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RIEMANN'S CONTRIBUTION TO DIFFERENTIAL GEOMETRY

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SUMMARIES

In order to make a reasonable assessment of the significance of Riemann's role in the history of differential geometry, not unduly influenced by his reputation as a great mathematician, we must examine the contents of his geometric writings and consider the response of other mathematicians in the years immediately following their publication.

Pour juger adéquatement le rôle de Riemann dans le développement de la géométrie différentielle sans être influencé outre mesure par sa réputation de très grand mathématicien, nous devons étudier le contenu de ses travaux en géométrie et prendre en considération les réactions des autres mathématiciens au cours de trois années qui suivirent leur publication.

Um Riemann's Einfluss auf die Entwicklung der Differentialgeometrie richtig einzuschätzen, ohne sich von seinem Ruf als bedeutender Mathematiker übermässig beeindrucken zu lassen, ist es notwendig den Inhalt seiner geometrischen Schriften und die Haltung zeitgenössischer Mathematiker unmittelbar nach ihrer Veröffentlichung zu untersuchen.

On June 10, 1854, Georg Friedrich Bernhard Riemann read his probationary lecture, "Über die Hypothesen welche der Geometrie zu Grunde liegen," before the Philosophical Faculty at Göttingen [1]. His biographer, Dedekind [1892, 549], reported that Riemann had worked hard to make the lecture understandable to nonmathematicians in the audience, and that the result was a masterpiece of presentation, in which the ideas were set forth clearly without the aid of analytic techniques. Gauss was very impressed and, with uncharacteristic warmth, declared his regard for the depth of Riemann's ideas.

The lecture has come to be regarded as a milestone in the history of geometry. Clifford [1873a, 565] said, "It was Riemann ... who first accomplished the task of analysing all

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the assumptions of geometry, and showing which of them were independent." Turnbull [1929, 114] wrote, "In a few pages of epoch-making dissertation, [Riemann] not only contemplated geometry for space of any dimensions ... [but] showed that the earlier three types of geometry [Euclidean, spherical, and Lobachevski] were particular instances of a still more general geometry." Newman [1956, 1, 644-645] wrote of "Riemann's imperishable doctoral dissertation [sic]" that "the entire tone of the lecture, the attitude it expresses toward geometry is modern.

It is possible to take a more critical view. Gauss' remark was reported only indirectly, long after his death, and may be of limited value considering Gauss' age and ill health at the time. None of the other citations above predate Riemann's own death in 1866, by which time he was universally recognized for mathematical work in many fields. Apparently the only surviving contemporary comments on the lecture are two letters from Riemann to his family [Dedekind 1892, 547-548]. In the first, dated December 28, 1853, Riemann explained that he had prepared only the first two of three topics he submitted for the lecture and was taken by surprise when Gauss chose the third. The letter of June 26, 1854, indicated that Riemann had found it somewhat difficult to tear himself away from investigations in mathematical physics in order to prepare the lecture; but now he was happily through the ordeal. The tone of these letters, together with a quick look at the content of the lecture, may suggest that Riemann turned the topic around so that he could draw on his interest and expertise in analysis and mathematical physics, paying little more than lip service to the foundations of geometry.

A critic might further note that Riemann left no general treatise in which the ideas of the probationary lecture were thoroughly worked out in their analytic setting. In fact, he did little further work in geometry of any sort, and even the probationary lecture was not published during his lifetime. On the basis of these observations, one might suggest that Riemann's reputation as a geometer has been greatly enhanced by romantic sentiment, and that the true originators of "Riemannian" geometry were Helmholtz and Beltrami.

In order to evaluate such criticism, we must first look at the probationary lecture itself [Riemann 1868]. Riemann begins by saying that the reason for so much confusion concerning the hypotheses of geometry [2] is the failure to examine the underlying notion of "multiply extended magnitudes." His claim to have had no sources for his ideas except Gauss and the philosopher Herbart may be questioned; certainly he also drew on the multidimensional linear algebra of Jacobi [1834] and others. The nature of Herbart's influence is the subject of a forthcoming paper by E. Scholz. Many of Riemann's ideas have precedents not only in Gauss' geometric works [3] but also in the "geometrization" of complex numbers.

