Discrete Mathematics 30 (1980) 133–149.

THE ENUMERATION OF SEQUENCES WITH RESPECT TO STRUCTURES ON A BIPARTITION

D.M. JACKSON, B. JEFFCOTT and W.T. SPEARS

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont. N2L 3G1, Canada

Received 22 September 1978
Revised 16 August 1979

We give a method for enumerating sequences over a finite alphabet with respect to certain maximal configurations. The required generating functions are obtained as solutions of systems of linear equations. The method utilizes a combinatorial decomposition of sequences into maximal sub-configurations.

1. Introduction

A large class of enumeration problems concerns the enumeration of sequences over an alphabet $\mathcal{N}_n = \{1, \ldots, n\}$ subject to conditions specifying the number of

(C1) increasing subsequences of length p,

(C2) non-increasing subsequences of length q,

(C3) an increasing subsequence of length p, followed immediately by a non-increasing subsequence of length q,

(C4) maxima,

(C5) rises,

(C6) non-rises.

A *rise* is an element of the set $\pi_1^{(1)} = \{(i, j) \in \mathcal{N}_n^2 \mid i < j\}$, and a *non-rise* is an element of $\pi_2^{(1)} = \{(i, j) \in \mathcal{N}_n^2 \mid i \geq j\}$. A *maximum* is an element σ_i of $\sigma = \sigma_1 \cdots \sigma_l \in \mathcal{N}_n^2$ such that either $\sigma_{i-1} < \sigma_i \geq \sigma_{i+1}$, or $\sigma_{i-1} < \sigma_i$ where $i = l$. A subsequence $\alpha_1 \cdots \alpha_s$ of σ is taken throughout to be a sequence such that $\sigma_{t+i} = \alpha_i$ for $i = 1, 2, \ldots, s$ where $s + t \leq l$. The enumeration of alternating permutations (permutations $\sigma_1 \cdots \sigma_n$ of $\mathcal{N}_n$ such that $\sigma \geq \sigma_2 < \sigma_3 \geq \sigma_4 < \cdots$), considered by André [2, 3] and later by others, is an example of a problem in this class. André obtained the number, c_n, of such permutations as

$$\left[\frac{x^n}{n!}\right](\sec x + \tan x)$$

where $[x^n]f[x]$ denotes the coefficient of x^n in the formal power series $f(x)$ in the indeterminate x.

This class of problems contains an underlying bipartition $\Pi = \{\Pi_1, \Pi_2\}$ of $\mathcal{N}_n^2$

and the structures to be recognized (increasing subsequences, for example) are constructed from the blocks of this partition. In the subsequent analysis, theorems apply to an arbitrary bipartition, although it is convenient to recall the specific case $\Pi^{(1)}=\{\pi_1^{(1)},\pi_2^{(1)}\}$ for concrete applications and examples. Conditions C1 to C6, which are stated in terms of $\Pi^{(1)}$, have analogous restatements in terms of the arbitrary bipartition Π.

It seems likely that any general methods for enumerating sequences will be of value more generally in combinatorial enumeration since sequences may be used as devices for encoding more complex combinatorial structures. The encoding of planar maps (Cori and Richard [6]) and self-avoiding walks on infinite square grids are examples of the use of sequences for such purposes. Contributions to the area of sequence enumeration have been made by Cartier and Foata [5], Foata and Schützenberger [7], Stanley [15], Gessel [8], Jackson and Goulden [11], Reilly [14] and others. The methods employed in this paper are similar to those employed in Jackson and Goulden [11].

The class of problems described above seems to be a reasonable area in which to search for evidence for a general theory of sequence enumeration because of the abundance of published results on specific cases. Where such problems have been addressed in the past, they have usually been treated by the "classical method", perhaps the most familiar of all enumerative methods. The method rests, in principle, on the discovery of a convenient decomposition of the set of sequences to be enumerated. This is then used to develop a recurrence equation for the required quantity. The classical method employs a variety of inversion formulae and special results on combinatorial numbers (Euler numbers, Genocchi numbers, and Motzkin numbers for example). The disadvantages of such an approach are well-known and apparent to any one who has used it. In the first place it is not always an easy matter to find a decomposition, and the special case analysis which this entails is often prohibitive. In the second place, even when a recurrence equation has been constructed, it is seldom possible to perceive a direct route to the solution. It requires considerable dexterity, and the familiarity which this implies, to recognize the appropriate inversion formula to apply and to recall specific properties of combinatorial numbers for manipulative purposes. Finally, slight variations of a given problem often unset the analysis so much that there is little hope of deriving a solution of the variant from the original.

The purpose of this paper is twofold. First we present a single theorem, based on a straightforward decomposition of the sequence monoid, which is sufficient for enumerating sequences subject to the conditions C1–C6. The theorem unifies many of the existing results. Second, we use a formalism which has the two important properties of being algebraically convenient while remaining combinatorially motivated. The methods are such that the required generating functions are constructed directly from the generating functions for subproblems which are easier to derive. Explicit coefficients may be computed by constructing recurrence equations. The method may therefore be regarded, superficially at least, as the classical method operated in reverse. We construct the generating functions by combinatorial means and use recurrence equations for purely computational purposes: the classical method derives recurrence equations by combinatorial means, and these are then used to obtain the generating functions.

The main counting proposition for sequences is given in Section 2, together with a means for specializing the results for sequences to permutations. Sequences with a fixed pattern are considered in Section 3, and the main counting theorem is given in Section 4. This theorem develops the required generating function as the solution of a system of simultaneous linear equations. Specialisations of the main theorem for less refined decompositions are also given in Section 4. The general method is extended in Section 5 to enumerate sequences over another alphabet. These new sequences may be represented by rectangular matrices, called r-arrays. Examples of this extension are given in Section 6. We have selected examples which already have been treated by binomial posets. We have not attempted to list all of its applications of the results presented here. However, the examples which are given are representative of a broad spectrum of applications. Other developments of the theory together with examples have been given by Jackson and Goulden [11, 12].

2. Preliminaries

Let $\sigma = \sigma_1 \cdots \sigma_l \in \mathcal{N}_n^+$ where $\mathcal{N}_n^+ = \mathcal{N}_n^* - \varepsilon$. Then σ has *length* $\lambda(\sigma) = l$. If σ has i_j occurrences of j, for $j = 1, 2, \ldots, n$ then $\tau(\sigma) = (i_1, \ldots, i_n)$ is the *type* of σ. Since a permutation has type $(1, 1, \ldots, 1)$, results for permutations may be deduced immediately from the corresponding ones for sequences. Let $\Pi = \{\pi_1, \pi_2\}$ be a partition of $\mathcal{N}_n^2$ and let $\sigma = \sigma_1\sigma_2 \cdots \sigma_l \in \mathcal{N}_n^*$ be such that $(\sigma_i, \sigma_{i+1}) \in \pi_1$, for $i = 1, 2, \ldots, l-1$. Then σ is called a *π_1-path of (vertex) length l*, or equivalently, a *π_1-path of edge length $l-1$*. π_2-paths are defined similarly. A *(π_1, π_2)-structure of type (p, q)* is a sequence of length $p+q+1$ consisting of a π_1-path of edge length p followed by a π_2-path of edge length q. For example, the sequence 123511 is a $(\pi_1^{(1)}, \pi_2^{(1)})$-structure of type $(3, 2)$, where $\pi_1^{(1)}$ is the set of rises in $\mathcal{N}_n^2$ and $\pi_2^{(1)}$ is the set of non-rises in $\mathcal{N}_n^2$. A π_1-path, π_2-path or a (π_1, π_2)-structure is termed *maximal* if it is not contained properly in another π_1-path, π_2-path or (π_1, π_2)-structure, respectively. A π_1-path is a (π_1, π_2)-structure of type $(p, 0)$ for some p, and a *π_2-path is a* (π_1, π_2)-structure of type $(0, q)$ for some q. Accordingly we term π_1-path and π_2-paths *degenerate* (π_1, π_2)-structures. Any sequence may be decomposed uniquely into an ordered set $(s_1, s_2, \ldots, s_r)$ of (π_1, π_2)-structures in which only s_1 and s_r are degenerate, and $r \geq 2$. Of course s_1 and s_r may be empty.

