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The Othello game on an $n \times n$ board is PSPACE-complete

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Abstract

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Given an arbitrary position of the Othello game played on an $n \times n$ board, the problem of determining the winner is shown to be PSPACE-complete. It can be reduced from generalized geography played on bipartite graphs with maximum degree 3.

1. Introduction

The complexity of generalized versions of popular games and puzzles has been studied. The problems of determining the winner are shown to be exponential-time complete for generalized Chess [4], Checkers [10], Go [9] and Shogi [1]; the problems of determining the winner are PSPACE-complete for generalized Hex [3, 8] and Gomoku [7]; and the problem of determining whether there is a solution in generalized Hi-Q (peg-solitaire) [12] is shown to be NP-complete.

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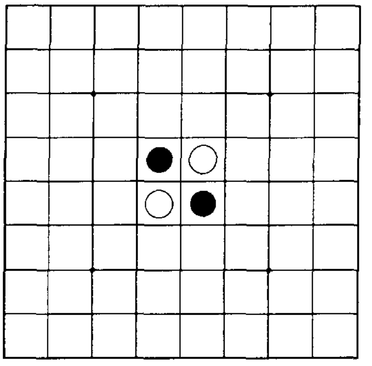


Fig. 1. Initial position of the Othello game.

The generalizations are natural in the sense that the board is extended by planar $n \times n$ locations, the spirit of the game is preserved, an arbitrary position of each game or puzzle is given, and no further rules are modified.

We consider the game of Othello, one of the popular games in the world. Othello is played by two players, *Black* and *White*, on a board of 8×8 locations called *squares*. Initially, black and white markers are placed on the board as shown in Fig. 1. The two players play alternately and Black plays first. Each player in turn *moves* by placing a marker of his color on a vacant square so that the opponent's markers are placed between the player's two markers either horizontally, vertically or diagonally. The enclosed opponent's colored markers are changed to the color of the player's markers. If there is no square upon which a player can place his marker in order to put his opponent's in between, then the player should pass his move. If both the players pass, then the game is over and the winner is the player having more markers of his color on the board.

The *generalized Othello problem* is one to determine whether Black can win in a given arbitrary Othello position on an $n \times n$ board. In this paper we will show that the generalized Othello problem is PSPACE-complete. In order to show that the problem is PSPACE-hard, we establish a polynomial-time reduction from a restricted version of generalized geography. Generalized geography [11, 6, 5] is a game played by two players, $\exists$-player and $\forall$-player, on the nodes of a given directed graph. The $\exists$-player is the first player and first places a marker on a given distinguished node. Then players alternately put a marker on any unmarked node v to which there is an edge from the last node played to v. The first player who cannot move loses.

Theorem 1.1 (Lichtenstein and Sisper [6]). *Generalized geography played on a given bipartite graph with maximum degree* 3 *is* PSPACE-*complete.*

2. Construction

The object of this paper is to prove the following theorem.

Theorem 2.1. *The generalized Othello problem is* PSPACE-*complete.*

Proof. For a given $n \times n$ board, the game must end after at most n^2 steps by two players. Thus the problem can be computed by an alternating Turing machine [2] within polynomial time. By [2], the generalized Othello problem is in PSPACE.

We show a polynomial-time reduction from generalized geography played on a bipartite graph with maximum degree 3. Let $G=(X, Y, E)$ be a bipartite graph with nodes as follows:

(type 1) both indegree and outdegree one,
(type 2) indegree two and outdegree one, or
(type 3) indegree one and outdegree two,

and let $x \in X$ be a distinguished node. Without loss of generality, we may assume that x is of type 3.

From G and x, we construct an Othello position such that the $\exists$-player has a winning strategy in generalized geography if and only if Black has a winning strategy from the constructed position of the generalized Othello game. Black plays first in the constructed game. The overall position on the board is shown in Fig. 2. There is a large region of unguaranteed white territory which is a collection of white markers. The territory is so large that the player who obtains the territory as his own will win the game. Note that the lower right corner of the board is blank. We call this corner δ. Also note that the left-adjacent squares of the unguaranteed white territory are blank.

