}, \tag{2}$$
$$\operatorname{reg} H^{n-1}_{P_+}(M)_j \leqslant \max\bigl\{\operatorname{reg} H^n_{P_+}(U)_j, \operatorname{reg} K_j + 1\bigr\}. \tag{3}$$

Let $F = \bigoplus_{i=1}^k P(-a_i, -b_i)$, then

$$H^n_{P_+}(F)_j = \bigoplus_{i=1}^{k} \bigoplus_{|a|=-n-j+b_i} P_0(-a_i) z^a.$$

Therefore, $\operatorname{reg} H^n_{P_+}(F)_j = \max_i\{a_i\}$. By Proposition 2.7, the functions $f_{n,M}$ and $f_{n,U}$ are bounded above, so that, by the inequalities (2) and (3), $f_{n-1,M}$ is bounded above. To complete our proof, for $i > 1$ we see that

$$H^{n-i}_{P_+}(M)_j \cong H^{n-i+1}_{P_+}(U)_j.$$

Thus $f_{n-i,M} = f_{n-i+1,U}$ for $i > 1$. By induction on $i > 1$ all $f_{i,M}$ are bounded above, as required. □

3. The Hilbert function of the components of the top local cohomology of a hypersurface ring

Let R be a hypersurface ring. In this section we want to show that the Hilbert function of the P_0-module $H^n_{P_+}(R)_j$ is a nonincreasing function in j. Let $f \in P$ be a bihomogeneous form of degree (a, b). Write

$$f = \sum_{\substack{|\alpha|=a\\|\beta|=b}} c_{\alpha\beta} x^\alpha y^\beta \quad \text{where } c_{\alpha\beta} \in K.$$

We may also write $f = \sum_{|\beta|=b} f_\beta y^\beta$ where $f_\beta \in P_0$ with $\deg f_\beta = a$. The monomials y^β for which $|\beta| = b$ are ordered lexicographically induced by $y_1 > y_2 > \cdots > y_n$. We consider the hypersurface ring $R = P/fP$. From the exact sequence

$$0 \to P(-a, -b) \xrightarrow{f} P \to P/fP \to 0,$$

we get an exact sequence of P_0-modules

$$\bigoplus_{|c|=-n-j+b} P_0(-a)z^c \xrightarrow{f} \bigoplus_{|c|=-n-j} P_0 z^c \to H^n_{P_+}(R)_j \to 0.$$

We also order the bases elements z^c lexicographically induced by $z_1 > z_2 > \cdots > z_n$. Applying f to the bases elements we obtain $fz^c = \sum_{|\beta|=b} f_\beta z^{\beta-c}$, where $z^{\beta-c} = 0$ if $c \nleq \beta$ componentwise. With respect to these bases the map of free P_0-modules is given by a $\binom{-j-1}{n-1} \times \binom{-j+b-1}{n-1}$ matrix which we denote by U_j. This matrix also describes the image of this map as submodule of the free module F_j where $F_j = \bigoplus_{|c|=-n-j} P_0 z^c$, so that $H^n_{P_+}(R)_j$ is just $\operatorname{Coker} f = F_j/U_j$. Note that $H^n_{P_+}(R)_j = 0$ for all $j > -n$.

Let B_d denote the set of all monomials of degree d in the indeterminates $z_1, \ldots, z_n$. Let $h = \sum_{v \in B_{-n-j}} h_v v \in U_j$ where $h_v \in P_0$ for all v. Then $h_u u$ is called the *initial term* of h if $h_u \neq 0$ and $h_v = 0$ for all $v > u$, and we set $\operatorname{in}(h) = h_u u$. The polynomial $h_u \in P_0$ is called the *initial coefficient* and the monomial u is called the *initial monomial* of h.

Now for a monomial $u \in B_{-n-j}$ we denote $U_{j,u}$ the set of elements in U_j whose initial monomial is u, and we denote by $I_{j,u}$ the ideal generated by the initial coefficients of the elements in $U_{j,u}$.

Note that

$$U_j \setminus \{0\} = \bigcup_{u \in B_{-n-j}} U_{j,u}.$$

We fix the lexicographical order introduced above, and let $\mathrm{in}(U_j)$ be the submodule generated by $\{\mathrm{in}(h)\colon\ h \in U_j\}$. Then

$$\mathrm{in}(U_j) = \bigoplus_{u \in B_{-n-j}} I_{j,u}u. \tag{4}$$

Proposition 3.1. *With the above notation we have*

$$I_{j,u} = I_{j-1,z_1u} \quad \textit{for all } j \leqslant -n \textit{ and } u \in B_{-n-j}.$$

Proof. Let $h_0 \in I_{j,u}$. Then there exists $h \in U_j$ such that $h = h_0u + \text{lower terms}$. We set $k = -n - j + b$, for short. Since h is in the image of f, we may also write $h = \sum_{|c|=k} f_c f z^c$ where $f_c \in P_0$ and $f z^c = \sum_{\beta \leqslant c} f_\beta z^{c-\beta}$. We define $g = \sum_{|c|=k} f_c f z^{c+e_1}$ where $f z^{c+e_1} = \sum_{\beta \leqslant c+e_1} f_\beta z^{c+e_1-\beta}$ and $e_1 = (1, 0, \ldots, 0)$. We see that $g \in U_{j-1}$. We may write

$$g = \sum_{|c|=k} f_c \sum_{\beta \leqslant c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f_c \sum_{\substack{\beta \nleqslant c \\ \beta \leqslant c+e_1}} f_\beta z^{c+e_1-\beta}.$$

Thus we conclude that $g = z_1h + h_1$ where

$$h_1 = \sum_{|c|=k} f_c \sum_{\substack{\beta \nleqslant c \\ \beta \leqslant c+e_1}} f_\beta z^{c+e_1-\beta}.$$

We now claim that h_1 does not contain z_1 as a factor. For each $\alpha \in \mathbb{N}^n$ we denote by $\alpha(i)$ the ith component of α. Assume that $(c + e_1 - \beta)(1) > 0$ for some β appearing in the sum of h_1. Then $c(1) \geqslant \beta(1)$. Moreover, if $i > 1$, then $(c + e_1 - \beta)(i) \geqslant 0$ implies that $c(i) \geqslant \beta(i)$. Hence $c(i) \geqslant \beta(i)$ for all i, a contradiction. It follows that $\mathrm{in}(g) = \mathrm{in}(h)z_1$. Therefore $h_u \in I_{j-1,z_1u}$.