Let $x_1, \ldots, x_n$ be commutative indeterminates and let $\mathbf{X} = \operatorname{diag}(x_1, \ldots, x_n)$ be the diagonal matrix with elements $x_1, \ldots, x_n$. Let $\boldsymbol{\omega} = \mathbf{XJ}$, where $\mathbf{J}$ is the $n \times n$ matrix of all 1's, and $\mathbf{e} = [\delta_{ij}]_{n \times n}$ denote the $n \times n$ identity matrix. Let $\mathbf{a} = \mathbf{X}\mathcal{I}(\pi_1)$

and $\mathbf{b} = \mathbf{X}\mathcal{I}(\pi_2)$ where $\mathcal{I}(\pi_k)$ is an $n \times n$ matrix such that

$$[\mathcal{I}(\pi_k)]_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \pi_k, \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2$. If $\mathbf{C} = [c_{ij}]_{s\times t}$ and $\mathbf{D} = [d_{ij}]_{s\times t}$, then $\mathbf{C}^{\mathbf{D}}$ denotes the expression

$$\prod_{\substack{1\le i\le s\\ 1\le j\le t}} c_{ij}^{d_{ij}}.$$

If $\mathbf{c} = (c_1, \ldots, c_s)^T$ then$[\mathbf{C}:\mathbf{c}]_k$ denotes the $s \times t$ matrix formed from $\mathbf{C}$ by replacing column k of $\mathbf{C}$ with $\mathbf{c}$. If $f(\boldsymbol{x})$ is a formal power series in $x_1, \ldots, x_n$ then $[\boldsymbol{x}^{\mathbf{i}}]$ denotes the coefficient of $x_1^{i_1} \cdots x_n^{i_n}$ in $f(\boldsymbol{x})$.

Let $\sigma = \sigma_1 \cdots \sigma_l \in \mathcal{N}_n^+$, and $\kappa(\sigma) = (j_1, \ldots, j_{l-1})$ where $(\sigma_i, \sigma_{i+1}) \in \pi_{j_i}$ for $i = 1, \ldots, l-1$. We call $\pi_{j_1}\pi_{j_2} \cdots \pi_{j_{l-1}}$ the *partition sequence* of σ.

Proposition 2.1. *Let* $\mathbf{M}_1 = \mathcal{I}(\pi_1)$ *and* $\mathbf{M}_2 = \mathcal{I}(\pi_2)$, *and* $\sigma \in \mathcal{N}_n^*$. *Then*

$$[\mathbf{X}\mathbf{M}_{k_1}]_{\sigma_1,\sigma_2}[\mathbf{X}\mathbf{M}_{k_2}]_{\sigma_2,\sigma_3}, \ldots [\mathbf{X}\mathbf{M}_{k_{l-1}}]_{\sigma_{l-1},\sigma_l} x_l = \begin{cases} \boldsymbol{x}^{\tau(\sigma)} & \text{if } \boldsymbol{k} = \kappa(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Straightforward. $\square$

Proposition 2.2. *Let* g_{ij} *where* $i, j \geq 1$, u *and* v *be commutative indeterminates. Then the number of sequences in* $\mathcal{N}_n^*$ *of type* $\mathbf{i}$, *with an initial maximal* π_2-*path of edge length* p *and a terminal maximal* π_1-*path of edge length* q, *and* m_{ij} *maximal* (π_1, π_2)-*structures of type* (i, j) *for* $i, j \geq 1$ *is*

$$[\boldsymbol{x}^{\mathbf{i}}\mathbf{g}^{m}u^{p}v^{p}]\Phi(\boldsymbol{x}, u, v, \mathbf{g})$$

where

$$\Phi(\boldsymbol{x}, u, v, \mathbf{g}) = \operatorname{trace}(\mathbf{e} - u\mathbf{b})^{-1}\Big\{\mathbf{e} - \sum_{i,j\geq 1} g_{ij}\mathbf{a}^{i}\mathbf{b}^{j}\Big\}^{-1}(\mathbf{e} - v\mathbf{a})^{-1}\boldsymbol{\omega}$$

and $\mathbf{g}$ *is the matrix whose* (i,j)-*element is* $g_{i,j}$, $i, j \geq 1$.

Proof. Let π_1 and π_2 be non-commutative undeterminates. Then

$$(1 - \pi_2)^{-1}\Big\{1 - \sum_{i,j\geq 1} \pi_1^{i}\pi_2^{j}\Big\}^{-1}(1 - \pi_1)^{-1} = \sum_{s\in\{\pi_1,\pi_2\}^*}$$

where $\{\pi_1, \pi_2\}^*$ is the set of all sequences on the set of symbols π_1 and π_2.

From Proposition 2.1 we have

$$\sum_{\substack{\sigma\in\mathcal{N}_n^*\\ \lambda(\sigma)=l\\ \kappa(\sigma)=\boldsymbol{k}}} \boldsymbol{x}^{\tau(\sigma)} = \operatorname{trace} \mathbf{X}\mathbf{M}_{k_1}\mathbf{X}\mathbf{M}_{k_2} \cdots \mathbf{X}\mathbf{M}_{k_l}\mathbf{X}\mathbf{J}$$

where $\boldsymbol{k}=(k_1,\ldots,k_{l-1})$ and $\mathbf{J}$ is the matrix of all 1's.

It follows (see Lemma 3.8), Jackson and Goulden [11]) that

$$\sum_{\sigma\in\mathcal{N}^+}\boldsymbol{x}^{\tau(\sigma)}=\operatorname{trace}(\mathbf{e}-\mathbf{b})^{-1}\left\{\mathbf{e}-\sum_{i,j\geq 1}\mathbf{a}^i\mathbf{b}^j\right\}^{-1}(\mathbf{e}-\mathbf{a})^{-1}\boldsymbol{\omega}.$$

We may now mark the maximal (π_1,π_2)-structures, π_1-paths and π_i-paths as follows. Let u^i, v^i mark the initial maximal π_i-path of edge length i and terminal maximal π_1-path of edge length i. Let g_{ij} mark the maximal (π_1,π_2)-structures of type (i,j). Inserting these into the expression for $\sum_{\sigma\in\mathcal{N}_n^+}\boldsymbol{x}^{k(\sigma)}$ we have

$$\operatorname{trace}(\mathbf{e}-u\mathbf{b})^{-1}\left\{e-\sum_{i,j\geq 1}g_{ij}\mathbf{a}^i\mathbf{b}^j\right\}^{-1}(\mathbf{e}-v\mathbf{a})^{-1}\boldsymbol{\omega}=\sum_{\sigma\in\mathcal{N}_n^+}\boldsymbol{x}^{\tau(\sigma)}u^{\nu(\sigma)}v^{\mu(\sigma)}\mathbf{g}^{\mathbf{m}(\sigma)}$$

where $\nu(\sigma)$ is the edge length of the initial maximal π_2-path, $\mu(\sigma)$ is the edge length of the terminal maximal π_1-path and $\mathbf{m}(\sigma)=[m_{ij}(\sigma)]$ where $m_{ij}(\sigma)$ is the number of maximal (π_1,π_2)-structures of type (i,j) in σ. The result follows. □

In the next section we obtain an explicit expression for $\Phi(\boldsymbol{x},u,v,\mathbf{g})$ in terms of the π_1-path enumerators. The following result gives the π_1-path enumerators in terms of the incidence matrix $\mathcal{I}(\pi_1)$ of π_1, and a method for reducing to permutations. The results are taken from Jackson and Goulden [11], but are included here for completeness.

