NORTH-HOLLAND

On k-EP Matrices

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ABSTRACT

The concept of k-EP matrix is introduced. Relations between k-EP and EP matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be k-EP$_r$. © 1998 Elsevier Science Inc.

1. INTRODUCTION

Let $C_{n\times n}$ be the space of $n \times n$ complex matrices of order n. Let C_n be the space of complex n-tuples. For $A \varepsilon C_{n\times n}$, let A^T, A^*, $A^\dagger$, $R(A)$, $N(A)$, and (A) denote the transpose, conjugate transpose, Moore-Penrose inverse, range space, null space, and rank of A respectively. We denote a solution X of the equation $AXA = A$ by A^-. Throughout let k be a fixed product of disjoint transpositions in $S_n = \{1, 2, \dots, n\}$, and K be the associated permutation matrix. A matrix $A = (a_{ij}) \varepsilon C_{n\times n}$ is k-hermitian if $a_{ij} = \bar{a}_{k(j),k(i)}$ for $i, j = 1$ to n. A theory for k-hermitian matrices is developed in [3]. In this paper, we introduce the concept of k-EP matrices as a generalization of k-hermitian and EP matrices and extend many of the basic results on k-hermitian [3] and EP matrices [1, 4, 6, 7]. A matrix $A \varepsilon C_{n\times n}$ is EP if $N(A) = N(A^*)$. Relations between k-EP and EP matrices are discussed.

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2. k-EP Matrices

In this section we present equivalent characterizations of a k-EP matrix. Necessary and sufficient conditions are determined for a matrix to be k-EP$_r$ (k-EP and of rank r). As an application, it is shown that the class of all k-EP matrices having the same range space forms a group under multiplication. For $x = (x_1, x_2, \dots, x_n)^T \varepsilon C_n$, let us define the function $\mathscr{k}(x) = (x_{k(1)}, x_{k(2)}, \dots, x_{k(n)})^T \varepsilon C_n$. Since k is involutory, it can be verified that the associated permutation matrix K satisfy the following properties:

$$K = K^T = K^{-1} \quad \text{and} \quad \mathscr{k}(x) = Kx, \tag{P.1}$$

$$(KA)^{\dagger} = A^{\dagger}K \text{ and } (AK)^{\dagger} = KA^{\dagger} \quad \text{for} \quad A \varepsilon C_{n\times n}$$

$$\text{(by [2, p. 182])} \tag{P.2}$$

DEFINITION 2.1. A matrix $A \varepsilon C_{n\times n}$ is said to be k-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^*\mathscr{k}(x) = 0$ or equivalently $N(A) = N(A^*K)$. Moreover, A is said to be k-EP$_r$ if A is k-EP and of rank r.

In particular, when $k(i) = i$ for each $i, j = 1$ to n, then the associated permutation matrix K reduces to the identity matrix and Definition 2.1 reduces to $N(A) = N(A^*)$, which implies that A is an EP matrix [7]. If A is nonsingular, then A is k-EP for all transpositions k in S_n.

REMARK 2.2. We note that a k-hermitian matrix A is k-EP. For, if A is k-hermitian, then by [3, Result 2.1], $A = KA^*K$. Hence $N(A) = N(KA^*K) = N(A^*K)$, which implies A is k-EP. However, the converse need not be true.

EXAMPLE 2.3. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

For a transposition $k = (1\ 2)$, the associated permutation matrix

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$KA^*K = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \neq A.$$

Therefore, A is not k-hermitian. Since A is a nonsingular matrix, A is a k-EP matrix. Thus, the set of k-EP matrices contains the set of k-hermitian matrices.