Lemma 2.2. *Suppose that α', β', γ' and δ are left blank. If Black could place his marker on at least one of α, β and γ in his turn, then Black wins the game. If White could place white markers on all of α, β and γ, then White wins the game.*

Proof. Assume that Black places his marker on α (β, γ) during his turn. Then Black can put his marker on α' (β', γ', resp.), and thus on δ in at most three moves by Black. White cannot prevent these moves by Black. Once Black places his marker on δ, he can change the white markers of the unguaranteed white territory into black ones by putting black markers along the leftmost edge of the territory one by one in the lower left-hand corner first. Since Black obtains the unguaranteed white territory, Black wins the game.

Assume that White places three markers on α, β, γ. This prevents Black from putting a black marker on δ, and White can obtain the unguaranteed white territory as his own. Thus White wins the game. □

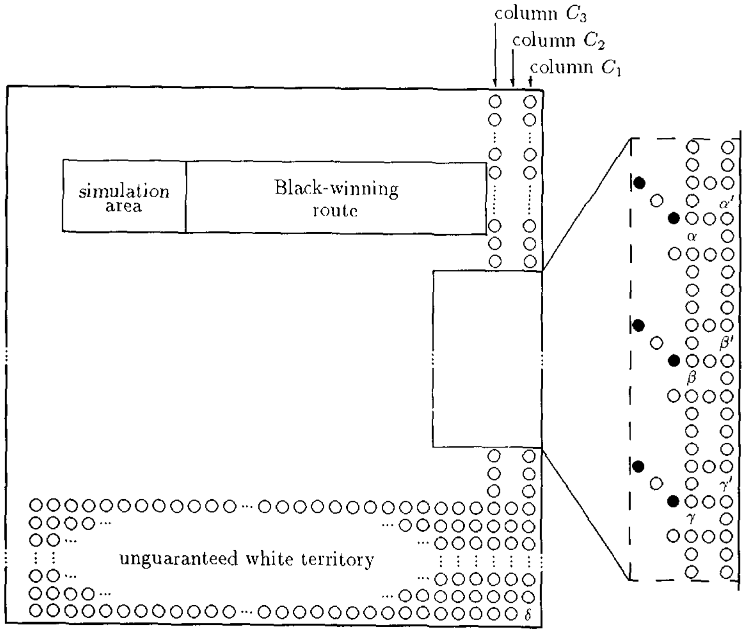
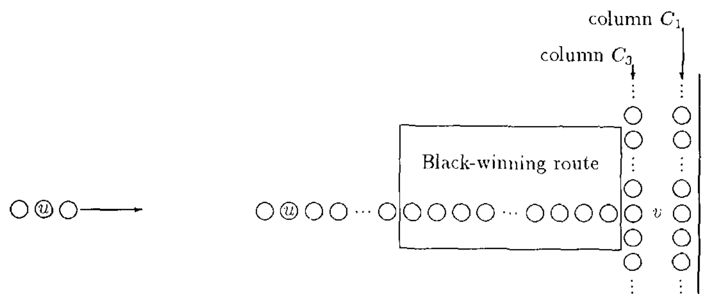


Fig. 2. Constructed Othello position.



(a) (b)

Fig. 3. Abbreviation.

We call the vertical line furthest to the right of the board *column* C_1, the left-adjacent line to C_1 *column* C_2, and the left-adjacent line to C_2 *column* C_3. The *Black-winning route* consists of some horizontal lines of white markers and connects the simulation area with C_3. We call each of those horizontal lines of white markers a *Black-winning line*. We use an abbreviation shown in Fig. 3(a) for Fig. 3(b). The arrow of Fig. 3(a) used in the construction is directed to the right of the board.

A *critical square* contains a white marker on a Black-winning line and there is a white marker on the left of the square. For example, the square of a white marker u of Fig. 3 is critical. A Black-winning line may be continued to the left of the left-adjacent square of u. A *normal play* by Black is a move to change one or more white markers of critical squares into black. A *normal play* by White is a move to change the black markers of critical squares into white.