Proposition 2.3. *The number of π_1-paths of type $\boldsymbol{i}$ and vertex length $k+1$ is $[\boldsymbol{x}^{\boldsymbol{i}}]\gamma_{k+1}(\pi_1,\mathbf{x})$ where $\gamma_0(\pi_1,\mathbf{X})=1$ and $\gamma_{k+1}(\pi_1,\mathbf{X})=\operatorname{trace}\mathbf{a}^k\boldsymbol{\omega}$, $k\geq 0$.*

Proof. Put $g_{ij}=0$, $i,j\geq 1$ and $u=0$ in Proposition 2.2, and the result follows. □

For brevity we will denote $\gamma_k(\pi_1,\mathbf{X})$ by $\gamma_k(\pi_1)$, except in situations where it becomes important to specify the actual indeterminates.

For the most frequently encountered bipartition, namely $\Pi^{(1)}=\{\pi_1^{(1)},\pi_2^{(1)}\}$, the path enumerators have a particularly simple form. This is given in the next proposition.

Proposition 2.4.

$$\sum_{k\geq 0}\gamma_k(\pi_1^{(1)})x^k=\prod_{i=1}^{n}(1+xx_i)\quad\textit{and}\quad\sum_{k\geq 0}\gamma_k(\pi_2^{(1)})x^k=\prod_{i=1}^{k}(1-xx_i)^{-1}$$

Proof. Immediate. □

The next lemma facilitates the specialisation from sequences to permutations for problems involving the special bipartition $\Pi^{(1)}$.

Lemma 2.5. *Let $\phi(\gamma_1, \gamma_2, \ldots)$ be a power series in $\gamma_1, \gamma_2, \ldots$ where $\gamma_k \equiv \gamma_k(\pi_1^{(1)})$. Then*

$$[\boldsymbol{x}]\phi(\gamma_1, \gamma_2, \ldots) = \left[\frac{x^n}{n!}\right]\phi\left(\frac{x^n}{1!}, \frac{x^2}{2!}, \ldots\right).$$

Proof.

$$[\boldsymbol{x}]\phi(\gamma_1, \gamma_2, \ldots) = [\boldsymbol{x}]\phi(\gamma_1, \gamma_2, \ldots, \gamma_n, 0, 0, \ldots) = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}}[\boldsymbol{x}]\gamma_1^{i_1} \cdots \gamma_n^{i_n}$$

where $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$. But

$$[\boldsymbol{x}]\gamma_1^{i_1} \cdots \gamma_n^{i_n} = n!\,(1!^{i_1}\,2!^{i_2} \cdots n!^{i_n})^{-1}$$

where $i_1 + 2i_2 + 3i_3 + \cdots + ni_n = n$. Thus

$$[\boldsymbol{x}]\phi(\gamma_1, \gamma_2, \ldots) = \left[\frac{x^n}{n!}\right]\sum_{\boldsymbol{i}} a_{\boldsymbol{i}}\left(\frac{x}{1!}\right)^{i_1}\left(\frac{x^2}{2!}\right)^{i_2} \cdots \left(\frac{x^n}{n!}\right)^{i_n}$$

and the result follows. □

We note that the enumeration of sequences in $\mathcal{N}_n^*$ when the conditions C1, C2, ..., C6 of Section 1 are applied may be obtained by setting g_{ij}, u, v to the appropriate values or to the appropriate power series. Accordingly the class of enumeration problems described in Section 1 may be regarded as those which entail the enumeration of sequences with respect to (π_1, π_2)-structures.

3. Sequences with a fixed pattern

We consider first the enumeration of sequences in $\mathcal{N}_n^*$ with a fixed partition sequence. The result is used in later sections.

Lemma 3.1. *Let $\Pi = \{\pi_1, \pi_2\}$ be an arbitrary bipartition of $\mathcal{N}_r^2$ and let $p_1, \ldots, p_k \geq 1$. Let $p_1 + \cdots + p_j = s_j$, $j = 1, 2, \ldots, k$ and $s_0 = 0$. Then the number of sequences in $\mathcal{N}_n^*$ with partition sequence*

$$\pi_1^{p_1-1}\pi_2\pi_1^{p_2-1}\pi_2 \cdots \pi_1^{p_{k-1}-1}\pi_2\pi_1^{p_k-1}$$

and type $\boldsymbol{i}$ is

$$[\boldsymbol{x}^{\boldsymbol{i}}]\det \mathbf{Q}$$

where $\mathbf{Q} = [q_{ij}]_{k\times k}$ and

$$q_{ij} = \begin{cases} \gamma_{s_j - s_{i-1}} & \text{if } j \geq i, \\ 1 & \text{if } j+1 = i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let

$$\xi_j = H_j(p_j, \ldots, p_k) = \text{trace}\, \mathbf{a}^{p_j-1}\mathbf{b} \ldots \mathbf{a}^{p_{k-1}-1}\mathbf{b}\mathbf{a}^{p_k-1}\boldsymbol{\omega}, \quad k \geq j \geq 1.$$

From Proposition 2.1 the desired generating function is ξ_1. Now

$$\begin{aligned}\xi_j &= \text{trace}\, \mathbf{a}^{p_j-1}(\boldsymbol{\omega}-\mathbf{a})\mathbf{a}^{p_{j+1}-1}\mathbf{b} \cdots \mathbf{b}\mathbf{a}^{p_k-1}\boldsymbol{\omega} \\ &= \gamma_{p_j} H_{j+1}(p_{j+1}, \ldots, p_k) - H_{j+1}(p_j + p_{j+1}, p_{j+2}, \ldots, p_k)\end{aligned}$$

whence, by iterating this, we have

$$\xi_j = \sum_{i=j}^{k} \gamma_{s_i - s_{j-1}} \xi_{i+1} (-1)^{i-j} \quad \text{for } j = 1, 2, \ldots, k$$

where

$$\xi_{k+1} = 1.$$

Let $\boldsymbol{\xi} = (\xi_2, \xi_3, \ldots, \xi_{k+1})^{\mathrm{T}}$, $\boldsymbol{c} = (\xi_1, 0, \ldots, 0)$ and $\mathbf{Q}' = [q'_{ij}]_{k \times k}$ where $q'_{ij} = (-1)^{j-i} q_{i,j}$, $1 \leq i, j \leq k$. Then $\boldsymbol{\xi}$ satisfies the system of linear equations

$$\mathbf{Q}'\boldsymbol{\xi} = \boldsymbol{c}.$$

Thus by Cramer's Rule, we have

$$1 = \xi_{k+1} = \frac{\det[\mathbf{Q}' : \boldsymbol{c}]_k}{\det \mathbf{Q}} = \frac{\xi_1}{\det \mathbf{Q}'}$$

where $\xi_1 = \det \mathbf{Q}' = \det \mathbf{Q}$, and the result follows. □

We remark that this lemma is equivalent to

$$\text{trace}\,(\mathbf{a}^{p_1-1}\mathbf{b} \cdots \mathbf{b}\mathbf{a}^{p_{k-1}-1}\mathbf{b}\mathbf{a}^{p_k-1}\boldsymbol{\omega}) = \|\gamma_{s_j - s_{i-1}}\|$$

where $s_0 = 0$, $p_1 + \cdots + p_j = s_j$ for $j = 1, 2, \ldots, k$ and $\gamma_i = 0$ if $i < 0$.