The lecture as published has three sections, treating in turn topological notions, metric relations, and applications to physical space.

In Section I Riemann introduces the idea of a continuous manifold as a collection of objects in which it is possible to proceed from one to another along a continuous path. Physical space and color [4] are cited as examples. Without some standard by which to make measurements, the only quantitative relation is set inclusion. Riemann cites the lack of research into the notion of what we now call topological manifolds as the reason for rather disappointing progress in the theories of differential equations and multivalued analytic functions [1868, 274]. (The notion of a Riemann surface, the natural domain of a multivalued function, was clearly and thoroughly articulated in Riemann's dissertation of 1851.)

Next, manifolds of *n* dimensions are defined inductively:

> *If in [a continuous manifold] one passes from a certain [point] in a definite way to another, the [points] passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forwards or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, than all the [points] so obtained form a doubly extended manifoldness. In a similar manner one obtains a triply extended manifoldness, if one imagines a doubly extended one passing over in a definite way to another entirely different; and it is easy to see how this construction may be continued.* ([Riemann 1868, 275]; translation by Clifford [1873b, 58])

A modern definition of an *n*-dimensional manifold is a space which can be covered, in a consistent way, by neighborhoods homeomorphic to regions of *n*-space. Riemann's definition is vague and awkward by comparison, but it has an important advantage in being constructive rather than analytic. The technical details which Riemann gives later in the probationary lecture and in [1876] demonstrate that the analysis was certainly within his grasp. We may regard his choice of an intuitive definition as an indication of the desire to be intelligible to his general scholarly audience. Even most of the mathematicians may have found the idea of *n*-space rather difficult, for although important work in multivariate algebra had been going on for some years, the study of geometry in *n*-space had hardly begun [5].

Coordinate ideas are introduced somewhat indirectly when Riemann proposes to resolve any continuous manifold into a one-dimensional manifold and a manifold of fewer dimensions than the original (and thus, where possible, to determine the finite dimension of the manifold):

> *[L]et us take a continuous function of position within the given manifoldness, which, moreover, is not constant throughout any part of that manifoldness. Every system of points where the function has a constant value, forms then a continuous manifoldness of fewer dimensions than the given one. These manifoldnesses pass over continuously into one another as the function changes; we may therefore assume that out of one of them the others proceed, and speaking generally this may occur in such a way that each point passes over into a definite point of the other; the cases of exception (the study of which is important) may here be left unconsidered.... By repeating then this operation n times, the determination of position in an n-ply extended manifoldness is reduced to n determinations of quantity....* [Riemann 1868, 275-276; Clifford 1873b, 58-59]

Section II, in which metric relations are developed, provides the foundation of Riemannian differential geometry. Riemann apologizes for the necessity of abstract formulas, promising that geometric interpretations will follow. He notes that his ideas are based on Gauss' *Disquisitiones generales circa superficies curvas.*

The first problem is to determine the length of a curve in a way which does not depend on its position (that is, by some method other than moving a standard curve onto it). A curve is determined when the coordinates, $x_1, \ldots, x_n$, of a point are given as functions of a single variable. Riemann considers an "element" of a curve in which the increments dx_i are in fixed proportion, thus seeking, at each point of the manifold, an expression for ds in terms of the x_i and dx_i. Half a page of intuitive argument leads to the verbal expression of the basic equation of Riemannian geometry:

> *ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x.* [Riemann 1868, 278; Clifford 1873b, 61]

Apart from the generalization to n dimensions, this was anticipated by Gauss, who noted in the abstract of *Superficies curvas* that "the very nature of [a] curved surface is given by means of the expression of any linear element in the form $(Edp^2 + 2Fdp \cdot dq + Gdq^2)^{1/2}$" [Gauss 1827, 91].