Lemma 2.3. *Black or White loses the game if he does not play normally.*

Proof. Assume that Black does not play normally. Then White can win the game by putting white markers on α, β, γ in White's next three moves. At least three moves are required for Black to place his marker on one of α, β and γ to prevent these moves by White, since (1) Black makes a white marker on a critical square into black, (2) Black makes a white marker on column C_3 into black and (3) Black puts his marker on one of α, β and γ. During Black's three moves, White can put his three markers on α, β and γ. From Lemma 2.2, White can win the game.

Suppose that White does not play normally. By the previous normal play by Black, there is a black marker on a critical square. Assume that a white marker u of Fig. 3 was changed to black previously. Black can win the game by the following moves: (1) Black puts his marker on v of Fig. 3(b) which is on column C_2; then the markers positioned between u and v become black. Since there is a white marker to the right of v and a white marker to the left of u, White cannot change the black marker on column C_3 in the next move by White. (2) Black puts his marker on at least one of α, β and γ. No two moves by White can prevent Black from making one of these three moves. By Lemma 2.2, Black can win the game. □

From the above lemma, every Black and White move in the constructed game is forced during the course of the simulation of the geography game. Each move of generalized geography is simulated by some pairs of Black and White moves. In what follows we construct Othello positions shown in Figs. 4–7 for each node of the geography graph. We call the Othello position a *configuration* for the node. Each configuration contains *entry* square(s) (shown by P and P') and *exit* square(s) (shown by Q and Q'). The entries and the exits are left blank initially, unless stated otherwise. If there are no markers on entries and exits, then the configuration is called *initial*. If there is a white marker on one of the entries then the configuration is *activated*. Assume that it is Black's turn to move from the configuration just after it is activated. At most one configuration is activated during the simulation of the geography game. Starting from an activated configuration, if no normal play is possible after some nomal plays made by Black and White, then the configuration is called *deactivated*.

We present configurations and normal plays in each of the configurations below. The reader may proceed by examining the following:

(1) Once a configuration is activated, normal plays are only those explained below.

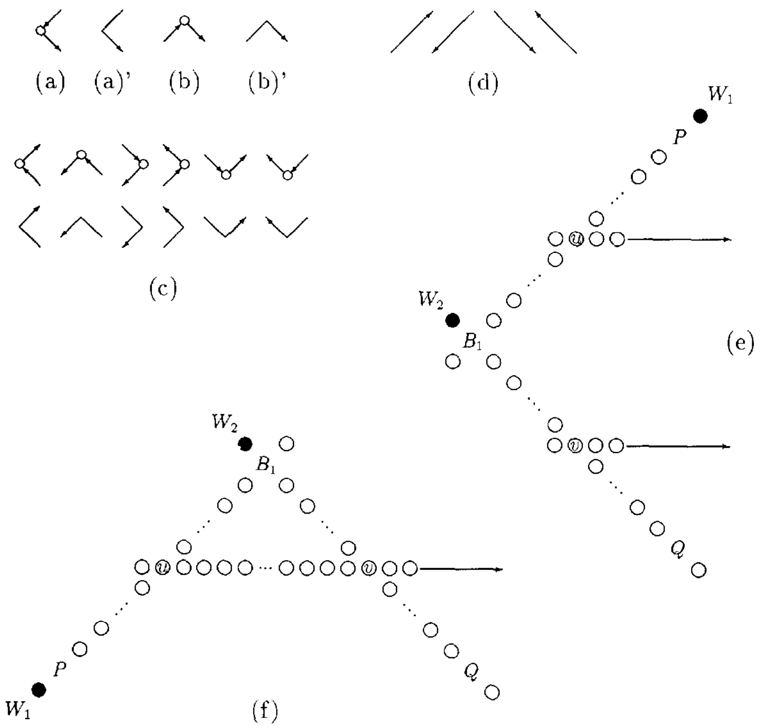


Fig. 4. Construction for a type 1 node, or for a connector.

(2) A configuration is deactivated by a normal move by White which causes a white marker to be placed on one exit.