Conversely, suppose $h_0 \in I_{j-1,z_1u}$. Then there exists $g \in U_{j-1}$ such that $g = h_0z_1u + \text{lower terms}$. We may write $g = \sum_{|c|=k} f'_c f z^{c+e_1}$ where $f'_c \in P_0$ and $f z^{c+e_1} = \sum_{\beta \leqslant c+e_1} f_\beta z^{c+e_1-\beta}$. Thus

$$g = \sum_{|c|=k} f'_c \sum_{\beta \leqslant c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f'_c \sum_{\substack{\beta \nleqslant c \\ \beta \leqslant c+e_1}} f_\beta z^{c+e_1-\beta}.$$

As above we see that $g = z_1 f' +$ lower terms, where $f' = \sum_{|c|=k} f'_c f z^c$. We see that $f' \in U_j$, and that $\operatorname{in}(f')z_1 = \operatorname{in}(g) = h_0 z_1 u$. Therefore, $\operatorname{in}(f') = h_0 u$, and hence $h_0 \in I_{j,u}$. □

Let M and N be graded P_0-modules. We denote by $\operatorname{Hilb}(M) = \sum_{i\in\mathbb{Z}} \dim_K M_i t^i$ the Hilbert-series of M. We write $\operatorname{Hilb}(M) \leqslant \operatorname{Hilb}(N)$ when $\dim_K M_i \leqslant \dim_K N_i$ for all i.

Let F be a free P_0-module with basis $\beta = \{u_1, \ldots, u_r\}$. Let U be a graded submodule of F. For $f \in U$, we write $f = \sum_{i=1}^r f_i u_i$ where $f_i \in P_0$. We set $\operatorname{in}(f) = f_j u_j$ where $f_j \neq 0$ and $f_i = 0$ for all $i < j$. We also set $\operatorname{in}(U)$ be the submodule of F generated by all $\operatorname{in}(f)$ such that $f \in U$. Let I be a homogeneous ideal of P_0. We say that set of homogeneous elements of P_0 forms a K-basis for P_0/I if its image forms a K-basis for P_0/I. Now we can state the following result which is related to a theorem of Macaulay [2, Theorem 4.2.3], see also [2, Corollary 4.2.4]. For the convenience of the reader we include its proof.

Lemma 3.2. *With notation as above we have*

$$\operatorname{Hilb}(F/U) = \operatorname{Hilb}\bigl(F/\operatorname{in}(U)\bigr).$$

Proof. As in (4) we have $\operatorname{in}(U) = \bigoplus_{i=1}^r I_{u_i} u_i$ where I_{u_i} is the ideal generated by all $f_i \in P_0$ such that there exists $f \in F$ with $\operatorname{in}(f) = f_i u_i$. Thus we have $F/\operatorname{in}(U) = \bigoplus_{i=1}^r P_0/I_{u_i}$. For each j let β_j be a set of homogeneous elements $h_{ij} \in P_0$ which forms a K-basis of P_0/I_{u_j}. Then $\beta = \{\beta_1 u_1, \ldots, \beta_r u_r\}$ is a homogeneous K-basis of $F/\operatorname{in}(U)$. To complete our proof we will show that β is also a K-basis of F/U. We first show that the elements of β in F/U are linearly independent. Suppose that in F/U, we have $\sum_{i,j} a_{ij} h_{ij} u_j = 0$ with $a_{ij} \in K$. Thus $\sum_{j=1}^r (\sum_i a_{ij} h_{ij}) u_j \in U$. We set $h_j = \sum_i a_{ij} h_{ij}$, so that $h_1 u_1 + \cdots + h_r u_r \in U$. If all $h_j = 0$, then $a_{ij} = 0$ for all i and j, as required. Assume that $h_j \neq 0$ for some j, and let k be the smallest integer such that $h_k \neq 0$. It follows that $h_k u_k + h_{k+1} u_{k+1} + \cdots \in U$, so that $h_k \in I_k$, and hence $\sum_i a_{ik} h_{ik} = 0$ modulo I_k. Since h_{ik} are part of a K-basis of P_0/I_k, it follows that $a_{ik} = 0$ for all i, and hence $h_k = 0$, a contradiction.

Now we want to show that each element in F/U can be written as a K-linear combination of elements of β. Let $f + U \in F/U$ where $f \in F$. Thus there exists $f_i \in P_0$ such that $f = \sum_{i=1}^r f_i u_i$. Since $f_1 + I_{u_1} \in P_0/I_{u_1}$, there exists $\lambda_{i1} \in K$ such that $f_1 + I_{u_1} = \sum_i \lambda_{i1}(h_{i1} + I_{u_1})$, so that $f_1 = \sum_i \lambda_{i1} h_{i1} + h_{u_1}$ for some $h_{u_1} \in I_{u_1}$. Hence

$$f = \sum_i \lambda_{i1} h_{i1} u_1 + h_{u_1} u_1 + \sum_{i=2}^r f_i u_i.$$

We set

$$f' = f - \sum_i \lambda_{i1} h_{i1} u_1 = h_{u_1} u_1 + \sum_{i=2}^r f_i u_i.$$

Since $h_{u_1} \in I_{u_1}$, there exist $g_2, \ldots, g_r \in P_0$ such that $h_{u_1} u_1 + \sum_{i=2}^r g_i u_i \in U$. Therefore, $h_{u_1} u_1 = -\sum_{i=2}^r g_i u_i$ modulo U. Hence it follow that

$$f' = -\sum_{i=2}^{r} g_i u_i + \sum_{i=2}^{r} f_i u_i = \sum_{i=2}^{r} f_i' u_i \quad \text{modulo } U.$$

Here $f_i' = -g_i + f_i$ for $i = 2, \dots, r$. By induction on the number of summands, we may assume that $\sum_{i=2}^{r} f_i' u_i$ is a linear combination of elements of β modulo U. Since f differs from f' only by a linear combination of elements of β, the assertion follows. □

Now we are able to prove that the Hilbert-series of the P_0-module $H^n_{P_+}(R)_j$ is a nonincreasing function in j.