THEOREM 2.4. *For $A \varepsilon C_{n\times n}$ the following are equivalent:*

(1) A *is* k-EP.
(2) KA *is* EP.
(3) AK *is* EP.
(4) $A^\dagger$ *is* k-EP.
(5) $N(A) = N(A^\dagger K)$.
(6) $N(A^*) = N(AK)$.
(7) $R(A) = R(KA^*)$.
(8) $R(A^*) = R(KA)$.
(9) $KA^\dagger A = AA^\dagger K$.
(10) $A^\dagger AK = KAA^\dagger$.
(11) $A = KA^*KH$ *for a nonsingular* $n \times n$ *matrix* H.
(12) $A = HKA^*K$ *for a nonsingular* $n \times n$ *matrix* H.
(13) $A^* = HKAK$ *for a nonsingular* $n \times n$ *matrix* H.
(14) $A^* = KAKH$ *for a nonsingular* $n \times n$ *matrix* H.
(15) $C_n = R(A) \oplus N(AK)$.
(16) $C_n = R(KA) \oplus N(A)$.

Proof. The proof for the equivalence of (1), (2), and (3) runs as follows:

A is k-EP	$\Leftrightarrow$	$N(A) = N(A^*K)$	(by Definition 2.1)
	$\Leftrightarrow$	$N(KA) = N(KA)^*$	[by (P.1)]
	$\Leftrightarrow$	KA is EP	(by definition of EP matrix)
	$\Leftrightarrow$	$K(KA)K^*$ is EP	(by [1, Lemma 3])
	$\Leftrightarrow$	AK is EP	[by (P.1)].

Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.

(2) $\Leftrightarrow$ (4):

$$\begin{aligned} KA \text{ is EP} &\Leftrightarrow (KA)^{\dagger} \text{ is EP} && \text{(by [2, p. 163])} \\ &\Leftrightarrow A^{\dagger}K \text{ is EP} && \text{[by (P.2)]} \\ &\Leftrightarrow A^{\dagger} \text{ is } k\text{-EP} && \text{[by equivalence of (1) and (3)} \\ & && \text{applied to } A^{\dagger}]. \end{aligned}$$

Thus equivalence of (2) and (4) is proved.

(1) $\Leftrightarrow$ (5):

$$\begin{aligned} A \text{ is } k\text{-EP} &\Leftrightarrow N(A) = N(A^*K) && \text{(by Definition 2.1)} \\ &\Leftrightarrow N(A) = N(KA)^* && \text{[by (P.1)]} \\ &\Leftrightarrow N(A) = N(A^{\dagger}K) && \text{[by (P.2)]}. \end{aligned}$$

Thus equivalence of (1) and (5) is proved.

Now we shall prove the equivalence of (1), (6), and (7) using $\rho(A) = \rho(A^*) = \rho(A^*K) = \rho(AK)$ in the following way:

$$\begin{aligned} A \text{ is } k\text{-EP} &\Leftrightarrow N(A) = N(A^*K) \\ &\Leftrightarrow N(A) \subseteq N(A^*K) \\ &\Leftrightarrow A^*K = A^*KA^-A && \text{(by [2, p. 21])} \\ &\Leftrightarrow A^* = A^*KA^-AK && \text{[by (P.1)]} \\ &\Leftrightarrow A^* = A^*K^{-1}A^-AK \\ &\Leftrightarrow A^* = A^*(AK)^-AK && \text{[by (P.2)]} \\ &\Leftrightarrow N(AK) \subseteq N(A^*) && \text{(by [2, p. 21])} \\ &\Leftrightarrow N(A^*) = N(AK) \\ &\Leftrightarrow R(A) = R(AK)^* \\ &\Leftrightarrow R(A) = R(KA)^* && \text{[by (P.1)]}. \end{aligned}$$

Thus (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) hold.

(1) $\Leftrightarrow$ (8):

$$\begin{aligned} A \text{ is } k\text{-EP} \quad &\Leftrightarrow \quad N(A) = N(A^*K) \\ &\Leftrightarrow \quad N(A) = N(KA)^* \\ &\Leftrightarrow \quad R(A^*) = R(KA). \end{aligned}$$

Thus equivalence of (1) and (8) is proved.