4. The main theorem

In this section we consider the evaluation of trace $(\mathbf{e} - u\mathbf{b})^{-1} Q^{-1} (\mathbf{e} - v\mathbf{a})^{-1} \omega$ where $Q = -\sum_{i,j \geq 0} g_{ij} \mathbf{a}^i \mathbf{b}^j$, and $q_{00} = -1$. By setting $g_{0j} = 0$ for $j \geq 1$ and $g_{i,0} = 0$ for $i > 1$ we may deduce an expression for $\Phi(\boldsymbol{x}, u, v, \mathbf{g})$ defined in Proposition 2.2.

Lemma 4.1. *Let $\Pi = \{\pi_1, \pi_2\}$ be an arbitrary bipartition of $\mathcal{N}_n^2$. Let $Q = -\sum_{i,j \geq 0} g_{ij} \mathbf{a}^i \mathbf{b}^j$ and $R = (\mathbf{e} - v\mathbf{a}) Q (\mathbf{e} - u\mathbf{b})$ where $g_{00} = -1$ and $\{g_{ij} \mid i, j \geq 0\}$ are commutative indeterminates. Let $\xi_j = \text{trace}\, \mathbf{b}^j R^{-1} \boldsymbol{\omega}$ and $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots)^{\mathrm{T}}$. Then $\boldsymbol{\xi}$ satisfies the system of linear equations*

$$\mathbf{A}\boldsymbol{\xi} = \boldsymbol{c}$$

where $\mathbf{A}=[a_{jk}]$ *and*

(1) $a_{jk} = \text{trace}\ a^jF^{-1}F_k\boldsymbol{\omega} + \zeta(j,k)(-1)^k\gamma_{j-k}(\pi_1)$ *and*

$$\zeta(j,k)=\begin{cases}1 & \text{if } k\le j,\\ 0 & \text{otherwise.}\end{cases}$$

(2) $\mathbf{c}=(c_0,c_1,\ldots)^{\mathrm{T}}$ *and* $c_j = \text{trace}\ \mathbf{a}^jF^{-1}\boldsymbol{\omega}$.

(3) $F \equiv F(u,v,\mathbf{g}) = -(\mathbf{e}+u\mathbf{a})(\mathbf{e}-v\mathbf{a})\sum_{i,j\ge 0} g_{ij}(-1)^j\mathbf{a}^{i+j}$,

$$F_k \equiv F_k(u,v,\mathbf{g}) = -(\mathbf{e}-v\mathbf{a})(\mathbf{e}+v\mathbf{a})\sum_{\substack{i\ge 0\\ l\ge 1}}(-1)^{l-1}\mathbf{a}^{l+i-1}g_{i,l+k}$$

$$+\,u(\mathbf{e}-v\mathbf{a})\sum_{i\ge 0} g_{i,k}\mathbf{a}^i,\quad k\ge 0$$

Moreover, if $F_l=0$ *for* $l\ge N$, *then* $\boldsymbol{\xi}'=(\xi_0,\xi_1,\ldots,\xi_{N-1})^{\mathrm{T}}$ *satisfies the system of linear equations*

$$\mathbf{B}\boldsymbol{\xi}'=\boldsymbol{d}$$

where $\mathbf{B}=[a_{ij}]_{N\times N}$ *and* $\boldsymbol{d}=(c_0,c_1,\ldots,c_{N-1})^{\mathrm{T}}$.

Proof. Now

$$R=-(\mathbf{e}-v\mathbf{a})\left\{\sum_{i\ge 0} g_{i0}\mathbf{a}^i+\sum_{\substack{i\ge 0\\ j\ge 1}} g_{ij}\mathbf{a}^i\mathbf{b}^j\right\}+(\mathbf{e}-v\mathbf{a})u\sum_{i,j\ge 0} g_{ij}\mathbf{a}^i\mathbf{b}^j$$

But $\mathbf{b}^j=(-1)^j\mathbf{a}^j+\sum_{k=0}^{j-1}(-1)^k\mathbf{a}^k\boldsymbol{\omega}\mathbf{b}^{j-k-1}$. Substituting for $\mathbf{b}^j$ in R we have after routine manipulation

$$R=F+\sum_{k\ge 0}F_k\boldsymbol{\omega}\mathbf{b}^k$$

Thus for $j\ge 0$

$$\text{trace}\ \mathbf{a}^j\mathbf{F}^{-1}\boldsymbol{\omega}=\text{trace}\ \mathbf{a}^jF^{-1}RR^{-1}\boldsymbol{\omega}=\text{trace}\ \mathbf{a}^jF^{-1}\{F+\textstyle\sum_{k\ge 0}F_k\boldsymbol{\omega}\mathbf{b}^k\}R^{-1}\boldsymbol{\omega},$$

whence $c_j=\text{trace}\ \mathbf{a}^jR^{-1}\boldsymbol{\omega}+\sum_{l=0}^{\infty}d(j,l)\xi_l$ where $d(j,l)=\text{trace}\ a^jF^{-1}F_l\boldsymbol{\omega}$. But

$$\mathbf{a}^j=\sum_{k=0}^{j-1}(-1)^k\mathbf{a}^{j-k-1}\boldsymbol{\omega}\mathbf{b}^k+(-1)^j\mathbf{b}^j.$$

Thus trace $\mathbf{a}^jR^{-1}\boldsymbol{\omega}=\sum_{k=0}^{j}(-1)^k\xi_k\gamma_{j-k}$ where $\gamma_{j-k}\equiv\gamma_{j-k}(\pi_1)$, $\gamma_0=1$. Substituting this into the expression for c_j we have

$$c_j=\sum_{k=0}^{j}(-1)^k\gamma_{j-k}\xi_k+\sum_{l=0}^{\infty}d(j,l)\xi_l\quad\text{for}\quad j\ge 0$$

So $\boldsymbol{\xi}$ satisfies the system of linear equations $A\boldsymbol{\xi} = \mathbf{c}$.

If $F_l = 0$ for $l \geq N$, then $\mathbf{A}$ takes the form

$$\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

for some matrices $\mathbf{C}$, $\mathbf{D}$. $\mathbf{B}$ is $N \times N$ and $\mathbf{0}$ is a zero matrix. Thus $\boldsymbol{\xi}'$ satisfies $\mathbf{B}\boldsymbol{\xi}' = \boldsymbol{d}'$, which completes the proof. □

The following theorem yields the generating function for the enumeration of sequences with respect to (π_1, π_2)-structures as the solution of a system of linear equations.