Riemann notes that a change of variables may transform the expression $ds^2 = \Sigma g_{ij}dx_i dx_j$ into another of the same form; but not every such form is obtainable in this way, since there are $n(n + 1)/2$ coefficients g_{ij}, only n of which can be fixed by an appropriate choice of variables. In particular, not every manifold is flat.

Next Riemann introduces what are now called *geodesic normal coordinates*. The derivation is both intuitive and intrinsic.

> *Let us imagine that from any given point the system of shortest lines going out from it is constructed; the position of an arbitrary point may then be determined by the initial direction of the geodesic in which it lies, and by its distance measured along that line from the origin. It can therefore be expressed in terms of the ratios [of the initial values] dx_0 of the quantities dx in this geodesic, and of the length s of this line. Let us introduce now instead of the dx_0 linear functions $d\alpha$ of them, such that the initial value of the square of the line-element shall equal the sum of the squares of these expressions, so that the independent variables are now the length s and the ratios of the quantities $d\alpha$. Lastly, take instead of the $d\alpha$ quantities x_1, x_2, x_3, ..., x_n proportional to them, but such that the sum of their squares = s^2.* [Riemann 1868, 279; Clifford 1873b, 62]

Note that Riemann regarded the x_i's as infinitesimals. Nowhere did he elucidate the relation between x and dx, but he treated them as if they were independent infinitesimals, corresponding closely to dx and δx in Article 7 of Gauss' *Superficies curvas*. Without offering any reasons, he stated that in these special coordinates, ds^2 has a second-order term, Σdx_i^2, and no third-order term (in modern terminology, $g_{ij} = \delta_{ij}$ and $\Gamma_{ij} = 0$ at the origin). The fourth-order term has the form $\Sigma_{ijkl}(x_i dx_j - x_j dx_i)(x_k dx_l - x_l dx_k)$, and is thus a finite multiple (say, Q) of the squared area of the "infinitesmal triangle" with vertices at O, x, and dx. These statements are plausible but not obvious extensions of ideas in *Superficies curvas*.

Following this is a statement having no precedent in Gauss' work: "This quantity [Q] retains the same value so long as the two geodesics from O to x and from O to dx remain in the same surface-element" [Riemann 1868, 279; Clifford 1873b, 63]. The geometric terminology is symbolic rather than explanatory, and Riemann properly put the algebraic description first. Geometric intuition may serve to correlate ideas in this vastly more general setting, but only algebra and analysis are completely reliable guides.

Riemann's earlier remark that the metric is determined by $n(n + 1)/2$ more or less arbitrary functions is now restated in geometric form: the metric is determined when, at each point, the sectional curvature is known in each of $n(n - 1)/2$ two-dimensional directions, "wofern nur zwischen diesen Werthen keine identischen Relationen stattfinden" [1868, 280]. Riemann does not point out that this assertion is an approximate converse of Gauss' *Theorema Egregium*. The quoted passage has generally been interpreted to mean that the directions must be independent. Yet even with that perhaps charitable reading, the assertion must be qualified (some information about parallel translation is also required), and it is far more difficult to prove than one might suspect [6]. It seems likely that Riemann had at most an intuitive idea, especially since he did not even provide details for the special case of zero curvature in [1876] (discussed below), although he did give the correct formula for ds in a manifold of constant curvature [1868, 282].

For the remainder of the lecture Riemann returns to intuitive arguments. He reviews some well-known facts about the geometry of surfaces in space, gives two geometric characterizations of the Gauss curvature of a surface [7], and notes that curvature is invariant under bending (a weaker version of the *Theorema Egregium*).