(3) There is no normal play for Black from an initial configuration or from a deactivated configuration.

For a type 1 node in $X \cup Y$ of the form shown in Fig. 4(a) and (b), or for a connector shown in Fig. 4(a)′ and (b)′, construct configurations of Fig. 4(e) and (f), respectively. Suppose that there is a white marker on P and the configuration is activated. Black puts his marker on B_1 to change u to black, and then White on W_1 to change the color of u to white. Black then puts his marker on Q to change v to black, and White on W_2. By the last move of White, the marker on Q is changed into white. The configuration is deactivated and it is Black's turn to move. Similar constructions are possible corresponding to each type 1 node or each connector shown in Fig. 4(c) Fig. 4(d).

For a type 2 node in X of the form shown in Fig. 5(a), construct a configuration Fig. 5(b), where the two double-circled squares are left blank. Without loss of generality, we assume that the ∀-player comes from the upper left, and the configuration is activated by placing a white marker on P. Black puts his marker on B_1, then White on W_1, Black on B_2, White on W_2, Black on Q and White on W_3. Note that the

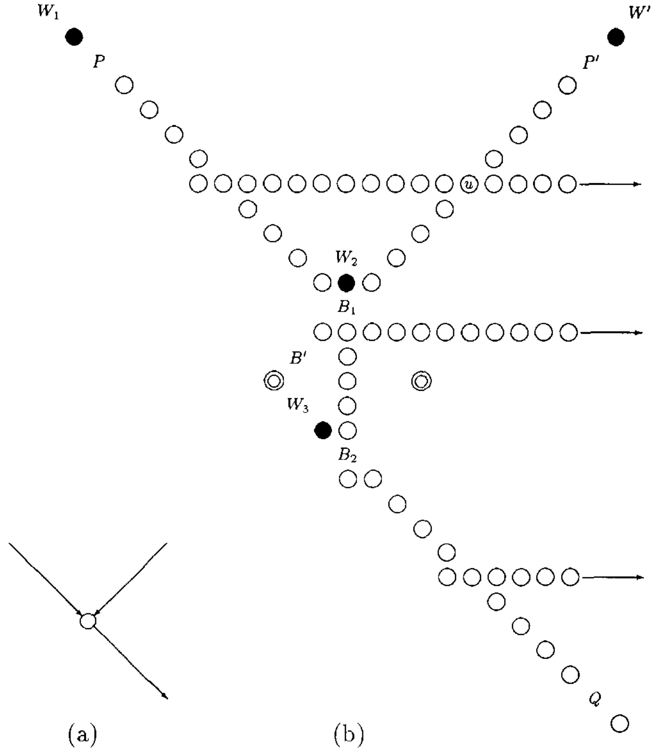


Fig. 5. Type 2 node.

configuration is deactivated and that a white marker is placed on the square Q. These moves simulate that the ∀-player places a marker on a type 2 node of X in generalized geography.

We say that a configuration is *revisited* if a white marker is placed on the blank entry of deactivated configuration. Assume that a deactivated configuration of Fig. 5(b) with no markers on double-circled squares is revisited by putting a white marker on P', and that it is Black's turn to move. Then Black wins the game, since Black puts his marker on B' to change u to black and since there will be no normal play by White. This implies in generalized geography that once the ∀-player placed a marker on a type 2 node of X, he cannot put a marker on the same node again.

For a type 2 node in Y of the form shown in Fig. 5(a), construct a configuration Fig. 5(b), where there are two white markers on the double-circled squares. We assume that the ∃-player comes from the upper left, and that a white marker is placed on P to activate the configuration. Black then places a black marker on B_1, White on W_1, Black on B_2, White on W_2, Black on Q and White on W_3. Then the configuration is deactivated and there is a white marker on Q. These moves correspond, in