Theorem 3.3. *Let $R = P/fP$ be a hypersurface ring. Then*

$$\operatorname{Hilb}\bigl(H^n_{P_+}(R)_{j-1}\bigr) \geqslant \operatorname{Hilb}\bigl(H^n_{P_+}(R)_j\bigr) \quad \textit{for all } j \leqslant -n.$$

Proof. Let $F_j = \bigoplus_{u \in B_{-n-j}} P_0 u$ where $u = z_1^{a_1} \dots z_n^{a_n}$ with $\sum_{i=1}^{n} a_i = -n - j$. In view of (4) we have $F_j / \operatorname{in}(U_j) = \bigoplus_{u \in B_{-n-j}} P_0 / I_{j,u}$. By Lemma 3.2 we know that F_j / U_j and $F_j / \operatorname{in}(U_j)$ have the same Hilbert function. Thus Proposition 3.1 implies that for all $j \leqslant -n$ we have

$$\begin{aligned}
\operatorname{Hilb}\bigl(H^n_{P_+}(R)_j\bigr) = \operatorname{Hilb}(F_j/U_j) &= \sum_i \dim_K \Biggl(\bigoplus_{u \in B_{-n-j}} P_0/I_{j,u}\Biggr)_i t^i \\
&= \sum_i \sum_{u \in B_{-n-j}} \dim_K (P_0/I_{j,u})_i t^i \\
&= \sum_i \sum_{u \in B_{-n-j}} \dim_K (P_0/I_{j-1,z_1 u})_i t^i \\
&= \sum_i \sum_{\substack{v \in B_{-n-j+1} \\ a_1 > 0}} \dim_K (P_0/I_{j-1,v})_i t^i \\
&\leqslant \sum_i \sum_{v \in B_{-n-j+1}} \dim_K (P_0/I_{j-1,v})_i t^i \\
&= \sum_i \dim_K \Biggl(\bigoplus_{v \in B_{-n-j+1}} P_0/I_{j-1,v}\Biggr)_i t^i = \operatorname{Hilb}\bigl(H^n_{P_+}(R)_{j-1}\bigr),
\end{aligned}$$

as desired. □

Corollary 3.4. *Let R be the hypersurface ring P/fP such that the P_0-module $H^n_{P_+}(R)_j$ has finite length for all j. Then*

$$\operatorname{reg} H^n_{P_+}(R)_{j-1} \geqslant \operatorname{reg} H^n_{P_+}(R)_j \quad \textit{for all } j \leqslant -n.$$

Proof. The assertion follows from the fact that

$$\operatorname{reg} H^n_{P_+}(R)_j = \deg \operatorname{Hilb}\bigl(H^n_{P_+}(R)_j\bigr). \qquad \square$$

Now one could ask when P_0-module $H^n_{P+}(R)_j$ is of finite length. To answer this question we need some preparation. Let A be a Noetherian ring and M be a finitely generated A-module with presentation

$$A^m \xrightarrow{\varphi} A^n \to M \to 0.$$

Let U be the corresponding matrix of the map φ and $I_{n-i}(U)$ for $i = 0, \dots, n-1$ be the ideal generated by the $(n-i)$-minors of matrix U. Then $\operatorname{Fitt}_i(M) := I_{n-i}(U)$ is called the ith Fitting ideal of M. We use the convention that $\operatorname{Fitt}_i(M) = 0$ if $n - i > \min\{n, m\}$, and $\operatorname{Fitt}_i(M) = A$ if $i \geqslant n$. In particular, we obtain $\operatorname{Fitt}_r(M) = 0$ if $r < 0$, $\operatorname{Fitt}_0(M)$ is generated by the n-minors of U, and $\operatorname{Fitt}_{n-1}(M)$ is generated by all entries of U. Note that $\operatorname{Fitt}_i(M)$ is an invariant on M, i.e., independent of the presentation. By [5, Proposition 20.7] we have $\operatorname{Fitt}_0(M) \subseteq \operatorname{Ann} M$ and if M can be generated by r elements, then $(\operatorname{Ann} M)^r \subseteq \operatorname{Fitt}_0(M)$. Thus we can conclude that $\sqrt{\operatorname{Fitt}_0(M)} = \sqrt{\operatorname{Ann} M}$. Therefore

$$\dim M = \dim A/\operatorname{Ann} M = \dim A/I_n(U). \tag{5}$$

Now we can state the following

Proposition 3.5. *Let R be the hypersurface ring P/fP, and $I(f)$ the ideal generated by all the coefficients of f. Then $\dim_{P_0} H^n_{P+}(R)_j \leqslant \dim P_0/I(f)$. In particular, if $I(f)$ is $\mathfrak{m}$-primary where $\mathfrak{m} = (x_1, \dots, x_n)$, then P_0-module $H^n_{P+}(R)_j$ is of finite length for all j.*

Proof. Note that $H^n_{P+}(R)_j = 0$ for $j > -n$. Therefore we may suppose that $j \leqslant -n$. As we have already seen, $H^n_{P+}(R)_j$ has P_0-presentation

$$P_0^{n_1}(-a) \xrightarrow{\varphi} P_0^{n_0} \to H^n_{P+}(R)_j \to 0,$$

where $n_0 = \binom{-j-1}{n-1}$ and $n_1 = \binom{-j+b-1}{n-1}$. In view of (5) we have $\dim_{P_0} H^n_{P+}(R)_j = \dim P_0/I_{n_0}(U_j)$ where U_j is the corresponding matrix of the map φ. By [9, Lemma 1.4] we have $\sqrt{I(f)} \subseteq \sqrt{I_{n_0}(U_j)}$. It follows that $\dim_{P_0} H^n_{P+}(R)_j \leqslant \dim P_0/I(f)$. Since $I(f)$ is $\mathfrak{m}$-primary it follows that $\dim P_0/I(f) = 0$. Therefore $\dim_{P_0} H^n_{P+}(R)_j = 0$, and hence $H^n_{P+}(R)_j$ has finite length, as required. $\square$

4. The regularity of the graded components of local cohomology for a special class of hypersurfaces

Let $A = \bigoplus_{i=0}^n A_i$ be a standard graded Artinian K-algebra, where K is a field of characteristic 0. We say that A has the weak Lefschetz property if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l} A_{i+1}$ has maximal rank for all i. This means the corresponding matrix has maximal rank, i.e., l is either injective *or* surjective. Such an element l is called a weak Lefschetz element on A. We also say that A has the strong Lefschetz property if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l^k} A_{i+k}$ has maximal rank for all i and k. Such an element l is called a strong Lefschetz element on A. Note that the set of all weak Lefschetz elements on A is a Zariski-open subset of the affine space A_1, and the same holds for the set of all strong Lefschetz elements on A. For an algebra A as above, we say that A has the strong Stanley property (SSP) if there exists $l \in A_1$ such that $l^{n-2i} : A_i \to A_{n-i}$ is bijective for $i = 0, 1, \ldots, [n/2]$. Note that the Hilbert function of standard graded K-algebra satisfying the weak Lefschetz property is unimodal. Stanley [10] and Watanabe [11] proved the following result: Let $a_1, \ldots, a_n$ be the integers such that $a_i \geqslant 1$ and assume as always in this section that $\operatorname{char} K = 0$. Then $A = K[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$ has the strong Lefschetz property.