Returning to the n-dimensional situation, Riemann states that a geodesic emanating from a point is determined by its original direction. This was already known for surfaces and is derived by straightforward analytic methods once it is recognized (as it was by Gauss) that the condition for a curve to be a geodesic can be written as a differential equation. For two-dimensional submanifolds the situation is not quite so simple as the following passage suggests:

> *According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the n-fold continuum at the given point in the given surface-direction.* [Riemann 1868, 281; Clifford 1873b, 64]

In properly chosen normal coordinates, the set of points described above is given by $x_3 = x_4 = \cdots = x_n = 0$. Calling this set a surface is thus consistent with Riemann's earlier suggestion that a submanifold is the collection of points at which a continuous function takes on a particular value. It is not too difficult to show that this is not one of the pathological cases left undiscussed (e.g., [Helgason 1962, 65]). The statement about curvature follows from the assertions made about the quantity Q. What is glossed over here is the important distinction between local and global geometries of surfaces. We will see that this leads to a serious problem in [Riemann 1876].

Next Riemann briefly discusses manifolds of constant curvature, noting that these are characterized by the fact that figures can be moved and turned arbitrarily in them. He gives the formula

$$(1/(1 + \alpha\Sigma x^2/4))(\Sigma dx^2)^{1/2}$$

for the increment of arc length in a manifold of constant curvature α. (This is the only displayed formula in the text.)

Section II closes with a geometric illustration using surfaces of constant curvature, all tangent along a given circle. The right cylinder with that circle as directrix represents surfaces of curvature zero. Inscribed in the cylinder are surfaces with constant positive curvature: the inscribed sphere and incomplete surfaces of revolution obtained by bending or rolling out portions of spheres of larger and smaller radii. Circumscribed about the cylinder are incomplete surfaces of revolution having constant negative curvature, which Riemann describes as resembling the inner surface of a ring (torus).

> *If we regard these surfaces as* locus in quo *for surface-regions moving in them, as Space is* locus in quo *for bodies, the surface-regions can be moved in all these surfaces without stretching. The surfaces with positive curvature can always be so formed that surface-regions may be moved arbitrarily about upon them without* bending, *namely (they may be formed) into sphere-surfaces; but not those with negative curvature. Besides this independence of surface-regions from position there is in surfaces of zero curvature also an independence of* direction *from position, which in the former surfaces does not exist.* ([Riemann 1868, 283; Clifford 1873b, 66]; emphasis added by Clifford)

The reference to bending is inconsistent with the otherwise intrinsic character of Riemann's observations and creates an artificial distinction between positively and negatively curved manifolds.

A completely satisfactory explanation of the last sentence quoted requires the notion of geodesic parallelism, later developed by Levi-Civita [1917]. In order for tangent vectors (directions) at one point of a surface or manifold to be compared to vectors at another point, there must be an unambiguous way of moving vectors along a curve. If the curve is not a geodesic, or if the manifold has dimension greater than 2, the solution is less obvious. Levi-Civita's criterion is expressed as an ordinary differential equation, closely related to the condition for a geodesic. In fact, a curve is a geodesic if and only if its tangent vector is geodesically parallel along the curve. Since Riemann certainly knew the condition for a geodesic, it is not impossible that he envisioned something like geodesic parallelism. Levi-Civita cited Riemann in the introduction to [1917], and Riemann's last statement above corresponds to the important theorem that a tangent vector translated parallel to itself around a closed curve necessarily returns to its original position only if the curve lies in a flat manifold. However, if the curves are restricted to geodesic polygons on surfaces, the statement follows easily from Gauss' theorem [1827, 54-55], which asserts that the angular excess or defect of a geodesic triangle is equal to the total curvature of the enclosed region [8].

In Section III Riemann turns first to the question of sufficient conditions for a Riemannian manifold to be Euclidean. He offers three sets of conditions, each of which he claims (without proof) to be sufficient.

1. The sectional curvature at every point, and in each of three (independent) surface-directions, must equal zero. This is equivalent to the requirement that the sum of angle measures in any triangle must be π.

2. If figures are free to move and turn in the space, without distortion, then the curvature must be constant; moreover, if all triangles have the same angle sum, then the space must be flat.

3. There must be a consistent method of measuring the direction of a curve, as well as its length, independently of its position within the manifold.