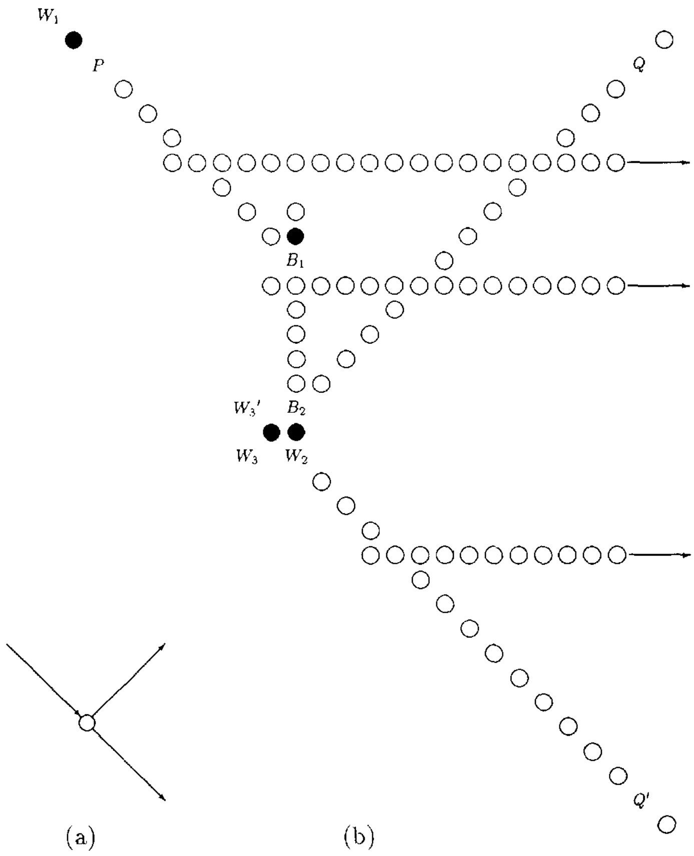


Fig. 6. Type 3 node of X.

generalized geography, to the ∃-player putting a marker on a type 2 node of Y. If another white marker is placed on P' to change the status of the configuration from deactivated into revisited, then Black puts his marker on B', White on W', and there will be no normal play by Black; thus White wins the game. A series of these moves simulate that after the ∃-player places a marker on a type 2 node of Y, he cannot place a marker any more on the same node in generalized geography.

For a type 3 node in X of the form shown in Fig. 6(a), construct Fig. 6(b). Black places a black marker on B_1, then White on W_1, Black on B_2, White on W_2 and Black can choose either Q or Q' to put a black marker: if Black chooses Q (Q') then White's normal play will be on W_3 (W_3', resp.). It is again Black's turn. This corresponds, in generalized geography, to the ∃-player selecting one of the nodes of Y to place a marker.

For a type 3 node in Y of the form shown in Fig. 7(a), construct Fig. 7(b). Black puts his marker on B_1, then White on W_1, Black on B_2 to change the white marker on

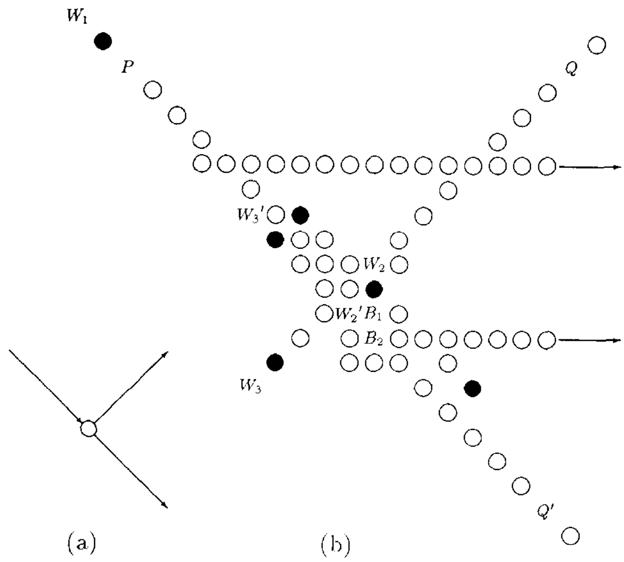


Fig. 7. Type 3 node of Y.