(3) $\Leftrightarrow$ (9):

$$\begin{aligned} AK \text{ is EP} \quad &\Leftrightarrow \quad (AK)(AK)^{\dagger} = (AK)^{\dagger}(AK) \qquad &&(\text{by } [2, \text{p. } 166]) \\ &\Leftrightarrow \quad (AK)\,(KA^{\dagger}) = (KA^{\dagger})\,(AK) \qquad &&[\text{by } (\text{P.2})] \\ &\Leftrightarrow \quad AA^{\dagger} = KA^{\dagger}AK \qquad &&[\text{by } (\text{P.1})] \\ &\Leftrightarrow \quad AA^{\dagger}K = KA^{\dagger}A. \end{aligned}$$

Thus equivalence of (3) and (9) is proved.

(9) $\Leftrightarrow$ (10): Since, by the property (P.1), $K^2 = I$, this equivalence follows by pre- and postmultiplying $KA^{\dagger}A = AA^{\dagger}K$ by K.

(2) $\Leftrightarrow$ (11):

$$KA \text{ is EP} \quad \Leftrightarrow \quad (KA)^* = (KA)H_1$$

for a nonsingular $n \times n$ matrix H_1 (by [2, p. 166])

$$\begin{aligned} &\Leftrightarrow \quad A^*K = KAH_1 \\ &\Leftrightarrow \quad KA^*K = AH_1 \\ &\Leftrightarrow \quad A = KA^*KH, \end{aligned}$$

where $H = H_1^{-1}$ is a nonsingular $n \times n$ matrix. Thus equivalence of (2) and (11) is proved.

(3) $\Leftrightarrow$ (12):

$$AK \text{ is EP} \quad \Leftrightarrow \quad (AK)^* = H_1(AK)$$

for a nonsingular $n \times n$ matrix H_1 (by [2, p. 166])

$$\Leftrightarrow \quad KA^* = H_1 AK$$

$$\Leftrightarrow \quad KA^*K = H_1 A$$

$$\Leftrightarrow \quad A = H_1^{-1} KA^*K$$

$$\Leftrightarrow \quad A = HKA^*K,$$

where $H = H_1^{-1}$ is a nonsingular $n \times n$ matrix. Thus equivalence of (3) and (12) is proved.

The equivalences (11) $\Leftrightarrow$ (13) and (12) $\Leftrightarrow$ (14) follow immediately by taking conjugate transpose and using $K = K^*$.

(13) $\Leftrightarrow$ (16):

$$A^* = HKAK \text{ for a nonsingular } n \times n \text{ matrix } H$$

$$\Leftrightarrow \quad A^*A = H(KA)(KA)$$

$$\Leftrightarrow \quad A^*A = H(KA)^2$$

$$\Leftrightarrow \quad \rho(A^*A) = \rho\big(H(KA)^2\big)$$

$$\Leftrightarrow \quad \rho(A^*A) = \rho\big((KA)^2\big).$$

Over the complex field, A^*A and A have the same rank. Therefore,

$$\rho\big((KA)^2\big) = \rho(A^*A) = \rho(A) = \rho(KA) \quad \Leftrightarrow \quad R(KA) \cap N(KA) = \{0\}$$

$$\Leftrightarrow \quad R(KA) \cap N(A) = \{0\}$$

$$\Leftrightarrow \quad C_n = R(KA) \oplus N(A).$$

Thus (13) $\Leftrightarrow$ (16) holds.

(14) $\Leftrightarrow$ (15): This can be proved along the same lines and using $\rho(AA^*) = \rho(A)$. Hence the proof is omitted.

(16) $\Rightarrow$ (1): If $C_n = R(KA) \oplus N(A)$, then $R(KA) \cap N(A) = \{0\}$. For $x \in N(A)$, $x \notin R(KA) \Leftrightarrow x \in R(KA)^{\perp} = N(KA)^* = N(A^*K)$. Hence $N(A) \subseteq N(A^*K)$ and $\rho(A) = \rho(A^*K) \Rightarrow N(A) = N(A^*K) \Rightarrow A$ is k-EP. Thus (1) holds. Similarly, we can prove (15) $\Rightarrow$ (1). ■

REMARK 2.5. In particular, when A is k-hermitian, Theorem 2.4 reduces to [3, Result 2.1]. We note that, without requiring A to be normal (refer to [3, Result 2.8]), A is k-hermitian $\Leftrightarrow A = KA^*K \Leftrightarrow A^\dagger = (KA^*K)^\dagger = K(A^\dagger)^*K \Leftrightarrow A^\dagger$ is k-hermitian.