Theorem 4.1. *Let $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$ and $n \geqslant 2$, and assume that* $\operatorname{char} K = 0$. *Then there exists a Zariski open subset $V \subset K^n$ such that for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in V$ one has*

$$\operatorname{reg} H^n_{P+}\big(P/f_\lambda^r P\big)_j = -n - j + r - 1.$$

Proof. We first prove the theorem in the case that $f = f_{(1,\ldots,1)} = \sum_{i=1}^n x_i y_i$, and set $R = P/f^r P$. From the exact sequence

$$0 \to P(-r, -r) \xrightarrow{f^r} P \to R \to 0,$$

we get an exact sequence of P_0-modules,

$$\bigoplus_{|b|=-n-j+r} P_0(-r) z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0. \tag{6}$$

Note that $H^n_{P+}(R)_j$ is generated by elements of degree 0 and the ideal generated by the coefficients of f is $\mathfrak{m}$-primary. By Proposition 3.5, we need only to show that

(a) $\big[H^n_{P+}(R)_j\big]_{-n-j+r-1} \neq 0$, and

(b) $\big[H^n_{P+}(R)_j\big]_{-n-j+r} = 0$.

Let $k = -n - j$ for short. For the proof of (a), we take the $(k + r - 1)$th component of the exact sequence (6), and obtain the exact sequence of K-vector spaces

$$\bigoplus_{\substack{|a|=k-1\\|b|=k+r}} K x^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r-1\\|b|=k}} K x^a z^b \to \big[H^n_{P_+}(R)_j\big]_{k+r-1} \to 0.$$

We set

$$V_{\alpha,\beta} := \bigoplus_{\substack{|a|=\alpha\\|b|=\beta}} Kx^a z^b.$$

Hence one has $\dim_K V_{k-1,k+r} = \binom{n+k-2}{k-1}\binom{n+k+r-1}{k+r}$ which is less than $\dim_K V_{k+r-1,k} = \binom{n+k+r-2}{k+r-1}\binom{n+k-1}{k}$ for $n \geqslant 2$. Thus f^r is not surjective, so (a) follows. For the proof of (b), we take the $(k+r)$th component of the exact sequence (6), and obtain the exact sequence of K-vector spaces

$$\bigoplus_{\substack{|a|=k\\|b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r\\|b|=k}} Kx^a z^b \to \left[H^n_{P_+}(R)_j\right]_{k+r} \to 0.$$

Note that $\dim_K V_{k,k+r} = \dim_K V_{k+r,k}$. We will show that f^r is an isomorphism, then we are done. We fix $c \in \mathbb{N}_0^n$ such that $c = (c_1, \dots, c_n)$ where $c_i \geqslant 0$. We set

$$V^c_{\alpha,\beta} := \bigoplus_{\substack{|a|=\alpha\\|b|=\beta\\a+b=c}} Kx^a z^b \quad \text{and} \quad A^c_i := \bigoplus_{\substack{|a|=i\\a\leqslant c}} Kx^a.$$

We define $\varphi : V^c_{k,k+r} \to A^c_k$ by setting $\varphi(x^a z^b) = x^a$. Note that φ is an isomorphism of K-vector spaces. Let $A^c = \bigoplus_{i=0}^{|c|} A^c_i$. We can define an algebra structure on A^c. For $x^s, x^t \in A^c$ we define

$$x^s x^t = \begin{cases} x^{s+t}, & \text{if } s+t \leqslant c,\\ 0, & \text{if } s+t \nleqslant c.\end{cases}$$

A K-basis of A^c is given by all monomials x^a with $a \leqslant c$. It follows that

$$A^c = K[x_1, \dots, x_n]\big/\big(x_1^{c_1+1}, \dots, x_n^{c_n+1}\big).$$

Now we see that the map

$$V_{k,k+r} = \bigoplus_{|c|=2k+r} V^c_{k,k+r} \xrightarrow{f^r} \bigoplus_{|c|=2k+r} V^c_{k+r,k} = V_{k+r,k}$$

is an isomorphism if and only if restriction map $f' := f^r|_{V^c_{k,k+r}} : V^c_{k,k+r} \to V^c_{k+r,k}$ is an isomorphism for all c with $|c| = 2k+r$.

For each such c we have a commutative diagram

$$\begin{array}{ccc} V^c_{k,k+r} & \xrightarrow{f'} & V^c_{k+r,k} \\ \downarrow & & \downarrow \\ A^c_k & \xrightarrow{l^r} & A^c_{k+r} \end{array}$$

with $l = x_1 + x_2 + \cdots + x_n \in A^c_1$ and where $A^c_k \xrightarrow{l^r} A^c_{k+r}$ is multiplication by l^r in the K-algebra A^c. Since the socle degree of A^c equals $s = 2k + r$, we have $k + r = s - k$. Therefore the multiplication map $l^r : A^c_k \to A^c_{s-k}$ with $r = s - 2k$ is an isomorphism by the strong Stanley property of the algebra A^c, see [11, Corollary 3.5].

Now if we replace f by f_λ, then the corresponding linear form in the above commutative diagram is the form $l_\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$. It is known that the property of l_λ to be a weak Lefschetz element is an open condition, that is, there exists a Zariski open set $V \subset K^n$ such that l_λ is a weak Lefschetz element. This open set is not empty since $\lambda = (1, \ldots, 1) \in V$. Since any weak Lefschetz element satisfies (SSP), we can replace in the above proof f by f_λ for each $\lambda \in V$, and obtain the same conclusion. □

Remark 4.2. It is now the time that to show Theorem 2.4 may fail without the assumption that $\dim_{P_0} M/P_+M \leqslant 1$. In case of Theorem 4.1 we have $M = R = P/f^r_\lambda P$, and so $M/P_+M = P_0$. Therefore in that case $\dim_{P_0} M/P_+M = \dim_{P_0} P_0 = n \geqslant 2$, and in fact $f_{n,R}$ is not bounded.