B_1 to black, and White can choose either W_2 or W_2' to make the three black markers including B_1 and B_2 into white: if White chooses W_2 then Black places his marker on Q, and White on W_3; and if White chooses W_2' then Black on Q' and White on W_3'. The configuration is deactivated and it is again Black's turn. This corresponds, in the geography game to the ∀-player selecting one of the nodes of X to place a marker.

Assume that there is an edge from node y to node z in the geography graph. In order to connect the exit of the configuration for y with the entry of the configuration for z, either we identify the exit of u with the entry of z, or we may use connectors shown in Fig. 4. To identify the exit with the entry, it is necessary that a direction of the line to the exit is orthogonal to a direction of the line from the entry. To connect the exit with the entry using connectors, it is further required that the exit and the entry are in the same parity positions. Two squares located in (x_1, y_1) and (x_2, y_2) in the system of coordinates are said to be in the same parity positions if

$$x_1 + y_1 \equiv x_2 + y_2 \pmod 2.$$

If their parities are different, then connect the parity changer of Fig. 8 to obtain the same parities.

Each edge of the geography graph is constructed by a series of white markers placed in a diagonal line, and a Black-winning line is formed by white markers placed in

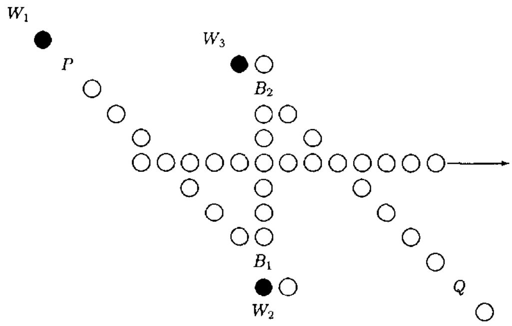


Fig. 8. Parity changer.

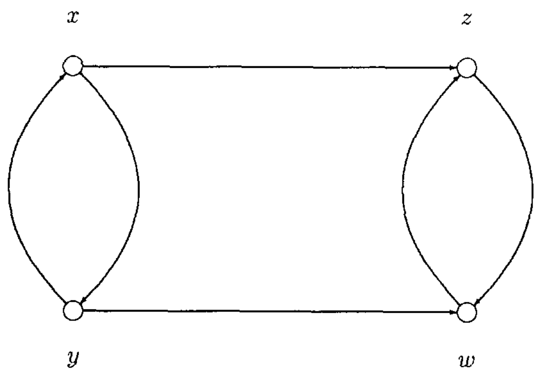


Fig. 9. Example of bipartite graph.

a horizontal line. We note that two diagonal lines of white markers which are vertical may cross each other in the construction, and that both, a diagonal line of white markers and a Black-winning line, may cross on the board.

We have assumed that the distinguished node x of G is in X and is of type 3. A white marker is placed on P of Fig. 6(b) of the configuration for x, and the configuration is activated. By a sequence of normal plays by Black and White, the constructed Othello game simulates the moves of the geography game.

Suppose that the $\exists$-player has a winning strategy in a given geography game. According to the winning strategy, Black can win the Othello game. White will lose the game either because he does not play normally or because he enters a revisited configuration for a node of X.

Assume that the $\forall$-player has a winning strategy in the geography game. At the beginning of the simulation of a move of the geography game, only one configuration is activated or revisited, and the other configurations are either initial or deactivated.