When $k(i) = i$ for each $i, j = 1$ to n, then Theorem 2.4 reduces to [1, Theorem 1], [6, Theorem 1], and [4, Theorem 1].

It is well known that a complex normal matrix is EP. However, a normal matrix need not be k-EP [refer to Example 2.6(iii)].

EXAMPLE 2.6. For $k = (1\ 2)$,

(i) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is EP as well as k-EP.

(ii) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is k-EP but not EP.

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is hermitian, normal, and EP, but not k-EP and hence not k-hermitian.

This motivates the following result:

THEOREM 2.7. *Let $A \in C_{n \times n}$. Then any two of the following conditions imply the other one*:

(1) *A is EP.*
(2) *A is k-EP.*
(3) *$R(A) = R(KA)$.*

Proof. First we prove that whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds; then by [1, Theorem 1], A EP implies $R(A) = R(A^*)$. Now by Theorem 2.4, A is k-EP $\Leftrightarrow R(A^*) = R(KA)$. Therefore, A is k-EP $\Leftrightarrow R(A) = R(KA)$. This completes the proof of [(1) and (2)] $\Rightarrow$ (3) and [(1) and (3)] $\Rightarrow$ (2).

Now let us prove [(2) and (3)] $\Rightarrow$ (1): Since A is k-EP, by Theorem 2.4, KA is EP. Hence, $R(KA) = R(KA)^*$. By using (3), we have $R(A) = R(KA) = R(KA)^* = R(A^*K) = R(A^*)$. Again by [1, Theorem 1], A is EP. Thus (1) holds. ■

COROLLARY 2.8. *If $A \in C_{n \times n}$ is normal and AA^* is k-EP, then A is k-EP.*

Proof. Since A is normal, $[A$ is EP and AA^* is k-EP$] \Leftrightarrow R(AA^*) = R(KAA^*) \Rightarrow R(A) = R(KA)$. That A is k-EP then follows from Theorem 2.7. ■

COROLLARY 2.9. *Let $E = E^* = E^2 \in C_{n \times n}$ be a hermitian idempotent that commutes with K, the permutation matrix associated with a fixed product of disjoint transpositions k is S_n. Then $H_k(E) = \{A : A$ is k-EP and $R(A) = R(E)\}$ forms a maximal subgroup of $C_{n \times n}$ containing E as identity.*

Proof. Since $EK = KE$, by (P.1) and (P.2) we have $E = KEK$ and $EE^\dagger = E^2 = E = (KE)(EK) = (KE)(KE)^\dagger$; hence $R(E) = R(KE)$. Since E is hermitian it is automatically EP, and by Theorem 2.7, E is k-EP. Thus $E \in H_k(E)$. For $A \in H_k(E)$, $[A$ is k-EP and $R(A) = R(E) = R(KE) \Rightarrow [AA^\dagger = EE^\dagger = E$ and $AA^\dagger = E = (KE)(KE)^\dagger = KEE^\dagger K^\dagger = KAA^\dagger K^\dagger = (KA)(KA)^\dagger]$. Therefore $R(A) = R(KA)$. Hence by Theorem 2.7, A is EP. Thus $H_k(E) = H(E) = \{A : A$ is EP and $R(A) = R(E)\}$. By [5, Theorem 2.1], $H_k(E)$ forms a maximal subgroup of $C_{n \times n}$ containing E as identity. ■

REMARK 2.10. For $0 \neq E \neq I_n$, by [5, Corollary 2.3], $H_k(E)$ is a non-abelian group if and only if $n > 2$.

For $A \in C_{n \times n}$, there exist unique k-hermitian matrices P and Q such that $A = P + iQ$, where $P = \frac{1}{2}(A + KA^*K)$ and $Q = (1/2i)(A - KA^*K)$ (refer to [3, Result 2.11]).

In the following theorem, an equivalent condition for a matrix A to be k-EP is obtained in terms of P, the k-hermitian part of A.