Now in Theorem 4.1, we want to compute the Hilbert function of the P_0-module $H^n_{P_+}(R)_j$.

Corollary 4.3. *With the assumption of Theorem* 4.1*, we have*

$$\dim_K \big(H^n_{P_+}(R)_j\big)_i = \begin{cases} \binom{n+i-1}{i}\binom{-j-1}{-n-j}, & \text{if } i \leqslant r, \\ \binom{n+i-1}{i}\binom{-j-1}{-n-j} - \binom{n+i-r-1}{i-r}\binom{-j+r-1}{-n-j+r}, & \text{if } r \leqslant i \leqslant -n-j+r-1. \end{cases}$$

Proof. We set $-n - j = k$, for short. We take ith component of exact sequence (6), and obtain the exact sequence of K-vector space

$$\bigoplus_{\substack{|a|=i-r \\ |b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=i \\ |b|=k}} Kx^a z^b \to \big[H^n_{P_+}(R)_j\big]_i \to 0.$$

If $i \leqslant r$, from the above exact sequence we see that

$$\dim_K \big(H^n_{P_+}(R)_j\big)_i = \dim_K V_{i,k} = \binom{n+i-1}{i}\binom{-j-1}{-n-j}.$$

Now let $r \leqslant i \leqslant -n-j+r-1$. First one has $\dim_K V_{i-r,k+r} < \dim_K V_{i,k}$. We claim that f^r is injective, then we are done. We see that the map

$$V_{i-r,k+r} = \bigoplus_{|c|=i+k} V^c_{i-r,k+r} \xrightarrow{f^r} \bigoplus_{|c|=i+k} V^c_{i,k} = V_{i,k},$$

where $f^r(V^c_{i-r,k+r}) \subset V^c_{i,k}$ is injective if and only if restriction map $f' := f^r|_{V^c_{i-r,k+r}} : V^c_{i-r,k+r} \to V^c_{i,k}$ is injective for all c with $|c| = i+k$.

For each such c we have a commutative diagram

$$\begin{array}{ccc} V^c_{i-r,k+r} & \xrightarrow{f'} & V^c_{i,k} \\ \downarrow & & \downarrow \\ A^c_{i-r} & \xrightarrow{l^r} & A^c_i \end{array}$$

with $l = x_1 + x_2 + \cdots + x_n \in A^c_1$. Since $i < -n-j+r$, then $i < |c| - (i-r)$ and by the weak Lefschetz property the algebra A^c is unimodal. Therefore $\dim_K A^c_{i-r} \leqslant \dim_K A^c_i$. The strong Lefschetz property implies that the map l^r is injective, and hence f' is injective, as required. □

Corollary 4.4. *Assume that* char $K = 0$. *Then with the notation of Theorem* 4.1*, we have*

$$\operatorname{reg} H^{n-1}_{P+}\bigl(P/f^r_\lambda P\bigr)_j = -n-j+r+1.$$

Proof. We consider the exact sequence of P_0-modules

$$0 \to H^{n-1}_{P+}(R)_j \to \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P+}(R)_j \to 0, \qquad (7)$$

where $R = P/f^r_\lambda P$. It follows that $H^{n-1}_{P+}(R)_j$ is the second syzygy module of $H^n_{P+}(R)_j$. Let

$$\cdots \to \bigoplus_{j=1}^{t_2} P_0(-a_{1j}) \to \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \to H^{n-1}_{P+}(R)_j \to 0$$

be the minimal graded free resolution of $H^{n-1}_{P+}(R)_j$. We combine two above resolutions, and obtain a graded free resolution for $H^n_{P+}(R)_j$ of the form

$$\cdots \to \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \xrightarrow{d_0} \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P+}(R)_j \to 0.$$

We choose a basis element $h \in \bigoplus_{j=1}^{t_1} P_0(-a_{0j})$ of degree a_{0j}. Thus

$$d_0(h) = \sum_{|b|=-n-j+r} h_b z^b,$$

where $h_b \in P_0$ with $\deg h_b = a_{0j} - r$. Because the free resolution is minimal, at least one $h_b \neq 0$, so that $r < a_{0j}$ and hence $r - 1 \leqslant a_{0j} - 2$. Thus we have

$$\operatorname{reg} H^n_{P+}(R)_j = \max_{i,j}\{0, r-1, a_{ij} - i - 2\} = \max_{i,j}\{a_{ij} - i - 2\}.$$

Theorem 4.1 implies that

$$\operatorname{reg} H^{n-1}_{P+}(R)_j = \max_{i,j}\{a_{ij} - i\} = -n - j + r + 1. \qquad \square$$

Corollary 4.5. *Assume that* $\operatorname{char} K = 0$. *Then with the notation of Theorem* 4.1 *the* P_0*-module* $H^{n-1}_{P+}(P/f_\lambda^r P)_j$ *has a linear resolution.*

Proof. Taking the kth component of the exact sequence (7), we obtain the exact sequence of K-vector spaces

$$0 \to \big[H^{n-1}_{P_+}(R)_j\big]_k \to \bigoplus_{\substack{|a|=k-r\\ |b|=-n-j+r}} K x^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k\\ |b|=-n-j}} K x^a z^b \to \big[H^n_{P_+}(R)_j\big]_k \to 0.$$

For k we distinguish several cases. Let $k = -n - j + r + 1$. One has

$$\dim_K V_{k-r,-n-j+r} > \dim_K V_{k,-n-j}.$$

This implies that

$$\big[H^{n-1}_{P_+}(R)_j\big]_k \neq 0 \quad \text{for all } k \geqslant -n - j + r + 1,$$

since $H^{n-1}_{P_+}(R)_j$ is torsion-free.