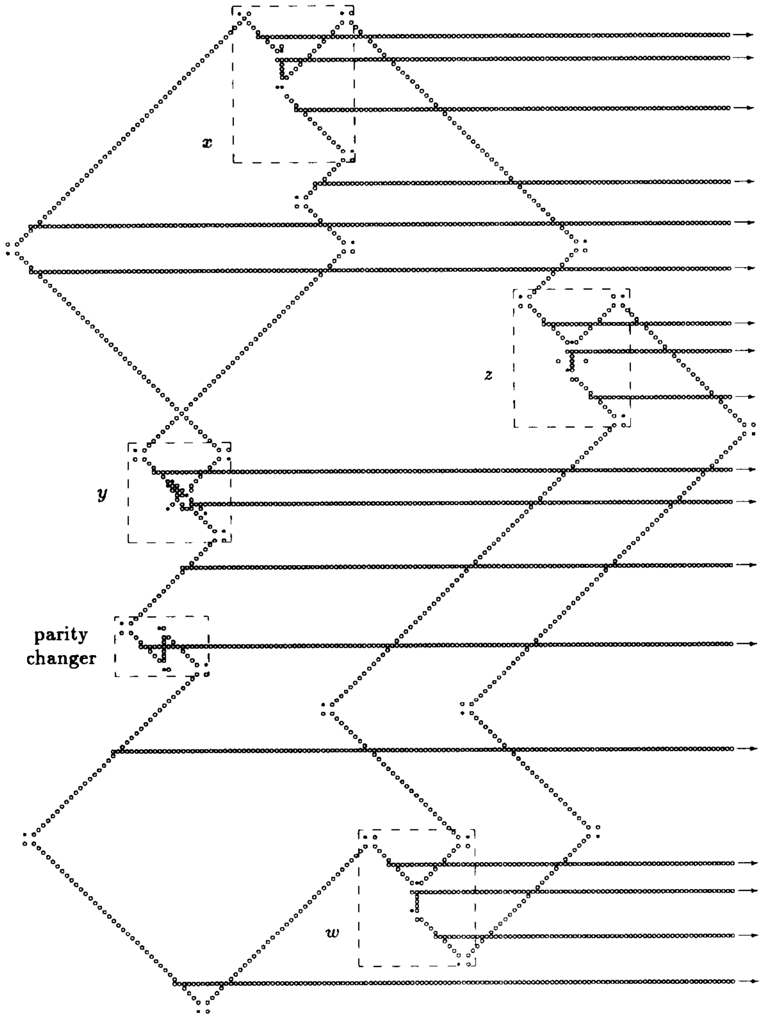


Fig. 10. Example of construction.

Since Black's play from initial configurations or from deactivated configurations is not normal, Black either simulates the geography game or does not play normally. Thus White can win the constructed game according to the winning strategy of the geography game.

As an example of the construction, consider a geography game on a bipartite graph shown in Fig. 9 with distinguished node x. The corresponding Othello position will be the one shown in Fig. 10.

The construction of the Othello position can be performed within polynomial time. This proves the theorem. □

References

[1] H. Adachi, H. Kamekawa on S. Iwata, Shogi $n \times n$ board is complete in exponential time, *Trans. IEICE* **J70-D** (1987) 1843–1852 (in Japanese).
[2] A.K. Chandra, D.C. Kozen and L.J. Stockmeyer, Alternation, *J. ACM* **28** (1981) 114–133.
[3] S. Even and R.E. Tarjan, A combinatorial problem which is complete in polynomial space, *J. ACM* **23** (1976) 710–719.
[4] A.S. Fraenkel and D. Lichtenstein, Computing a perfect strategy for $n \times n$ chess requires time exponential in n, *J Combin. Theory Ser. A.* **31** (1981) 199–214.
[5] D. Lichtenstein, Planar formulae and their uses, *SIAM J. Comput.* **11** (1982) 329–343.
[6] D. Lichtenstein and M. Sisper, GO is polynomial-space hard, *J. ACM* **27** (1980) 393–401.
[7] S. Reisch, Gobang ist PSPACE-vollständig, *Acta Inform.* **13** (1980) 59–66.
[8] S. Reisch, Hex ist PSPACE-vollständig, *Acta Inform.* **15** (1981) 167–191.
[9] J.M. Robson, The complexity of GO, in: *Proc. IFIP* 1983 (1983) 413–417.
[10] J.M. Robson, N by N checkers is EXPTIME complete, *SIAM J. Comput.* **13** (1984) 252–267.
[11] T.J. Schaefer, On the complexity of some two-person perfect-information games, *J. Comput. System Sci.* **16** (1978) 185–225.
[12] R. Uehara and S. Iwata, Generalized Hi-Q is NP-complete, *Trans. IEICE* **E73** (1990) 270–273.