Theorem 4.2. (Main theorem: maximal (π_1, π_2)-structures). *Let $\Pi = \{\pi_1, \pi_2\}$ be an arbitrary bipartition of $\mathcal{N}_n^2$. Then the number of sequences in $\mathcal{N}_n^+$ with an initial maximal π_2-path of edge length p, m_{ij} instances of maximal (π_1, π_2)-structures of type (i, j) for $i, j \geq 1$, a terminal maximal π_1-path of edge length q, and type $\mathbf{t}$ is*

$$[u^p v^q \mathbf{x}^{\mathbf{t}} \mathbf{g}^{\mathbf{m}}]\Phi(\mathbf{x}, u, v, \mathbf{g})$$

where $\Phi(\mathbf{x}, u, v, \mathbf{g}) = \xi_0$ and $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots)^{\mathrm{T}}$ satisfies the system of linear equations

$$\mathbf{A}\boldsymbol{\xi} = \boldsymbol{c}$$

in which

(i) $$[\mathbf{A}]_{jk} = \operatorname{trace} \mathbf{a}^j F^{-1} F_k \boldsymbol{\omega} + \zeta(j, k)(-1)^k \gamma_{j-k}(\pi_1) \quad \textit{and}$$

$$\zeta(j, k) = \begin{cases} 1 & \textit{if } k \leq j, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) $$\boldsymbol{c} = (c_0, c_1, \ldots)^{\mathrm{T}} \quad \textit{and} \quad c_j = \operatorname{trace} \mathbf{a}^j F^{-1} \boldsymbol{\omega}.$$

(iii) $$F = (\mathbf{e} + u\mathbf{a})(\mathbf{e} - v\mathbf{a})\left\{\mathbf{e} - \sum_{i,j \geq 1} g_{ij} \mathbf{a}^{i+j}(-1)^j\right\}.$$

$$F_0 = -(\mathbf{e} - v\mathbf{a})\left\{u\mathbf{e} + (\mathbf{e} + u\mathbf{a}) \sum_{i,k \geq 1} g_{ik}(-1)^{k-1}\mathbf{a}^{i+k-1}\right\}.$$

$$F_l = (\mathbf{e} - v\mathbf{a})\left\{u \sum_{i \geq 1} g_{il}\mathbf{a}^i - (\mathbf{e} + u\mathbf{a}) \sum_{i,k \geq 1} g_{i,l+k}(-1)^{k-1}\mathbf{a}^{i+k-1}\right\}, \quad l > 0$$

Moreover, if $F_l = 0$ for $l \geq N$, then $\boldsymbol{\xi}' = (\xi_0, \ldots, \xi_{N-1})^{\mathrm{T}}$ satisfies the system of linear equations

$$\mathbf{B}\boldsymbol{\xi}' = \boldsymbol{d}$$

where $\mathbf{B} = [a_{ij}]_{N \times N}$ and $d = (c_0, \ldots, c_{N-1})^{\mathrm{T}}$.

Proof. Set $g_{0,0} = -1$, $g_{i,0} = 0$ for $i \geq 1$ and $g_{0,j} = 0$ for $j \geq 1$, and the theorem follows from Lemma 4.1 and Proposition 2.2. □

As a preliminary example of the use of this theorem we enumerate all sequences in $\mathcal{N}_n^*$. Accordingly, set $u = v = g_{ij} = 1$ for $i, j \geqslant 1$. It follows immediately that $F = \mathbf{e}$, $F_0 = -\mathbf{e}$ and $F_l = \mathbf{0}$ for $l \geqslant 1$. Thus the system of linear equations is 1×1 and has the form

$$a_{00}\xi_0 = c_0$$

where $a_{00} = \gamma_0 + \text{trace}\, F^{-1}F\boldsymbol{\omega} = 1 - \text{trace}\, \boldsymbol{\omega} = 1 - \gamma_1$ and $c_0 = \text{trace}\, F^{-1}\boldsymbol{\omega} = \text{trace}\, \boldsymbol{\omega} = \gamma_1$. Thus $\xi_0 = \gamma_1(1-\gamma_1)^{-1}$ so the generating function for all sequences in N_n^* is $1 + \xi_0 = (1-\gamma_1)^{-1} = \{1 - (x_1 + \cdots + x_n)\}^{-1}$ as required. From Lemma 2.5, the number of permutations in N_n is $[x^n/n!](1-x)^{-1} = n!$, as required.

Theorem 4.2 may be specialised in a number of ways. For example, by setting $g_{ij} = g_i h_j$ we may obtain the enumeration of sequences with respect to maximal π_1-paths of length $i \geqslant 1$ marked by g_i and maximal π_2-paths of length $j \geqslant 1$ marked by h_j. The next corollary deals with the enumeration of sequences with respect to maximal π_1-paths alone.

Corollary 4.3 (Maximal π_1-paths). *Let $G(x) = 1 + g_1 x + g_2 x^2 + \cdots$ where g_i marks maximal π_1-paths of vertex length i. Then the number of sequences in $\mathcal{N}_n^*$ with m_i maximal π_1-paths of vertex length i for $i = 1, 2, \ldots,$ and with type $\mathbf{i}$ is*

$$[\mathbf{x}^{\mathbf{i}}\mathbf{g}^{\mathbf{m}}]\left\{\sum_{j \geqslant 0} h_j \gamma_j(\pi_1)\right\}^{-1}$$

where $G^{-1}(x) = \sum_{i \geqslant 0} h_i x^i$, $\mathbf{g} = (g_1, g_2, \ldots)$ and $\mathbf{i} = (i_1, i_2, \ldots)$.

Proof. In Theorem 4.2 set $u = g_1$ and $g_{i,j} = g_{i+1} g_1^{i+1}$. In addition replace v^i by g_{i+1} in the power series expansion of ξ_0, given by Theorem 4.1, in v. The result follows. □

Corollary 4.3 has been given by Jackson and Aleliunas [10] and a related result is given by Gessel [8]. Clearly, Theorem 4.2 may be used to derive the generating function for the enumeration of sequences in N_n^* with respect to rises and non-rises given by Carlitz [4]. A more general expression is given by Jackson [9].

A more complex example is given by the following corollary.

Corollary 4.4. *The number of permutations in $\mathcal{N}_n$, for the case $\Pi = \Pi^{(1)}$, with no maximal (π_1, π_2)-structures of type (2, 2) is $[x^n/n!]\eta_0$ where $(1 + x^4 - x^6)^{-1} = \sum_{i=0}^{\infty} \alpha_i x^i$ and η_0 is shown in Fig. 1.*

Proof. In Theorem 4.2 set

$$g_{ij} = \begin{cases} 0 & \text{if } i = 2 \text{ and } j = 2, \\ 1 & \text{otherwise} \end{cases}$$

$$\begin{vmatrix} \sum_{i=0}^{\infty} \alpha_i \frac{x^{i+1}}{(i+1)!} & \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+3}}{(i+3)!} - \frac{x^{i+5}}{(i+5)!} \right\} & -\sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+3}}{(i+3)!} - \frac{x^{i+4}}{(i+4)!} \right\} \\ \sum_{i=0}^{\infty} \alpha_i \frac{x^{i+2}}{(1+2)!} & -1 + \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+4}}{(i+4)!} - \frac{x^{i+6}}{(i+6)!} \right\} & -\sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+4}}{(i+4)!} - \frac{x^{i+5}}{(i+5)!} \right\} \\ \sum_{i=0}^{\infty} \alpha_i \frac{x^{i+3}}{(i+3)!} & -x + \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+5}}{(i+5)!} - \frac{x^{i+7}}{(i+7)!} \right\} & 1 - \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+5}}{(i+5)!} - \frac{x^{i+6}}{(i+6)!} \right\} \end{vmatrix}$$

$$\begin{vmatrix} 1 - \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+1}}{(i+1)!} + \frac{x^{i+4}}{(i+4)!} - \frac{x^{i+6}}{(i+6)!} \right\} & \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+3}}{(i+3)!} - \frac{x^{i+5}}{(i+5)!} \right\} & -\sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+3}}{(i+3)!} - \frac{x^{i+4}}{(i+4)!} \right\} \\ x - \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+2}}{(i+2)!} + \frac{x^{i+5}}{(i+5)!} - \frac{x^{i+7}}{(i+7)!} \right\} & -1 + \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+4}}{(i+4)!} - \frac{x^{i+6}}{(i+6)!} \right\} & -\sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+4}}{(i+4)!} - \frac{x^{i+5}}{(i+5)!} \right\} \\ \frac{x^2}{2!} - \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+3}}{(i+3)!} + \frac{x^{i+6}}{(i+6)!} - \frac{x^{i+8}}{(i+8)!} \right\} & -x + \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+5}}{(i+5)!} - \frac{x^{i+7}}{(i+7)!} \right\} & 1 - \sum_{i=0}^{\infty} \alpha_i \left\{ \frac{x^{i+5}}{(i+5)!} - \frac{x^{i+6}}{(i+6)!} \right\} \end{vmatrix}$$