THEOREM 2.11. *For $A \in C_{n \times n}$, A is k-EP $\Leftrightarrow N(A) \subset N(P)$, where P is the k-hermitian part of A.*

Proof. If A is k-EP, then by Theorem 2.4, KA is EP. Since K is nonsingular, we have $N(A) = N(KA) = N(KA)^* = N(A^*K) = N(KA^*K)$. Then for $x \in N(A)$, both $Ax = 0$ and $KA^*Kx = 0$, which implies that $Px = \frac{1}{2}(A + KA^*K)x = 0$. Thus, $N(A) \subseteq N(P)$. Conversely, let $N(A) \subseteq N(P)$; then $Ax = 0$ implies $Px = 0$ and hence $Qx = 0$. Therefore, $N(A) \subseteq N(Q)$.

Thus, $N(A) \subseteq N(P) \cap N(Q)$. Since both P and Q are k-hermitian, by [3, Result 2.1], $P = KP^*K$ and $Q = KQ^*K$. Hence $N(P) = N(KP^*K) = N(P^*K)$ and $N(Q) = N(KQ^*K) = N(Q^*K)$. Now $N(A) \subseteq N(P) \cap N(Q) = N(P^*K) \cap N(Q^*K) \subseteq N(P^* - iQ^*)K$. Therefore, $N(A) \subseteq N(A^*K)$ and $\rho(A) = \rho(A^*K)$. Hence, $N(A) = N(A^*K)$. Therefore, A is k-EP. Hence the theorem. ∎

Toward characterizing a matrix being k-EP$_r$, we first prove two lemmas.

LEMMA 2.12. *Let*

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where D is an $r \times r$ nonsingular matrix. Then the following are equivalent:

(1) *B is k-EP$_r$.*
(2) *$R(KB) = R(B)$.*
(3) *BB^* is k-EP$_r$.*
(4) $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, *where K_1 and K_2 are permutation matrices of order r and $n - r$ respectively.*
(5) *$k = k_1 k_2$, where k_1 is the product of disjoint transpositions on $S_n = \{1, 2, \dots, n\}$ leaving $(r + 1, r + 2, \dots, n)$ fixed, and k_2 is the product of disjoint transpositions leaving $(1, 2, \dots, r)$ fixed.*

Proof. Since B is EP$_r$, the equivalence of (1) and (2) follows from Theorem 2.7.

(2) ⇔ (3) follows from Theorem 2.4.

(2) ⇔ (4): Let us partition

$$K = \begin{bmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{bmatrix},$$

where K_1 is $r \times r$. Then

$$\begin{aligned} R(KB) = R(B) \quad &\Leftrightarrow \quad (KB)(KB)^{\dagger} = BB^{\dagger} \\ &\Leftrightarrow \quad KBB^{\dagger}K = BB^{\dagger} \\ &\Leftrightarrow \quad KBB^{\dagger} = BB^{\dagger}K \\ &\Leftrightarrow \quad K\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}K \\ &\Leftrightarrow \quad \begin{bmatrix} K_1 & 0 \\ K_3^T & 0 \end{bmatrix} = \begin{bmatrix} K_1 & K_3 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = K. \end{aligned}$$

Thus, equivalence of (2) and (4) holds. The equivalence of (4) and (5) is clear from the definition of k. ■

REMARK 2.13. In Lemma 2.12, the condition that D is nonsingular cannot be relaxed, as illustrated by the following example:

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is not EP. For

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad KB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is EP. Hence B is k-EP_1. But

$$K \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

LEMMA 2.14. *A matrix $A \in C_{n\times n}$ if k-EP_r if and only if there exist a unitary matrix U and an $r \times r$ nonsingular matrix F such that*

$$A = KU\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Proof. Let us assume that A is k-EP$_r$. Then by Theorem 2.4, $C_n = R(KA) \oplus N(A)$. Choose an orthonormal basis $\{x_1, x_2, \dots, x_n\}$ of $R(KA) = R(A^*)$, and extend it to a basis $\{x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_n\}$ of C_n where $\{x_{r+1}, \dots, x_n\}$ is an orthonormal basis of $N(A)$.

