Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 390 (2004) 145–154
www.elsevier.com/locate/laa

The weighted logarithmic matrix norm and bounds of the matrix exponential

Guang-Da Hu [a,*], Mingzhu Liu [b]

[a] *Department of Control Science and Engineering, Harbin Institute of Technology, P.O. Box 329, Xi Da Zhi Street, Harbin 150001, China*
[b] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*

Received 6 November 2003; accepted 17 April 2004

Submitted by H. Schneider

Abstract

In this note the weighted logarithmic matrix norm is defined. The weighted logarithmic matrix norm is less than or equal to 2-logarithmic matrix norm. The bounds of the matrix exponential are obtained using the weighted logarithmic norm, which are sharper than those based on the 2-logarithmic matrix norm. Numerical examples are given to illustrate the results of the note.

AMS classification: 65F35; 15A60

Keywords: The weighted logarithmic matrix norm; Lyapunov equation; The matrix exponential

1. Introduction

The logarithmic norm of a matrix A (or the measure of a matrix) is defined by

$$\mu[A] = \lim_{\Delta \to 0_+} \frac{\|I + \Delta A\| - 1}{\Delta}. \tag{1}$$

for matrix norm $\|\cdot\|$ induced by a vector norm in $\mathscr{R}^n$. For the usual 1-, 2- and ∞-matrix norms, the following formulae are well-known [3,4]:

* Corresponding author.
E-mail addresses: ghu@hit.edu.cn, ghuca@yahoo.ca (G.-D. Hu), mzliu@hit.edu.cn (M. Liu).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2004.04.015

$$\mu_1[A] = \max_j \left[a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}| \right], \quad \mu_2[A] = \lambda_{\max}\left(\frac{A + A^{\mathrm{T}}}{2}\right),$$

and

$$\mu_\infty[A] = \max_i \left[a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}| \right],$$

where, $A = (a_{ij}) \in \mathscr{R}^{n\times n}$ and A^{T} denotes the transpose of A.

Bounds of the matrix exponential have been discussed in [8] based on Jordan normal form. But it needs a lot of computation to obtain Jordan normal form. It is useless to estimate the matrix exponential using the bounds in [8]. The usual logarithmic matrix norms above can be applied to estimate bounds of the matrix exponential [3,4]. Along the line of [3,4], the weighted logarithmic norm is defined for any real matrix in the present note which is a continuation of [7]. Then bounds of the matrix exponential are given using the weighted logarithmic matrix norm. Other applications of logarithmic matrix norm include: numerical analysis of ordinary differential equations [2,3], stability and numerical analysis of neutral delay differential equations [1,6], circuit analysis [4] and estimates for solutions of Lyapunov equation [5].

An outline of the present note is as follows: In Section 2, a class of weighted logarithmic norm of any real matrix is constructed and its properties are discussed. In Section 3, the weighted logarithmic matrix norm is applied to estimate the matrix exponential. In Section 4, two examples are given to illustrate the results of Sections 3 and 4.

2. The weighted logarithmic matrix norm

In the note, $(\cdot,\cdot)$ denotes an inner product on $\mathscr{R}^n$ and $\|\cdot\|$ the corresponding inner product norm. Let H be a symmetric positive definite matrix, the function $(\cdot,\cdot)_{(H)}$ defined on $\mathscr{R}^n$ by $(x, y)_{(H)} = y^{\mathrm{T}}Hx$ is said to be weight H inner product in order to distinguish from the standard (or Euclidean) inner product $(x, y)_{(I)} = y^{\mathrm{T}}x$, where I is the unit matrix.

Lemma 2.1 [3]. *For any inner product on $\mathscr{R}^n$, and the corresponding inner product norm $\|\cdot\|$, we have*

$$\mu[A] = \max_{x\neq 0} \frac{(Ax, x)}{\|x\|^2}.$$

Definition 2.1. For any vector x and any matrix A, the weight H vector norm, weight H matrix norm, and weight H logarithmic matrix norm is defined, respectively, by

$$\|x\|_{(H)} = \sqrt{x^{\mathrm{T}} H x}, \qquad \|A\|_{(H)} = \max_{x \in R^n, x \neq 0} \frac{\|Ax\|_{(H)}}{\|x\|_{(H)}},$$

and from Lemma 2.1

$$\mu_{(H)}[A] = \max_{x \neq 0} \frac{(Ax, x)_{(H)}}{\|x\|^2{}_{(H)}}.$$

Now we present two formulae for the weight H logarithmic matrix norm and weight H matrix norm, respectively.