Let $k = -n-j+r$. Then $\dim_K V_{k-r,-n-j+r} = \dim_K V_{k,-n-j}$, so that $[H^{n-1}_{P_+}(R)_j]_k = 0$. Finally let $k < -n - j + r$. We claim that

$$\dim_K V_{k-r,-n-j+r} = \binom{n+k-r-1}{k-r}\binom{-j+r-1}{-n-j+r}$$

is less than

$$\dim_K V_{k,-n-j} = \binom{n+k-1}{k}\binom{-j-1}{-n-j}.$$

Indeed,

$$\binom{n+k-r-1}{k-r}\binom{-j+r-1}{-n-j+r} < \binom{n+k-1}{k}\binom{-j-1}{-n-j} \quad \text{if and only if}$$
$$\prod_{i=1}^{r} \frac{-j+r-i}{-n-j+r-i+1} < \prod_{i=1}^{r} \frac{n+k-i}{k-i+1}.$$

Since

$$\frac{-j+r-i}{-n-j+r-i+1} < \frac{n+k-i}{k-i+1} \quad \text{for all } i=1,\dots,r \qquad \text{if and only if}$$
$$k(n-1) < (-n-j+r)(n-1),$$

the claim is clear. Thus the regularity of $H^{n-1}_{P+}(R)_j$ is equal to the least integer k such that $[H^{n-1}_{P_+}(R)_j]_k \neq 0$. This means that P_0-module $H^{n-1}_{P+}(R)_j$ has a linear resolution, and its resolution is the form

$$\cdots \to P_0^{\beta_3}(n+j-r-2) \to P_0^{\beta_2}(n+j-r-1) \to H^{n-1}_{P_+}(R)_j \to 0. \qquad \square$$

Combining the above resolution with the exact sequence

$$0 \to H^{n-1}_{P+}(R)_j \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^n_{P+}(R)_j \to 0,$$

we obtain a graded free resolution for $H^n_{P+}(R)_j$ of the form

$$\cdots \to P_0^{\beta_3}(n+j-r-2) \to P_0^{\beta_2}(n+j-r-1) \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^n_{P+}(R)_j \to 0.$$

In this resolution we know already the Betti numbers

$$\beta_0 = \binom{-j-1}{-n-j} \quad \text{and} \quad \beta_1 = \binom{-j+r-1}{-n-j+r}.$$

Next we are going to compute the remaining Betti numbers and also the multiplicity of $H^n_{P_+}(R)_j$. For this we need to prove the following extension of the formula of Herzog and Kühl [2].

Proposition 4.6. *Let M be a finitely generated graded Cohen–Macaulay P_0-module of codimension s with minimal graded free resolution*

$$0 \to P_0^{\beta_s}(-d_s) \to \cdots \to P_0^{\beta_1}(-d_1) \to P_0^{\beta_0} \to M \to 0.$$

Then

$$\beta_i = (-1)^{i+1}\beta_0 \prod_{j\neq i} \frac{d_j}{(d_j-d_i)}.$$

Proof. We consider the square matrix A of size s and the following $s \times 1$ matrices of X and Y:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & d_2^{s-1} & \cdots & d_s^{s-1} \end{pmatrix}, \qquad X = \begin{pmatrix} -\beta_1 \\ \beta_2 \\ \vdots \\ (-1)^s \beta_s \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -\beta_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

With similar arguments as in the proof of Lemma 1.1 in [8] one has

$$\sum_{i=1}^{s} (-1)^i \beta_i d_i^k = \begin{cases} 0 & \text{for } 1 \leqslant k < s, \\ (-1)^s s! e(M) & \text{for } k = s. \end{cases}$$

Note that $\sum_{i=1}^{s} (-1)^i \beta_i = \beta_0$. Thus we can conclude that $AX = Y$. Now we can apply Cramer's rule for the computation of β_i. We replace the ith column of A by Y, then we expand the determinant $|A|$ of A along to the Y, we get $\beta_i = -\beta_0 |A'|/|A|$ where A' is the matrix

$$\begin{pmatrix} d_1 & \cdots & d_{i-1} & d_{i+1} & \cdots & d_s \\ d_1^2 & \cdots & d_{i-1}^2 & d_{i+1}^2 & \cdots & d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & \cdots & d_{i-1}^{s-1} & d_{i+1}^{s-1} & \cdots & d_s^{s-1} \end{pmatrix},$$

of size $s - 1$. A is a Vandermonde matrix whose determinant is $\prod_{1 \leqslant j < i \leqslant s} (d_i - d_j)$. We also note that

$$|A'| = \prod_{j \neq i} d_j \prod_{\substack{1 \leqslant t < k \leqslant s \\ t \neq i}} (d_k - d_t),$$

so the desired formula follows. □

We also have the following generalization of a formula of Huneke and Miller [7].

Proposition 4.7. *With the assumption of Proposition* 4.6*, we have*

$$e(M) = \frac{\beta_0}{s!} \prod_{i=1}^{s} d_i.$$

Proof. We consider the square matrix

$$M = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} & \beta_s d_s \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 & \beta_s d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^s & \beta_2 d_2^s & \cdots & \beta_{s-1} d_{s-1}^s & \beta_s d_s^s \end{pmatrix} \tag{8}$$

of size s.

We will compute the determinant $|M|$ of M in two different ways. First we replace the last column of M by the alternating sum of all columns of M. The resulting matrix will be denoted by M'. It is clear that $|M| = (-1)^s|M'|$. Moreover, due to [8, Lemma 1.1], the last column of M' is the transpose of the vector $(0, \dots, 0, (-1)^s se(M))$. Thus if we expand M' with respect to the last column we get

$$|M| = (-1)^s|M'| = s!e(M)|N|,$$

where N is the matrix

$$N = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_{s-1} d_{s-1}^{s-1} \end{pmatrix}$$

of size $s-1$. Thus

$$|M| = s!e(M) \prod_{i=1}^{s-1} \beta_i \prod_{i=1}^{s-1} d_i \left| V(d_1, \dots, d_{s-1}) \right|, \tag{9}$$

where $V(d_1, \dots, d_{s-1})$ is the Vandermonde matrix of size $s-1$ whose determinant is $\prod_{1 \leqslant j < i \leqslant s-1}(d_i - d_j)$. On the other hand, directly from (8) we get

$$|M| = \prod_{i=1}^{s} \beta_i \prod_{i=1}^{s} d_i \left| V(d_1, \dots, d_s) \right|, \tag{10}$$

where $V(d_1, \dots, d_s)$ is the Vandermonde matrix of size s whose determinant is $\prod_{1 \leqslant j < i \leqslant s}(d_i - d_j)$. In view of (9) and (10) we get the desired formula. □