Fig. 1.

and $u = v = 1$. Thus, from Theorem 4.2 we have immediately $F(\mathbf{a}) = (\mathbf{e} + a^4 - \mathbf{a}^6)$, $F_0(\mathbf{a}) = -(\mathbf{e} + \mathbf{a}^3 - \mathbf{a}^5)$, $F_1(\mathbf{a}) = \mathbf{a}^2(\mathbf{e} - \mathbf{a}^2)$, $F_2(\mathbf{a}) = -(\mathbf{e} - \mathbf{a})\mathbf{a}^2$, and $F_l(\mathbf{a}) = 0$ for $l \geqslant 3$. Thus the system is 3×3. Now

$$\operatorname{trace} \mathbf{a}^j F^{-1} F_0 \boldsymbol{\omega} = -\sum_{i=0}^{\infty} \alpha_i \{\gamma_{i+j+1} + \gamma_{i+j+4} - \gamma_{i+j+6}\},$$

$$\operatorname{trace} \mathbf{a}^j F^{-1} F_1 \boldsymbol{\omega} = \sum_{i=0}^{\infty} \alpha_i \{\gamma_{i+j+3} - \gamma_{i+j+5}\},$$

$$\operatorname{trace} \mathbf{a}^j F^{-1} F_2 \boldsymbol{\omega} = -\sum_{i=0}^{\infty} \alpha_i \{\gamma_{i+j+3} - \gamma_{i+j+4}\},$$

$$\operatorname{trace} \mathbf{a}^j F^{-1} \boldsymbol{\omega} = \sum_{i=0}^{\infty} \alpha_i \gamma_{i+j+1}.$$

Let $II = II^{(1)}$. The result follows from Lemma 2.5. □

5. r-Arrays

We now extend the main theorem to the enumeration of r-arrays and permutation r-arrays over $\mathcal{N}_n$. Let $\mathbf{c}_j = (\sigma_{1j}, \ldots, \sigma_{rj}) \in \mathcal{N}_n^r$ for $j = 1, 2, \ldots, l$. Then $\sigma = [\sigma_{ij}]_{r \times l}$ is called an *r-array* over $\mathcal{N}_n$ with columns $c_1, c_2, \ldots, c_l$. Equivalently we may regard an r-array as a sequence over the alphabet $\mathcal{N}_n^r$. If each row of σ is a permutation on $\mathcal{N}_l$, then σ is called a *permutation r-array*.

Let $W = \pi_{p_1} \cdots \pi_{p_r}$ be a fixed word in $\{\pi_1, \pi_2\}^*$ and let $\Delta_W = \{\delta_1, \delta_2\}$ be the partition induced by W on the set of all ordered pairs of columns of length r as follows. Let $\mathbf{c} = (\sigma_1, \ldots, \sigma_r) \in \mathcal{N}_n^r$ and $\mathbf{c}' = (\sigma_1', \ldots, \sigma_r') \in \mathcal{N}_n^r$. Then $(\mathbf{c}, \mathbf{c}') \in \delta_1$ iff $(\sigma_i, \sigma_i') \in \pi_{p_i}$ for $i = 1, \ldots, r$. Otherwise $(\mathbf{c}, \mathbf{c}') \in \delta_2$. Accordingly maximal δ_1-paths, maximal δ_2-paths and maximal (δ_1, δ_2)-structures in r-arrays are defined. We refer to Δ_W as the *induced partition*, where the word W is understood from the context. An *r-array* $\sigma = (c_1, \ldots, c_l)$ has partition sequence $(\delta_{k_1}, \ldots, \delta_{k_{l-1}})$ iff $(c_i, c_{i+1}) \in \delta_{k_i}$ for $i = 1, 2, \ldots, l-1$. The *type* of σ is $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r)$ where $\boldsymbol{\mu}_i = (k_{i1}, \ldots, k_{in})$ is the type of $\sigma_{i1}\sigma_{i2} \cdots \sigma_{il}$ for $i = 1, 2, \ldots, r$.

For example, let $W = (\pi_1^{(1)})^3$. Then

$$\sigma = \begin{bmatrix} 2 & 1 & 3 & 4 & 7 & 8 \\ 1 & 2 & 5 & 8 & 9 & 8 \\ 4 & 4 & 5 & 9 & 11 & 8 \end{bmatrix}$$

is a 3-array with a maximal δ_1-path,

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 5 & 8 & 9 \\ 4 & 5 & 9 & 11 \end{bmatrix}$$

of edge length 3.

Our purpose is to enumerate *r-arrays* with respect to the numbers of δ_1-paths, δ_2-paths, (δ_1, δ_2)-structures, and type. We may specify the type of each row of an r-array and accordingly results for permutation r-arrays emerge as special cases. The following lemma extends all of the previous results on sequences to r-arrays.

Lemma 5.1. *Let $\Pi = \{\pi_1, \pi_2\}$ be an arbitrary bipartition of $\mathcal{N}_n^2$ and let $\Delta_W = \{\delta_1, \delta_2\}$ be the partition induced by $W = \pi_{p_1} \cdots \pi_{p_r} \in \{\pi_1, \pi_2\}^*$. If the number of sequences in $\mathcal{N}_n^+$ with an initial maximal π_2-path of edge length p, m_{ij} maximal (π_1, π_2)-structures of type (i, j) for $i, j \geq 1$, a terminal maximal π_1-path of edge length q, and with type $\mathbf{i}$ is*

$$[u^p v^q \mathbf{x}^{\mathbf{i}} \mathbf{g}^{\mathbf{m}}]\phi(\gamma(\pi_1, \mathbf{X}))$$

where

$$\gamma(\pi_1, \mathbf{X}) = \{\gamma_0(\pi_1, \mathbf{X}), \gamma_1(\pi_1, \mathbf{X}), \ldots\},$$

then the number of r-arrays on $\mathcal{N}_n$ with an initial maximal δ_2-path of edge length p, m_{ij} maximal (δ_1, δ_2)-structures of type (i, j) for $i, j \geq 1$, a terminal maximal δ_1-path of edge length q, and with type $(\boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(r)})$ is

$$[u^p v^q \mathbf{g}^{\mathbf{m}} \mathbf{y}_1^{\boldsymbol{\mu}^{(1)}} \cdots \mathbf{y}_n^{\boldsymbol{\mu}^{(r)}}]\phi(\gamma(\delta_1))$$

where

$$\gamma(\delta_1) = \{\gamma_0(\delta_1), \gamma_1(\delta_1), \gamma_2(\delta_1) \ldots\} \quad \text{and} \quad \gamma_k(\delta_1) = \prod_{i=1}^{r} \gamma_k(\pi_{p_i}, \mathbf{Y}^{(i)})$$

in which

$$\mathbf{y}_i = (y_{i1}, \ldots, y_{in}) \quad \text{and} \quad \mathbf{Y}^{(i)} = \operatorname{diag}(y_{i1}, \ldots, y_{in}).$$