The next paragraph concerns the empirical verification of metric assumptions about space. Here we find the famous observation that a line may be unbounded and yet have finite length. Most attempts to prove the parallel postulate had ruled out the hypothesis of the obtuse angle (equivalent to positive curvature) because it implies that lines have finite length. By pointing out that Euclid required only that a line can be continued indefinitely, Riemann showed that elliptic geometry (which is sometimes called "Riemannian geometry") was as viable as Euclidean and hyperbolic geometry.

Finally, Riemann asks whether the local assumptions of Riemannian geometry are in fact valid in physical space. In particular, the observation that matter and light are not infinitely divisible presents difficulties. However, Riemann concludes that these questions properly belong to the realm of physics rather than to the realm of mathematics.

Apparently Riemann made no effort to have his probationary lecture published. He did elaborate on some of the ideas, however, in his "Commentatio mathematica, qua respondere tentatur quaestioni ab Illustrissima Academia Parisiensi propositae" [1876]. First he reduced the proposed physical problem (concerning heat conduction) to an analytic one, which would be solved if it were known under what conditions a positive definite form $\Sigma b_{ij}dx_i dx_j$ could be transformed into Σdy_i^2 by a change of variables. In the second part of his response Riemann easily showed that a necessary condition is the vanishing of certain expressions $(ij,\ kl)$ (read $2R_{ijkl}$ in modern notation). He proposed to clarify the vanishing of these quantities by examining the expression,

$$\delta\delta\Sigma b_{ij}ds_i ds_j - 2d\delta\Sigma b_{ij}ds_i\delta s_j + dd\Sigma b_{ij}\delta s_i\delta s_j.$$

Under conditions [9] amounting to a choice of coordinates for which $\Gamma_{ij/k} = 0$, this can be expressed in terms of the $(ij,\ kl)$:

$$\sum_{\substack{i<j\\ k<l}} (ij,\ kl)(ds_i\delta s_j - ds_j\delta s_i)(ds_k\delta s_l - ds_l\delta s_k)$$

[1876, 403]. (The necessity of restricting the sum was noted by Weber in his explanatory notes [Riemann 1892, 411].) On dividing this expression by

$$(\Sigma b_{ij}ds_i ds_j)(\Sigma b_{ij}\delta s_i\delta s_j) - (\Sigma b_{ij}ds_i\delta s_j)^2$$

one obtains an expression which is invariant not only under a change of variables but also if ds, δs are replaced by linear combinations of them.

If the given positive definite form is the squared element of arc length in a Riemannian manifold, then the invariant expression above is the product of -2 and the sectional curvature of the manifold in the plane spanned by ds and δs. Riemann only mentioned these interpretations; Weber's notes filled in some of the details.

Riemann concluded the second section by claiming that it is not difficult to show (at least in the case $n = 3$), by standard methods, that if the curvature tensor vanishes then the form $\Sigma b_{ij} ds_i ds_j$ must be equivalent to Σdx_i^2. It is, however, quite difficult to show this. Perhaps Riemann reasoned as follows. Gauss had pointed out that the angular excess or defect of any (geodesic) triangle is equal to its total curvature; thus if the curvature of a surface is everywhere equal to zero, all triangles have angles summing to π, and the metric is Euclidean. In higher dimensions, if the sectional curvature is everywhere zero, then clearly the surface formed by geodesics through a point P and lying initially in a given plane has curvature equal to zero *at* P. It is in fact true that the curvature of this surface vanishes everywhere, but the proof is not trivial [6, 10]. To prove that the metric is Euclidean requires only the consideration of small triangles; possibly Riemann was thinking of some limit argument to complete the proof.

Riemann's solution was considered incomplete, and the Paris Academy eventually withdrew the problem without awarding the prize [1876, 391, footnote]. Dedekind planned to publish the solution from Riemann's papers, together with a commentary explaining its relation to "Über die Hypothesen," but was prevented from doing so by the press of other duties [Dedekind 1932, 442]. Eventually Weber obtained the original manuscript from the Paris Academy [Riemann 1892, preface to the first edition], and it first appeared in 1876, after "Über die Hypothesen" was well known.