Theorem 2.1. *For any real matrix A,*

$$\mu_{(H)}[A] = \lambda_{\max}\left(\frac{\tilde{A} + \tilde{A}^{\mathrm{T}}}{2}\right) \tag{2}$$

and

$$\|A\|_{(H)} = \sqrt{\lambda_{\max}(\tilde{A}^{\mathrm{T}} \tilde{A})}, \tag{3}$$

where $\tilde{A} = H_0 A H_0^{-1}$, $H_0 = \sqrt{H}$, and $\lambda_{\max}(F)$ stands for the maximal eigenvalue of a symmetric matrix F.

Proof. By means of $\tilde{A} = H_0 A H_0^{-1}$ and Definition 2.1, we have,

$$A = H_0{}^{-1}(H_0 A H_0{}^{-1}) H_0 = H_0{}^{-1} \tilde{A} H_0,$$

and

$$\mu_{(H)}[A] = \max_{x \neq 0} \frac{(Ax, x)_{(H)}}{\|x\|^2{}_{(H)}}. \tag{4}$$

Notice that

$$\|x\|_{(H)}{}^2 = x^{\mathrm{T}} H x \quad \text{and} \quad (Ax, x)_{(H)} = x^{\mathrm{T}} H A x = (x, Ax)_{(H)} = x^{\mathrm{T}} A^{\mathrm{T}} H x,$$

$$(Ax, x)_{(H)} = \frac{x^{\mathrm{T}}(A^{\mathrm{T}} H + HA)x}{2} = \frac{(H_0 x)^{\mathrm{T}}(\tilde{A}^{\mathrm{T}} + \tilde{A})(H_0 x)}{2} \tag{5}$$

hold. Let $y = H_0 x$, substituting (5) into (4), we obtain

$$\mu_{(H)}[A] = \max_{x\neq 0} \frac{x^{\mathrm{T}}(A^{\mathrm{T}}H + HA)x}{2\|x\|^2_{(H)}} = \max_{y\neq 0} \frac{y^{\mathrm{T}}(\tilde{A}^{\mathrm{T}} + \tilde{A})y}{2y^{\mathrm{T}}y} = \lambda_{\max}\left(\frac{\tilde{A} + \tilde{A}^{\mathrm{T}}}{2}\right).$$

In the same way, it is easy to prove

$$\|A\|_{(H)} = \sqrt{\lambda_{\max}(\tilde{A}^{\mathrm{T}}\tilde{A})}.$$

The proof is completed. □

Another expression of the weighted logarithmic matrix norm is given in [10].

Lemma 2.2 [10]. *For any real matrix A,*

$$\mu_{(H)}[A] = \max\{\lambda \mid \det(HA + A^{\mathrm{T}}H - 2\lambda H) = 0\}. \tag{6}$$

Remark 2.1. If $H = I$, we can obtain $\mu_2[A]$ from (2) or (6).

Remark 2.2. We now compare the formula (2) with the formula (6). Since H is invertible, the generalized eigenvalue problem in (6) is essentially the same as the eigenvalue problem in (2).

In the following, we will construct a weight H using Lyapunov equation such that $\mu_{(H)}[A] \leqslant \mu_2[A]$ for any real matrix A.

Definition 2.2. A real matrix A is said to be stable if the real parts of its eigenvalues are negative.

For a stable matrix, we obtain a negative weighted logarithmic matrix norm using the following lemma.

Lemma 2.3 [7]. *If a real matrix A is stable, then there is a weight H logarithmic matrix norm such that*

$$\mu_{(H)}[A] = -\frac{1}{\lambda_{\max}(H)}, \tag{7}$$

where the symmetric positive definite matrix H satisfies the following Lyapunov equation

$$A^{\mathrm{T}}H + HA = -2I. \tag{8}$$

When A is unstable, we have the following result.

Theorem 2.2. *If a real matrix A is unstable, then there is a weight H logarithmic norm of matrix A such that*

$$\mu_{(H)}[A] = \alpha - \frac{1}{\lambda_{\max}(H)}, \tag{9}$$

where, $\alpha > \max_j \Re\lambda_j(A)$ for $j = 1, \ldots, n$, and α is nonnegative, and the symmetric positive definite matrix H satisfies the following Lyapunov equation:

$$(A - \alpha I)^{\mathrm{T}} H + H(A - \alpha I) = -2I. \tag{10}$$

Proof. The matrix $A - \alpha I$ is stable since $\alpha > \max_j \Re\lambda_j(A)$ for $j = 1, \ldots, n$. We can apply Lemma 2.3 to the matrix $A - \alpha I$ and obtain

$$\mu_{(H)}[A - \alpha I] = -\frac{1}{\lambda_{\max}(H)},$$

where the symmetric positive definite matrix H satisfies (10). From $\alpha \geqslant 0$ and the properties of logarithmic matrix norm [3,4], we have (9). The proof is completed. □

We will give a relation between the 2-logarithmic matrix norm and the weighted logarithmic matrix norm derived from Lyapunov equation.