Now we are able to compute all Betti numbers and the multiplicity of $H^n_{P+}(R)_j$. We recall that its resolution is the form

$$\begin{aligned} 0 &\to P_0^{\beta_n}(j-r+1) \to P_0^{\beta_{n-1}}(j-r+2) \to \cdots \to P_0^{\beta_3}(n+j-r-2) \\ &\to P_0^{\beta_2}(n+j-r-1) \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^n_{P_+}(R)_j \to 0, \end{aligned}$$

$$\text{where} \quad \beta_0 = \binom{-j-1}{-n-j} \quad \text{and} \quad \beta_1 = \binom{-j+r-1}{-n-j+r}.$$

Corollary 4.8. *With the above notation we have*

$$\beta_i = \frac{(-1)^i r(n-1)!\beta_0\beta_1}{(i-2)!(n-i)!(-n-j+r+i-1)(n+j-i+1)} \quad \textit{for all } i \geqslant 2,$$

and

$$e\big(H^n_{P_+}(R)_j\big) = \frac{r(-j+r-1)!\beta_0}{n!(-n-j+r)!}.$$

Proof. The assertion follows from Propositions 4.7 and 4.6. □

5. Linear bounds for the regularity of the graded components of local cohomology for hypersurface rings

In this section for a bihomogeneous polynomial $f \in P$ we want to give a linear bound for the function $f_{i,R}(j) = \operatorname{reg} H^i_{P_+}(R)_j$ where $R = P/fP$. First we prove the following

Proposition 5.1. *Let R be the hypersurface ring P/fP where $f = \sum_{i=1}^n f_i y_i$ with $f_i \in P_0$. Suppose that* $\deg f_i = d$ *and that $I(f)$ is $\mathfrak{m}$-primary. Then there exists an integer q such that for $j \ll 0$ we have*

(a) $\operatorname{reg} H^n_{P_+}(R)_j \leqslant (-n-j+1)d + q$, *and*
(b) $\operatorname{reg} H^{n-1}_{P_+}(R)_j \leqslant (-n-j+1)d + q + 2$.

Proof. (a) From the exact sequence $0 \to P(-d,-1) \xrightarrow{f} P \to R \to 0$, we get exact sequence P_0-modules

$$\bigoplus_{|b|=-n-j+1} P_0(-d)z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0. \tag{11}$$

We first assume that $f_i = x_i$. Theorem 4.1 implies that $\operatorname{reg} H^n_{P_+}(R)_j = -n-j$. We set $k = -n-j$. Thus we can get the surjective map of K-vector spaces

$$\bigoplus_{\substack{|a|=k\\|b|=k+1}} Kx^a z^b \to \bigoplus_{\substack{|a|=k+1\\|b|=k}} Kx^a z^b.$$

Replacing x_i by f_i, we therefore get a surjective map

$$\bigoplus_{|b|=k+1} \big(I(f)^k\big)_{dk} z^b = \bigoplus_{\substack{|a|=k\\|b|=k+1}} Kf_1^{a_1}\dots f_n^{a_n} z^b$$
$$\xrightarrow{f} \bigoplus_{\substack{|a|=k+1\\|b|=k}} Kf_1^{a_1}\dots f_n^{a_n} z^b = \bigoplus_{|b|=k} \big(I(f)^{k+1}\big)_{d(k+1)} z^b.$$

Since $I(f)$ is $\mathfrak{m}$-primary by [3, Theorem 2.4] there exists an integer q such that

$$\operatorname{reg}\bigl(P_0/I(f)^{k+1}\bigr)=(k+1)d+q \quad \text{for } k\gg 0.$$

We set $l=(k+1)d+q$. Then for $l\gg 0$ we have

$$(P_0)_{l+1}=\bigl(I(f)^{k+1}\bigr)_{l+1}.$$

We take the $(l+1)$th component of the exact sequence (11) and consider the following diagram

$$\begin{array}{ccccccc}
\bigoplus_{|b|=k+1}(P_0)_{l-d+1}z^b & \longrightarrow & \bigoplus_{|b|=k}(P_0)_{l+1}z^b & \longrightarrow & [H^n_{P_+}(R)_j]_{l+1} & \longrightarrow & 0\\
\uparrow & & \| & & & & \\
\bigoplus_{|b|=k+1}(I(f)^k)_{l-d+1}z^b & \longrightarrow & \bigoplus_{|b|=k}(I(f)^{k+1})_{l+1}z^b & & \longrightarrow & & 0
\end{array}$$

in which left-hand vertical homomorphism is inclusion. Thus we conclude that

$$\bigl[H^n_{P_+}(R)_j\bigr]_{l+1}=0,$$

so that $\operatorname{reg} H^n_{P_+}(R)_j\leqslant l=(k+1)d+q$, as required.

For the proof (b), we notice that the exact sequence of P_0-modules of (11) breaks into two short exact sequence of P_0-modules

$$0\to K_j\to \bigoplus_{|b|=k} P_0z^b\to H^n_{P_+}(R)_j\to 0,$$

$$0\to H^{n-1}_{P_+}(R)_j\to \bigoplus_{|b|=k+1} P_0(-d)z^b\to K_j\to 0,$$

where $K_j=\operatorname{Im} f$. We see from the first of these sequences that $\operatorname{reg} K_j\leqslant \operatorname{reg} H^n_{P_+}(R)_j+1$. The second short exact sequence, together with part (a) of this theorem and the fact that $d\leqslant \operatorname{reg} K_j$ implies that

$$\operatorname{reg} H^{n-1}_{P_+}(R)_j\leqslant \max\{d,\operatorname{reg} K_j+1\}=\operatorname{reg} K_j+1\leqslant(-n-j+1)d+q+2,$$

as desired. □

Proposition 5.2. *Let* $\mathbb{N}^n_d=\{\beta\in\mathbb{N}^n\colon |\beta|=d\}$, $P_0=K[\{x_\beta\}_{\beta\in\mathbb{N}^n_d}]$ *and* $P=P_0[y_1,\ldots,y_n]$. *Let* $R=P/fP$ *where* $f=\sum_{|\beta|=d}x_\beta y^\beta$. *Then*

$$\operatorname{reg} H^n_{P_+}(R)_j\leqslant(-n-j+1)d-1.$$

Proof. We set $P_+ = (y_1, \dots, y_n)$ and $P_0 = K[x_1, \dots, x_m]$ where $m = \binom{n+d-1}{d}$, as usual. From the exact sequence

$$0 \to P(-1, -d) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-n-j+d} P_0(-1)\left(y^b\right)^* \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0\left(y^b\right)^* \to H^n_{P_+}(R)_j \to 0,$$

whose ith graded component is

$$\bigoplus_{\substack{|a|=i-1\\ |b|=-n-j+d}} Kx^a\left(y^b\right)^* \xrightarrow{f} \bigoplus_{\substack{|a|=i\\ |b|=-n-j}} Kx^a\left(y^b\right)^* \to H^n_{P_+}(R)_{(i,j)} \to 0. \tag{12}$$