A paper on minimal surfaces [1892, 301-333], based on work of 1860 and 1861, does not use the analytic techniques suggested in "Über die Hypothesen."

Because the audience included few geometers, and the material was presented orally and without technical details, it is not surprising that Riemann's lecture had little impact. More significant are the reactions of mathematicians who saw the written work after Riemann's death.

The first of these was Dedekind. On the publication of Riemann's *Gesammelte mathematische Werke* in 1876, Dedekind wrote that he had found a clear manuscript of the lecture among Riemann's papers and published it directly [Dedekind 1932, 421-423]. He had included a footnote indicating his intention to publish a separate paper with some of the analytic details [11]. His remarks of 1892 have already been noted.

Schering, who presented a memorial notice to the Göttingen Gesellschaft on June 19, 1867, had probably seen the lecture prior to its publication. He cited Riemann's concentration on the intrinsic properties of space and the distinction between infinite length and unboundedness of lines. He also recalled many discussions with Riemann concerning the nature of physical space [Schering 1909, 165-166].

Helmholtz, who learned of Riemann's lecture from Schering, requested and received a copy in May 1868 [Koenigsberger 1902, 254]. His paper, "Über die Thatsachen, die der Geometrie zum Grunde liegen" [1868], made numerous references to Riemann but was based primarily on work he had done in the preceding two years [12]. Helmholtz considered three-dimensional spaces in R^4 and found that constant curvature is a necessary and sufficient condition for the free movement of bodies in space. His results are less general than those of Riemann; in particular, they are not purely intrinsic. His popular lecture, "On the Origin and Significance of Geometrical Axioms" [Helmholtz 1870], presents a more thorough understanding of Riemann's ideas.

Clifford referred to Riemann's lecture in [1870] and later translated it into English [1873b]. In [Clifford 1870] he concentrated on the results on three-dimensional spaces of constant curvature, and on the relations between physical space and geometric axioms.

Lipschitz and Christoffel examined the analytic aspects of Riemann's work. In "Untersuchungen in Betreff der gantzen homogenen Functionen von *n* Differentialen," dated January 4, 1869, Lipschitz [1869, 71] credited Riemann's investigations with having aroused his own interest in the question of transformation of variables. He also expressed some doubt as to the proper interpretation of Riemann's statement that sectional curvature determines the metric [p. 73, footnote]. In [1870, 24] Lipschitz remarked that Riemann actually gave two different definitions for sectional curvature--as the curvature of a certain surface in the manifold and by means of the fourth-order term in ds^2. Until he discovered a formula equivalent to the one given by Riemann [1876] for the sectional curvature, Lipschitz was uncertain that these definitions were equivalent.

By contrast, Christoffel mentioned Riemann only in the last paragraph of his paper "Über die Transformation der homogenen Differentialausdrücke zweiten Grades" [1869]. Apparently he believed that Riemann had considered only a special case of no great interest for the problem to which he had addressed himself.

The most intriguing and instructive reaction may be that of Beltrami. Having met Riemann through Betti at Pisa, Beltrami shared some conversations with the two older men [Cremona 1900, xiv]. It is hard to believe that they did not discuss geometry. During the time that Riemann was in Pisa, Beltrami published a series of papers entitled "Ricerche di analisi applicata alla geometria" [1864-1865], and in [1865] he showed that surfaces whose geodesics are "linear" necessarily have constant curvature. However, Beltrami made no reference to Riemann in these papers; nor did he give Riemann's simple formula for the element of arc length in a space of constant curvature, not even in [1868a], in which he introduced as a model for hyperbolic geometry the interior of a circle with chords as geodesics. The fact that

Beltrami dealt primarily with imbedded surfaces suggests that until 1868 he was unaware of the general principles of Riemann's probationary lecture.

On June 9, 1868, Beltrami sent to Genocchi a translation of "Über die Hypothesen"; he said the original had only recently come to his attention. He described the great generality of Riemann's work as extremely interesting and said that he was encouraged by Riemann's ideas to pursue some of his own research [Loria 1901, 415].