Theorem 2.3. *For any real matrix A, the inequality*

$$\mu_{(H)}[A] \leqslant \mu_2[A] \tag{11}$$

holds, where $\mu_{(H)}[A]$ and H are given by (9) *and* (10) *in Theorem* 2.2, *respectively.*

The following lemma is useful to prove Theorem 2.3.

Lemma 2.4. *For a real stable matrix A, the inequality*

$$\mu_{(H)}[A] \leqslant \mu_2[A], \tag{12}$$

holds, where $\mu_{(H)}[A]$ and H are given by (7) *and* (8) *in Lemma* 2.3, *respectively.*

Proof. Let q be the maximal eigenvalue of matrix H and y the corresponding eigenvector, *i. e.*, $q = \lambda_{\max}(H)$ and $Hy = qy$. From (8),

$$y^{\mathrm{T}} A^{\mathrm{T}} H y + y^{\mathrm{T}} H A y = -2y^{\mathrm{T}} y.$$

It can be rewritten

$$q y^{\mathrm{T}} S y = -y^{\mathrm{T}} y,$$

where $S = \frac{(A^{\mathrm{T}}+A)}{2}$. Hence

$$\frac{y^{\mathrm{T}}Sy}{y^{\mathrm{T}}y} = -\frac{1}{q} = -\frac{1}{\lambda_{\max}(H)}. \tag{13}$$

According to Lemma 2.3 and (13), we have

$$\mu_{(H)}[A] = -\frac{1}{\lambda_{\max}(H)} = \frac{y^{\mathrm{T}}Sy}{y^{\mathrm{T}}y} \leqslant \max_{x\neq 0} \frac{x^{\mathrm{T}}Sx}{x^{\mathrm{T}}x} = \mu_2[A].$$

The proof is completed. □

Proof of Theorem 2.3. When A is stable, (11) holds with $\alpha = 0$ from Lemma 2.4. Now we consider the case when A is unstable. Since $A - \alpha I$ is stable, we can apply Lemma 2.4 to the matrix. According to Lemma 2.4 and the properties of logarithmic matrix norm [3,4], we obtain

$$-\frac{1}{\lambda_{\max}(H)} \leqslant \mu_2[A - \alpha I] = \mu_2[A] - \alpha.$$

Hence the inequality

$$\mu_{(H)}[A] = \alpha - \frac{1}{\lambda_{\max}(H)} \leqslant \mu_2[A]$$

holds. The proof is completed. □

Remark 2.3. Using Theorem 2.2, we can obtain a symmetric positive definite weight matrix H such that $\mu_{(H)}[A] \leqslant \mu_2[A]$ for any real matrix A. It will be interesting to investigate the minimum value of $\mu_{(H)}[A]$ among all possible α and H in (10).

3. Some bounds of the matrix exponential

We will need the following lemma to prove the results in the section.

Lemma 3.1 [3, 4]. *For any inner product on $\mathscr{R}^n$, and the corresponding inner product norm $\|\cdot\|$, we have*

$$\|\exp(At)\| \leqslant \exp(\mu[A]t), \tag{14}$$

where $\exp(At)$ *stands for the matrix exponential.*

When A is stable, for the 2-norm, we have the following theorem.

Theorem 3.1. *If a real matrix A is stable, we have*

$$\|\exp(At)\|_2 \leqslant \beta \exp\left(-\frac{t}{\lambda_{\max}(H)}\right), \tag{15}$$

where, H is given by Lyapunov equation (8) *in Lemma* 2.3 *and*

$$\beta = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}}. \tag{16}$$

Proof. Let $H_0 = \sqrt{H}$ and $\tilde{A} = H_0 A H_0{}^{-1}$, we have

$$\begin{aligned} &A = H_0{}^{-1}(H_0 A H_0{}^{-1})H_0 = H_0{}^{-1}\tilde{A}H_0, \\ &\exp(At) = H_0{}^{-1}\exp(\tilde{A}t)H_0, \end{aligned}$$

and

$$\|\exp(At)\|_2 = \|H_0{}^{-1}\exp(\tilde{A}t)H_0\|_2 \leqslant \|H_0{}^{-1}\|_2\|H_0\|_2\|\exp(\tilde{A}t)\|_2. \tag{17}$$