Proof. Let $\mathbf{a}_{p_i} = \mathbf{Y}^{(i)}\mathcal{I}(\pi_{p_i})$ and $\boldsymbol{\omega}_i = \mathbf{Y}^{(i)}\mathbf{J}$, for $i = 1, 2, \ldots, r$, and let $\mathbf{S} = \mathbf{a}_{p_1} \otimes \cdots \otimes \mathbf{a}_{p_r}$, $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 \otimes \cdots \otimes \boldsymbol{\omega}_r$, $\mathbf{T} = \boldsymbol{\Omega} - \mathbf{S}$. Now, from Proposition 2.2, with these new incidence matrices, the required generating function is given immediately by

$$\operatorname{trace} (\mathbf{e} - u\mathbf{T})^{-1}\left\{\mathbf{e} - \sum_{i,j \geq 1} \mathbf{S}^i \mathbf{T}^j\right\}^{-1}(\mathbf{e} - v\mathbf{S})^{-1}\boldsymbol{\Omega}.$$

But this is equal to $\phi(\gamma(\delta_1))$ where $\gamma_k(\delta_1) = \operatorname{trace} \mathbf{S}^{k-1}\boldsymbol{\Omega}$. Simplifying $\gamma_k(\delta_1)$ we have

$$\begin{aligned}
\gamma_k(\delta_1) &= \operatorname{trace} \mathbf{S}^{k-1}\boldsymbol{\Omega} \\
&= \operatorname{trace} (\mathbf{a}_{p_1} \otimes \cdots \otimes \mathbf{a}_{p_r})^{k-1}(\boldsymbol{\omega}_1 \otimes \cdots \otimes \boldsymbol{\omega}_r) \\
&= \operatorname{trace} ((\mathbf{a}_{p_1}^{k-1}\boldsymbol{\omega}_1) \otimes \cdots \otimes (\mathbf{a}_{p_r}^{k-1}\boldsymbol{\omega}_r)) \\
&= \prod_{i=1}^{r} \operatorname{trace} \mathbf{a}_{p_i}^{k-1}\boldsymbol{\omega}_i = \prod_{i=1}^{r} \gamma_k(\pi_{p_i}, \mathbf{Y}^{(i)})
\end{aligned}$$

and the proof is complete. □

A corresponding lemma for permutation r-arrays involving the bipartition $\Pi^{(1)}$ may now be given.

Lemma 5.2. *Let $\Delta_W = \{\delta_1, \delta_2\}$ be the partition induced by an arbitrary word of length r in $\{\pi_1^{(1)}, \pi_2^{(2)}\}$. If the number of sequences in $\mathcal{N}_n^+$ with an initial maximal $\pi_2^{(1)}$-path of edge length p, m_{ij} maximal $(\pi_1^{(1)}, \pi_2^{(1)})$-structures of type (i, j) for $i, j \geq 1$, a terminal maximal $\pi_1^{(1)}$-path of edge length q, and with type $\mathbf{i}$ is*

$$(u^p v^q \mathbf{g}^{\mathbf{m}} \mathbf{X}^{\mathbf{i}}]\phi(\pi_1, \mathbf{X})$$

where

$$\gamma(\pi_1, \mathbf{X}) = \{\gamma_0(\pi_1, \mathbf{X}), \gamma_1(\pi_1, \mathbf{X}), \ldots\},$$

then the number of permutation r-arrays on $\mathcal{N}_n$ with an initial maximal δ_2-path of edge length p, m_{ij} maximal (δ_1, δ_2)-structures of type (i, j) for $i, j \geq 1$ and a terminal maximal δ_1-path of edge length q is

$$\left[u^p v^q \mathbf{g}^{\mathbf{m}} \frac{y^n}{(n!)r}\right]\psi$$

where

$$\psi = \phi\left(1, \frac{y}{(1!)^r}, \frac{y^2}{(2!)^r}, \ldots\right)$$

Proof. Let

$$\phi(\gamma(\delta_1)) = \sum_{k_1, \ldots, k_r \geq 0} a(k_1, \ldots, k_r) y_1^{k_1} \cdots y_r^{k_r}$$

where $(k_1, \ldots, k_r)$ is the type of an r-array. But, from Lemma 7.1, $\phi(\gamma(\delta_1)) = \{\gamma_0(\delta_1), \gamma_1(\delta_1), \ldots\}$ where $\gamma_k(\delta_1) = \gamma_k(\pi_1^{(1)}, \mathbf{Y}^{(1)}) \cdots \gamma_k(\pi_1^{(1)}, \mathbf{Y}^{(r)})$. But each row of a permutation r-array is a permutation. Thus, from Lemma 2.4, we replace $\gamma_k(\pi_1^{(1)}, \mathbf{Y}^{(i)})$ and, by symmetry, $\gamma_k(\pi_2^{(1)}, \mathbf{Y}^{(i)})$ with $z_i^k/k!$, where $z_1, \ldots, z_r$ are indeterminates. Thus

$$\sum_{n \geq 0} b_n \frac{z_1^n}{n!} \cdots \frac{z_r^n}{n!} = \phi\left(1, \frac{z_1 \cdots z_r}{(1!)^r}, \frac{z_1^2 \cdots z_r^2}{(2!)^r}, \ldots\right)$$

where b_n is the number of required permutation r arrays on $\mathcal{N}_n$. Let $z_1 = y$, $z_2 = \cdots = z_r = 1$. Then

$$\sum_{n \geq 0} \frac{b_n}{(n!)^r} z^n = \phi\left(1, \frac{y}{(1!)^r}, \frac{y^2}{(2!)^r}, \ldots\right)$$

and the result follows. □

6. Applications

We now give some examples of the theory which has been developed so far.

Corollary 6.1. *The number of permutation r-arrays on $\mathcal{N}_n$ with no δ_1-paths of vertex length p, where $\{\delta_1, \delta_2\}$ is the partition induced by $(\pi_1^{(1)})^r$, is*

$$\left[\frac{x^n}{(n!)^r}\right]\left\{\sum_{k\geqslant 0}\left(\frac{x^{kp}}{(kp)!^r}-\frac{x^{kp+1}}{(kp+1)!^r}\right)\right\}^{-1}.$$

Proof. The enumerator for maximal π_1-paths of vertex length less than p is

$$G(x)=1+x+\cdots+x^{p-1}=(1-x^p)(1-x)^{-1},$$

so

$$G^{-1}=\sum_{k=0}^{\infty}(x^{kp}-x^{kp+1}).$$

Thus, from Corollary 4.3, the number of sequences in $\mathcal{N}_n^+$ with no π_1-paths of vertex length p, and having type $\boldsymbol{i}$ is

$$[\boldsymbol{x}^{\boldsymbol{i}}](G^{-1}\circ\gamma)=[\boldsymbol{x}^{\boldsymbol{i}}]\sum_{k=0}^{\infty}(\gamma_{kp}-\gamma_{kp+1})\}^{-1}$$

where $\gamma=\{\gamma_0(\pi_1), \gamma_1(\pi_1), \ldots\}$. The result follows directly from Lemma 5.2. □

The result of Abramson and Promislow [1] may be obtained by considering partitions induced by $(\pi_1^{(1)})^*$ and by considering the enumeration of sequences with respect to rises and non-rises.

The next example concerns permutation r-arrays with a periodic pattern.