Beltrami's "Theoria fondamentale degli spazî di curvatura costante" [1868b], dated August 1868, shows a dramatic change from his earlier work. The results of [1865] were transformed, by means of Riemannian methods, into intrinsic results for manifolds of arbitrary dimension. Beltrami derived Riemann's formula for the element of arc length and discussed in more detail the notions of normal coordinates and sectional curvature. The difference between the two papers of 1868 attests vividly to Riemann's influence on Beltrami.

By 1870 Riemann's probationary lecture was well-known, though not well understood. Hawkins [1980] has described the progress of geometry in the 1870s and 1880s. These later developments are beyond the scope of this paper. (A recent work of some interest is [Scholz 1980].)

It is clear that differential geometry underwent a remarkable change in the second half of the 19th century, a change which is properly described as revolutionary rather than evolutionary (see [Kuhn 1970]). The question is whether the crux of the change lies in Riemann's work or elsewhere. The evidence is strong (if perhaps not overwhelming) that publication of Riemann's probationary lecture did indeed establish what we now call Riemannian geometry.

Certainly some of Riemann's admirers have been overenthusiastic. For example, few who have read "Über die Hypothesen" have felt inclined to accept Dedekind's estimate of its clarity; Beltrami admitted that he found it "enigmatic" in places [Loria 1901, 415]. Nor was Clifford accurate in claiming that Riemann had completely analyzed the assumptions of geometry.

On the other hand, most of the critics' objections can be met. That Riemann was more interested in analysis and mathematical physics than in geometry cannot be denied, but it does not follow that he had no interest in geometry until forced to work up his probationary lecture. He had submitted the topic himself, although it was his third choice. It is not unreasonable to suppose that the links with analysis were on his mind from the beginning. There was, after all, a good deal of topology and generalized geometry in his dissertation on complex analysis. Moreover, the analytic techniques he suggested were not the sole content of the lecture; several different assumptions of geometry were examined, so that it is quite correct to say that the lecture dealt with the foundations of geometry.

Riemann's relief at being done with the lecture can be explained adequately by natural nervousness over an important public presentation; it does not necessarily imply distaste for the subject matter. Nor was his failure to publish his geometric investigations very surprising, since the discursive style of "Über die Hypothesen" made it a poor candidate for publication, especially while its author was relatively unknown. (Compare Klein's "Erlanger Programm," delivered in 1872 and first published, in Italian translation, in 1890.) Riemann clearly had many other interests and probably thought there was little to be gained from revising his probationary lecture. The "Commentatio mathematica" [1876] demonstrates that he did have the analytic details in hand. If the Paris Academy had awarded him the prize, presumably his response would have been published, and that might have generated interest in "Über die Hypothesen."

There is no doubt that important roots of Riemannian differential geometry are to be found in earlier works, particularly in Gauss' introduction of local coordinates and emphasis on intrinsic properties of a surface, in Dirichlet's tendency to "geometrize" analytic problems, and in the techniques developed for n-dimensional linear algebra. Still, Riemann presented a new viewpoint on geometry. Gauss had stressed the essential importance of intrinsic properties, but for Riemann intrinsic properties were the only ones considered; there was no recourse to measurements in a Euclidean space in which the manifold was immersed [13]. Others had used geometric methods in a variety of applications, but Riemann extended the notion of geometry itself. By pointing out that physical space might conceivably violate Euclidean axioms in several important ways, Riemann forced geometers to work within a much more general framework. He would have been delighted with the wedding of differential geometry and mathematical physics in relativity theory.

For "pure" differential geometry, the most significant change was a shift in the "disciplinary matrix," the collection of techniques considered appropriate for attacking geometric problems. This shift, which Thomas S. Kuhn [1970] treats as a basic ingredient of scientific revolution, is most easily seen in the works of Beltrami and Helmholtz. These two are the most likely alternatives, if one does not regard Riemann as the instigator of the change, and in both cases there was a change in approach just after the publication of Riemann's probationary lecture.