According to Lemma 3.1 and Lemma 2.3, we have

$$\begin{aligned} \|\exp(\tilde{A}t)\|_2 &= \max_{x\neq 0}\frac{x^{\mathrm{T}}(\exp\tilde{A}t)^{\mathrm{T}}\exp(\tilde{A}t)x}{x^{\mathrm{T}}x} \\ &= \max_{x\neq 0}\frac{x^{\mathrm{T}}(H_0\exp(At)H_0{}^{-1})^{\mathrm{T}}H_0\exp(At)H_0{}^{-1}x}{x^{\mathrm{T}}x} \\ &= \max_{z\neq 0}\frac{z^{\mathrm{T}}(\exp(At))^{\mathrm{T}}H\exp(At)z}{z^{\mathrm{T}}Hz} \\ &= \|\exp(At)\|_{(H)} \\ &\leqslant \exp\left(-\frac{t}{\lambda_{\max}(H)}\right), \end{aligned}$$

i.e.,

$$\|\exp(\tilde{A}t)\|_2 \leqslant \exp\left(-\frac{t}{\lambda_{\max}(H)}\right), \tag{18}$$

where $z = H_0{}^{-1}x$. On the other hand, since H_0 is a symmetric positive definite matrix,

$$\|H_0\|_2\|H_0{}^{-1}\|_2 = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}} = \beta \tag{19}$$

holds [9]. From (17)–(19), the inequality (15) holds. The proof is completed. □

By means of Theorem 3.1, we can obtain the following corollary.

Corollary 3.1. *If a real matrix A is stable and* $\mu_2[A] > 0$, *we have the following inequality*:

$$\| \exp(At) \|_2 \leqslant \beta \tag{20}$$

for $t \geqslant 0$, *where* β *is defined by* (16) *in Theorem* 3.1.

For any real matrix, we have the following result.

Theorem 3.2. *For any real matrix* A, *the inequality*

$$\| \exp(At) \|_2 \leqslant \beta \exp(\mu_{(H)}[A]t), \tag{21}$$

holds, where $\mu_{(H)}[A]$ *and* H *are given by* (9) *and* (10) *in Theorem* 2.2, *respectively, and*

$$\beta = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}}.$$

Proof. The proof of this theorem is similar to Theorem 3.1. So the proof is omitted. □

Remark 3.1. When A is stable, we have $0 > \max_j \Re\lambda_j(A)$ for $j = 1, \ldots, n$, and we can set $\alpha = 0$, the inequality (21) becomes (15). Hence Theorem 3.2 is an extension of Theorem 3.1.

Remark 3.2. If $\mu_{(H)}[A] < \mu_2[A]$, the inequality (21) is sharper than the following estimation:

$$\| \exp(At) \|_2 \leqslant \exp(\mu_2[A]t) \tag{22}$$

for $t \geqslant T$, where T is a sufficient large positive constant.

Remark 3.3. Since the bounds of the matrix exponential in [8] are concerned with Jordan normal form of the matrix, a lot of computation is required. If we know the Jordan normal form of a matrix, the matrix exponential can be obtained directly. It is useless to estimate the matrix exponential using the bounds in [8]. While only symmetric matrices are involved to obtain the bounds in Theorems 3.1 and 3.2.

4. Numerical examples

We will illustrate our results through two numerical examples and all computations are carried out by Matlab in the section.

Example 4.1. Let

$$A = \begin{bmatrix} -0.8 & 0.4 & 0.2 \\ 1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

The matrix is stable. We can estimate the bounds of its exponential by Theorem 3.1. We have $\mu_1[A] = 1.2$, $\mu_2[A] = 0.0359$, $\mu_\infty[A] = 0$ and the maximum real part of eigenvalues of A is -0.0566. After solving Lyapunov equation (8), we obtain

$$H = \begin{bmatrix} 5.0333 & 3.0266 & 6.9401 \\ 3.0266 & 2.5477 & 5.4324 \\ 6.9401 & 5.3424 & 13.2528 \end{bmatrix}.$$

Thence, $\mu_{(H)}[A] = -0.0514$ and $\beta = 8.1966$. According to Theorem 3.1, we have $\|\exp(At)\|_2 \leqslant 8.1966 \exp(-0.0514t)$ for $t \geqslant 0$. By $\mu_1[A]$, $\mu_2[A]$ or $\mu_\infty[A]$ we can not obtain a bound which tends to 0 as $t \to +\infty$ since the three usual logarithmic norms are nonnegative.