If (u, v) denotes the usual inner product on C_n and $1 \le i \le r < j \le n$ it follows that $x_i \in R(KA) = R(A^*) \Rightarrow x_i A^*y$. Therefore, $(x_i, x_j) = (A^*y, x_j) = (y, Ax_j) = 0$ [since $x_j \in N(A)$]. Hence, $\{x_1, x_2, \dots, x_n\}$ is an orthonormal basis of C_n. If we consider KA as the matrix of a linear transformation relative to any orthonormal basis of C_n, then

$$U^*KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix},$$

where F is $r \times r$ nonsingular matrix, whence

$$A = KU\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Conversely, if

$$A = KU\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*, \qquad U^*KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}.$$

But $N(KA) = N(KA)^*$, which implies KA is EP$_r$, and by Theorem 2.4, A is k-EP$_r$. ■

THEOREM 2.15. *Let $A \in C_{n \times n}$. Then A is k-EP$_r$ with $k = k_1k_2$ (where k_1 and k_2 are as in Lemma 2.12) if and only if A is unitarily k-similar to a diagonal block k-EP$_r$ matrix*

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where D is an $r \times r$ nonsingular matrix.

Proof. Since A is k-EP$_r$ by Lemma 2.14, there exist a unitary matrix U and a $r \times r$ nonsingular matrix F such that

$$A = (KUK)K\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Since $k = k_1 k_2$, the associated permutation matrix is

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Hence,

$$A = KUK\begin{bmatrix} K_1 F & 0 \\ 0 & 0 \end{bmatrix} U^* = KUK\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*, \qquad \text{where} \quad D = K_1 F.$$

Thus, A is unitarily k-similar to a diagonal block matrix

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where D is $r \times r$ nonsingular. Now, that B is k-EP$_r$ follows from Lemma 2.12.

Conversely, if

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

with D $r \times r$ nonsingular is k-EP$_r$, then again by using Lemma 2.12,

$$k = k_1 k_2 \quad \text{and} \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Since A is unitarily k-similar to

$$B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

there exists a unitary matrix U such that

$$A = KUK\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Since B is k-EP$_r$ by Theorem 2.4,

$$KB = K\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = U^*KAU$$

is EP_r. By [1, Lemma 2], KA is EP_r. Now, that A is k-EP follows from Theorem 2.4 and $\rho(A) = r$. Hence A is k-EP_r. The proof is complete. ■

The following [4, Theorem 1] can be deduced from Theorem 2.15 as a particular case for $k(i) = i$ for each $i, j = 1$ to n.

COROLLARY 2.16. *Let $A \in C_{n\times n}$. Then A is an EP_r matrix if and only if there are a unitary matrix U and a nonsingular $r \times r$ matrix D such that*

$$A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

To conclude, we note that the k-spectral property [3, p. 21] holds for k-EP matrices.

THEOREM 2.17. *If A is k-EP, then (λ, x) is a (k-eigenvalue, k-eigenvector) pair for A if and only if $(1/\lambda, \mathscr{k}(x))$ is a (k-eigenvalue, k-eigenvector) pair for $A^\dagger$.*

Proof.

(λ, x) is a (k-eigenvalue, k-eigenvector) pair for A

$$\begin{aligned}
&\Leftrightarrow \quad Ax = \lambda Kx && (\text{by } [3,\ p.\ 22]) \\
&\Leftrightarrow \quad KAx = \lambda x && [\text{by } (P.1)] \\
&\Leftrightarrow \quad (KA)^\dagger x = \frac{1}{\lambda}x && (\text{by } [2,\ p.\ 161]) \\
&\Leftrightarrow \quad A^\dagger Kx = \frac{1}{\lambda}x && [\text{by } (P.2)] \\
&\Leftrightarrow \quad A^\dagger \mathscr{k}(x) = \frac{1}{\lambda}K(\mathscr{k}(x)) && \\
&\Leftrightarrow \quad \left(\frac{1}{\lambda}, \mathscr{k}(x)\right) \text{ is a (k-eigenvalue, k-eigenvector) pair for } A^\dagger. &&
\end{aligned}$$

■

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