The analytic techniques involving the metric and curvature tensors produced a significant change in the type of geometric problem which could be attacked successfully, and thus altered the perception of what constitutes a reasonable geometric question. The Riemannian viewpoint dominated differential geometry until challenged by the school of Cartan, whose coordinate-free approach not only looks different but is better adapted to global questions.

NOTES

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1. This was one of the requirements for becoming an unpaid but official lecturer at the university. Riemann had earned a doctorate in 1851; his dissertation was the classic *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grosse* [1876, 3-45]. The term *Habilitationsschrift* has been applied both to this lecture and to the paper on trigonometric series [1876, 227-265], submitted in December 1853. Riemann, however, referred to them as *Probevorlesung* and *Habilitationsschrift,* respectively, in the letters quoted by Dedekind [1892, 547]. Both were first published, after Riemann's death, from clear manuscripts found by Dedekind [1932, 421].

2. Presumably he was referring to the question of the parallel postulate. Bolyai's work appeared in 1832; Lobachevski's was published in Kazan beginning in 1829, but it did not attract attention in western Europe until around 1840.

3. See [Dombrowski 1979] for an excellent study of Gauss' *Disquisitiones generales circa superficies curvas.*

4. Probably he had in mind the suggestion of Thomas Young (1773-1829) that all colors can be represented in terms of mixing red, green, and violet light. The investigations of Maxwell and Helmholtz were published in the 1860s.

5. Pioneering works in multivariate algebra include those by Jacobi [1834], Grassman [1844], and Cayley [1845]. Schläfli's more geometric *Theorie der vielfachen Kontinuität* was written in 1852 but was not published until 1901.

6. The equivalent analytic problem was addressed by Christoffel and Lipschitz in a series of papers appearing in *Crelle's Journal (Journal für reine und angewandte Mathematik)* beginning with volume 70 (1869). Spivak [1979] gives several proofs, spread over four chapters, in the zero-curvature case.

7. He cited definitions of the Gaussian curvature as the product of principal curvatures, and in terms of the angular excess or defect in geodesic triangles. Gauss' definition of curvature, as the ratio of the area of an infinitesimal region on the surface to the area of its image under the normal (or "Gauss") map, was the basis of Riemann's claim that his sectional curvature generalizes Gaussian curvature of surfaces.

8. This is Gauss' version of the Gauss-Bonnet formula for surfaces. Bonnet [1848] extended it to regions bounded by nongeodesic curves.

9. See [Riemann 1876, 402], the last three lines. The first condition should be $\delta'\Sigma b_{i,i'} ds_i \delta s_{i'} - \delta\Sigma b_{i,i'} ds_i \delta' s_{i'} - d\Sigma b_{i,i'} \delta s_i \delta' s_{i'} = 0$. As implied in [Riemann 1868], it is always

possible to choose coordinates so that $\Gamma_{ij/k} = 0$; but Riemann did not mention it here.

10. The ease with which one proves various theorems for curves does not always carry over to surfaces. For example, the existence of geodesic curves is a consequence of standard results on ordinary differential equations; totally geodesic surfaces are rare, except in manifolds of constant curvature (CF. [Cartan 1928, 119-121]).

11. "Hieraus erklärt sich die Form der Darstellung, in welcher die analytischen Untersuchungen nur angedeutet werden konnten; in einem besonderen Aufsatze gedenke ich demnächst auf dieselben zurückzukommen." Dugac [1976, 63] notes the existence of an unpublished manuscript, of about 100 pages, of Dedekind's analysis of Riemann's work. In [Riemann 1892] the note above was replaced by a reference to [Riemann 1876].

12. Hawkins [1980, 312-313] describes Helmholtz as responding directly to Riemann's paper on several points.

13. The problem of immersion and imbedding has continued to interest geometers. The fact that any Riemannian manifold can be imbedded in a sufficiently high-dimensional Euclidean space was proved by Nash [1956] and Whitney [1957, 113-124].

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