Example 4.2

$$A = \begin{bmatrix} 5 & 6 & 3 \\ -2 & 7 & 4 \\ 13 & 9 & -1 \end{bmatrix}.$$

The maximum real part of eigenvalues of A is $d = 12.6498$. The matrix is unstable. We can compute the bounds of its exponential by Theorem 3.2. We have $\mu_1[A] = 22$, $\mu_2[A] = 14.7033$, $\mu_\infty[A] = 21$. From Theorems 2.2 and 3.2, both $\mu_H[A]$ and β depend on α. The bounds of the matrix exponential also depend on α. Let $\alpha = d + 0.01$, $A - \alpha I$ is stable. From (10), we obtain

$$H = \begin{bmatrix} 20.7036 & 54.3665 & 20.4860 \\ 54.3665 & 143.8269 & 54.0920 \\ 20.4860 & 54.0920 & 20.4122 \end{bmatrix}.$$

Thence, $\mu_{(H)}[A] = 12.6544$ and $\beta = 60.3543$. We have $\|\exp(At)\|_2 \leqslant 60.3544 \exp(12.6544t)$ for $t \geqslant 0$.

Let $\alpha = \mu_2[A] = 14.7033$, $A - \alpha I$ is stable. After solving Lyapunov equation (10), we obtain

$$H = \begin{bmatrix} 0.2097 & 0.2198 & 0.1134 \\ 0.2198 & 0.5364 & 0.2015 \\ 0.1134 & 0.2015 & 0.1367 \end{bmatrix}.$$

Thence, $\mu_{(H)}[A] = 13.3445$ and $\beta = 4.0098$. We have $\|\exp(At)\|_2 \leqslant 4.0098 \exp(13.3445t)$ for $t \geqslant 0$.

Let $\alpha = 30$, $A - \alpha I$ is stable. We have by Lyapunov equation (10),

$$H = \begin{bmatrix} 0.0449 & 0.0085 & 0.0107 \\ 0.0085 & 0.0501 & 0.0112 \\ 0.0107 & 0.0112 & 0.0347 \end{bmatrix}.$$

Thence, $\mu_{(H)}[A] = 14.4682$ and $\beta = 1.5590$. We have $\|\exp(At)\|_2 \leqslant 1.5590 \exp(14.4682t)$ for $t \geqslant 0$. Since 12.6544, 13.3445 and 14.4682 are less than $\mu_2[A] = 14.7033$, the three bounds are sharper than $\|\exp(At)\|_2 \leqslant \exp(14.7033t)$ for $t \geqslant T$, where T is a sufficient large positive constant.

Remark 4.1. The above numerical examples show that the bounds derived by $\mu_{(H)}[A]$ are sharper than those by $\mu_2[A]$ for $t \geqslant T$, where T is a sufficient large positive constant.

Acknowledgement

We would like to thank the referee for the referee's report which improved Remarks 2.2 and 2.3.

References

[1] A. Bellen, N. Guglielmi, A.E. Ruehli, Methods for linear systems of circuit delay differential equations of neutral type, IEEE Trans. Circuits Systems I 46 (1999) 212–216.
[2] H. Brunner, A note on modified optimal linear methods, Math. Comp. 26 (1972) 625–631.
[3] K. Dekker, J.G. Verwer, Stability of Runge–Kutta Methods for Stiff Nonlinear Differential Equations, North-Holland, Amsterdam, 1984.
[4] C.A. Desoer, M. Vidyasagar, Feedback Systems: Input–Output Properties, Academic Press, New York, 1975.
[5] Y. Fang, K.A. Loparo, X. Feng, New estimates for solutions of Lyapunov equations, IEEE. Trans. Automat. Control 42 (1997) 408–411.
[6] G.-D. Hu, B. Cahlon, Estimations on numerically stable step-size for neutral delay differential systems with multiple delays, J. Comput. Appl. Math. 102 (1999) 221–234.
[7] G.-D. Hu, G.D. Hu, A relation between the weighted logarithmic norm of matrix and Lyapunov equation, BIT 40 (2000) 506–510.
[8] B. Kågström, Bounds and perturbation bounds for the matrix exponential, BIT 17 (1977) 39–57.
[9] P. Lancaster, The Theory of Matrices with Applications, Academic Press, Inc., Orlando, 1985.
[10] T. Ström, On logarithmic norms, SIAM J. Numer. Anal. 12 (1975) 741–753.