Here $(y^b)^* = z^b$ in the notation of Section 1. Now we exchange the role of x and y: We may write $f = \sum_{|\beta|=d} y^\beta x_\beta$ and set $Q_+ = (x_1, \dots, x_m)$ and $Q_0 = K[y_1, \dots, y_n]$. From the exact sequence

$$0 \to P(-d, -1) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-m-t+1} Q_0(-d)\left(x^b\right)^* \xrightarrow{f} \bigoplus_{|b|=-m-t} Q_0\left(x^b\right)^* \to H^m_{Q_+}(R)_t \to 0,$$

whose sth graded component is

$$\bigoplus_{\substack{|a|=s-d\\ |b|=-m-t+1}} Ky^a\left(x^b\right)^* \xrightarrow{f} \bigoplus_{\substack{|a|=s\\ |b|=-m-t}} Ky^a\left(x^b\right)^* \to H^m_{Q_+}(R)_{(s,t)} \to 0.$$

Applying the functor $\mathrm{Hom}_K(-, K)$ to the above exact sequence and due to the exact sequence (12) we have

$$0 \to H^m_{Q_+}(R)^*_{(s,t)} \to \bigoplus_{\substack{|a|=s\\ |b|=-m-t}} K\left(y^a\right)^* x^b \xrightarrow{f} \bigoplus_{\substack{|a|=s-d\\ |b|=-m-t+1}} K\left(y^a\right)^* x^b$$
$$\to H^n_{P_+}(R)_{(-m-t+1,-n-s+d)} \to 0.$$

Therefore

$$H^m_{Q_+}(R)^*_{(s,t)} \cong H^{n-1}_{P_+}(R)_{(-m-t+1,-n-s+d)}.$$

Thus we have

$$0 \to \big(H^{n-1}_{P_+}(R)_{-n-s+d}\big)_{-m-t+1} \to \bigoplus_{\substack{|a|=s\\|b|=-m-t}} K\big(y^a\big)^* x^b$$

$$\xrightarrow{f} \bigoplus_{\substack{|a|=s-d\\|b|=-m-t+1}} K\big(y^a\big)^* x^b \to \big(H^{n}_{P_+}(R)_{-n-s+d}\big)_{-m-t+1} \to 0.$$

We set $j = -n - s + d$. Proposition 5.1 implies that

$$\operatorname{reg} H^n_{P_+}(R)_j \leqslant (-n - j + 1)d + q \quad \text{for some } q.$$

Since $I(f) = (y_1, \dots, y_n)^d$, thus $\operatorname{reg}(P_0/I(f)^{k+1}) = (k+1)d - 1$. Hence in Proposition 5.1 we have $q = -1$. □

Now the main result of this section is the following

Theorem 5.3. *Let $P = K[x_1, \dots, x_m, y_1, \dots, y_n]$, and $f \in P$ be a bihomogeneous polynomial such that $I(f)$ is $\mathfrak{m}$-primary. Let $R = P/fP$. Then the regularity of $H^n_{P_+}(R)_j$ is linearly bounded.*

Proof. We may write $f = \sum_{|\beta|=d} f_\beta y^\beta$ and let $\deg f_\beta = c$. From the exact sequence

$$0 \to P(-c, -d) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-n-j+d} P_0(-c) z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0.$$

We first assume that $f_\beta = x_\beta$. Proposition 5.2 implies that $\operatorname{reg} H^n_{P_+}(R)_j \leqslant (-n - j + 1) \times d - 1$. We set $k = (-n - j + 1)d$. Thus we get the surjective map of K-vector spaces

$$\bigoplus_{\substack{|a|=k-1\\|b|=-n-j+d}} K x^a z^b \to \bigoplus_{\substack{|a|=k\\|b|=-n-j}} K x^a z^b.$$

We proceed as in the proof of Proposition 5.1, and we get $[H^n_{P_+}(R)_j]_{kd+q'+1} = 0$ for some q'. Therefore

$$\operatorname{reg} H^n_{P_+}(R)_j \leqslant (-n - j + 1)d^2 + q'. \qquad \square$$

Corollary 5.4. *With the assumption of Theorem* 5.3*, we have*

$$\operatorname{reg} H^{n-1}_{P_+}(R)_j \leqslant (-n-j+1)d^2 + q' + 2.$$

Proof. For the proof one use the same argument as in the proof of Proposition 5.1(b). □

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References

[1] M. Brodmann, R.Y. Sharp, Local Cohomology: An Algebraic Introduction with Geometric Applications, Cambridge Stud. Adv. Math., vol. 60, Cambridge Univ. Press, 1998.
[2] W. Bruns, J. Herzog, Cohen–Macaulay Rings, revised ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, 1998.
[3] S.D. Cutkosky, J. Herzog, N.V. Trung, Asymptotic behaviour of the Castelnuovo–Mumford regularity, Compos. Math. 118 (3) (1999) 243–261.
[4] M. Chardin, Some results and questions on Castelnuovo–Mumford regularity, preprint.
[5] D. Eisenbud, Commutative Algebra with a View to Algebraic Geometry, Springer, 1995.
[6] L. Gruson, R. Lazarsfeld, C. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72 (1983) 491–506.
[7] C. Huneke, M. Miller, A note on the multiplicity of Cohen–Macaulay algebras with pure resolution, Canad. J. Math. 37 (1985) 1149–1162.
[8] J. Herzog, H. Srinivasan, Bounds for multiplicities, Trans. Amer. Math. Soc. 350 (7) (1998) 2879–2902.
[9] M. Katzman, R.Y. Sharp, Some properties of top local cohomology modules, J. Algebra 259 (2003) 599–612.
[10] R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods 1 (1980) 168–184.
[11] J. Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, in: Commutative Algebra and Combinatorics, in: Adv. Stud. Pure Math., vol. 11, Kinokuniya/North-Holland, Amsterdam, 1987, pp. 303–312.