Corollary 6.2. *Let $\{\delta_1, \delta_2\}$ be the partition induced by an arbitrary word of length r in $\{\pi_1^{(1)}, \pi_2^{(1)}\}^*$. Let $f_{kr}(n)$ be the number of permutation r-arrays on $\mathcal{N}_n$ consisting only of maximal (δ_1, δ_2)-structures of type $(k-1, 1)$, and a terminal maximal δ_1-path of edge length l for some $l \leqslant k-1$. Then*

(i) $$\sum_{k\geqslant 0} f_{kr}(n)(-1)^{\lfloor (n-1)/k\rfloor}\frac{x^n}{(n!)^r}=\left(\sum_{j=1}^{\infty}\frac{x^j}{(j!)^r}\right)\left(\sum_{j=0}^{\infty}\frac{x^{kj}}{(kj!)^r}\right)^{-1},$$

(ii) $$\sum_{k\geqslant 0} f_{kr}(n)\frac{x^n}{(n!)^r}=\left(\sum_{j=1}^{\infty}\frac{x^j}{(j!)^r}(-1)^{\lfloor (j-1)/k\rfloor}\right)\left(\sum_{j=0}^{\infty}\frac{x^{kj}}{(kj!)^r}(-1)^j\right)^{-1}.$$

Proof. Let ξ_0 be the generating function for the number of sequences on $\mathcal{N}_n$ consisting only of maximal (π_1, π_2)-structures of type $(k-1, 1)$ and a terminal maximal π_1-path of edge length l for some $l \leqslant k-1$, and a specified number of (π_1, π_2)-structures. Let λ be an indeterminate marking (π_1, π_2)-structures of type $(k-1, 1)$. In Theorem 4.2, let $g_{k-1,1}=\lambda$, where k is fixed, and let all other g_{ij} be zero. Then

$$F_0 = -(\mathbf{e} - v\mathbf{a})\lambda\mathbf{a}^{k-1}, \qquad F_j = 0 \quad \text{for } j \geq 1, \qquad F = (\mathbf{e} - v\mathbf{a})(\mathbf{e} + \lambda\mathbf{a}^k).$$

Then ξ_0 satisfies $a_{00}\xi_0 = c_0$ where

$$a_{00} = 1 + \operatorname{trace}(\mathbf{e} + \lambda\mathbf{a}^k)^{-1}\lambda\mathbf{a}^{k-1}\boldsymbol{\omega}$$

whence

$$\xi_0 = \left(\sum_{j=1}^{\infty} \gamma_j(\pi_1)(-\lambda)^{\lfloor (j-1)/k \rfloor}\right)\left(\sum_{j=0}^{\infty} \gamma_{kj}(\pi_1)(-\lambda)^j\right)^{-1}.$$

Thus from Lemma 5.2 we have

$$\left(\sum_{j=1}^{\infty} \frac{x^j}{(j!)^r}(-\lambda)^{\lfloor j-1/k \rfloor}\right)\left(\sum_{j=0}^{\infty} \frac{x^j}{(kj)!^r}(-\lambda)^j\right) = \sum_{k \geq 0} f_{kr}(n)\lambda^{\lfloor (n-1)/k \rfloor}\frac{x^n}{n!^r}$$

since λ marks the number of (δ_1, δ_2)-structures. Setting λ to -1 and 1 we obtain (i) and (ii) respectively. □

Corollary 6.2(i) is a special case of Corollary 3.3 of Stanley [15] while Corollary 6.2(ii) is a special case of Corollary 3.5 of Stanley [15]. Stanley's results are more general since they contain an enumeration with respect to inversions as well, and they also serve as an example of the use of binomial posets.

Corollary 6.3. *Let $\{\delta_1, \delta_2\}$ be the partition induced by an arbitrary word of length r in $\{\pi_1^{(1)}, \pi_2^{(1)}\}^*$. Then the number of permutation r-arrays on $\mathcal{N}_k$ with partition sequence*

$$\delta_1^{p_1-1}\delta_2\delta_1^{p_2-1}\delta_2 \cdots \delta_2\delta_1^{p_{k-1}-1}\delta_2\delta_1^{p_k-1}$$

is

$$\left\| \binom{k - s_{i-1}}{s_j - s_{i-1}}^r \right\|$$

where

$$s_j = p_1 + \cdots + p_j \quad \text{for } j = 1, 2, \ldots, k \quad \text{and} \quad s_0 = 0.$$

Proof. From Lemma 3.1 and Lemma 5.2, the number of each permutation r-arrays is $[x^k/(k!)^r]\eta$ where

$$\begin{aligned}
\eta &= \left\| \frac{x^{s_j - s_{i-1}}}{(s_j - s_{i-1})!^r} \right\| \\
&= \left\| \binom{k - s_{i-1}}{s_j - s_{i-1}}^r \frac{(k - s_j)!^r}{(k - s_{i-1})!^r} \frac{x^{s_j}}{x^{s_{i-1}}} \right\| \\
&= \frac{x^k}{k!^r} \left\| \binom{k - s_{i-1}}{s_j - s_{i-1}}^r \right\| \qquad \text{since } s_k = k
\end{aligned}$$

and the result follows. □

For $r = 1$, this is a well-known result of MacMahon [13] and is a special case of a result of Stanley [15].

Acknowledgements

The authors thank A.C. Norman (Computer Laboratory, University of Cambridge) for assisting them in checking a number of the generating functions by expanding them with the algebraic manipulation system CAMAL. We have found the use of this system invaluable. This work was carried out while the authors were on leave at the Department of Pure Mathematics and Mathematical Statistics, at the University of Cambridge. The authors wish to thank Professor J.W.S. Cassels and Professor M.V. Wilkes for permitting them to make use of the facilities of the Department of Pure Mathematics and Mathematical Statistics, and the Computer Laboratory, respectively.

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] M. Abramson and D. Promislow, Enumeration of arrays by column rises, J. Combinatorial Theory (A) 24 (1978) 247–250.
[2] D. André, Developments de sec x et de tang x, C.R. Acad. Sci., Paris, 88 (1879) 965–967.
[3] D. André, Mémoire sur les permutations alterncés, J. Math. 7 (1881) 167–184.
[4] L. Carlitz, Enumeration of sequences by rises and falls; A refinement of the Simon Newcomb problem. Duke J. Math. 39 (1972) 267–280.
[5] P. Cartier and D. Foata, Problèmes Combinatoires de Commutation et Réarrangements, Lecture Notes in Mathematics 85 (Springer Verlag, Berlin, 1969).
[6] R. Cori and J. Richard, Enumeration des graphes planaires à l'aide des series formelles en variables non commutatives, Discrete Math. 2 (1972) 115–162.
[7] D. Foata and M. P. Schützenberger, Théorie Géométrique des Polynômes Eulérians, Lecture Notes in Mathematics 138 (Springer Verlag, Berlin, 1970).
[8] I.M. Gessel, Generating functions and enumeration of sequences, Ph.D. Thesis, Dept. of Mathematics, M.I.T. Cambridge, MA, 1977.
[9] D.M. Jackson, The unification of certain enumeration problems for sequences, J. Combinatorial Theory, 22 (1977) 92–96.
[10] D. M. Jackson and R. Aleliunas, Decomposition based generating functions for sequences, Canad. J. Math. 24 (1977) 971–1009.
[11] D.M. Jackson and I.P. Goulden, A formal calculus for the enumerative system of sequences: I. Combinatorial theorems. Studies Appl. Math. 61 (1979) 141–176.
[12] D.M. Jackson and I.P. Goulden, A formal calculus for the enumerative system of sequences: II. Applications. Studies Appl. Math. 61 (1979) 245–277.
[13] P.A. MacMahon, Combinatory Analysis, Vol. 1. (Chelsen, New York, 1960).
[14] J.W. Reilly, An enumerative combinatorial theory of formal power series, Ph.D. Thesis, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada, 1977.
[15] R.P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration. J. Combinatorial Theory (A) 20 (1976) 336–356.