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Statistical convergence of double-complex Picard integral operators

George A. Anastassiou [a,*], Oktay Duman [b]

[a] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA

[b] TOBB Economics and Technology University, Faculty of Arts and Sciences, Department of Mathematics, Söğütözü TR-06530, Ankara, Turkey

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ABSTRACT

In this work, we study the statistical approximation properties of the double-complex Picard integral operators. We also show that our statistical approach is more applicable than the classical one.

1. Introduction

The method of statistical approximation was first considered in the approximation theory by Gadjiev and Orhan [1] for positive linear operators. Later, this method was improved in approximation by more general linear operators (see [2–5]). The main purpose of the present work is to obtain some statistical approximation results for the bivariate complex Picard integral operators. The organization of this work is as follows. In the first section we recall some definitions and set the main notation used in the work, while, in the second section, we construct our complex Picard operators in two complex variables and investigate their geometric properties. In the last section, after giving an estimation via the concept of the second modulus of smoothness, we obtain a statistical approximation theorem for our operators.

In order to obtain some statistical approximation theorems we use the concept of A-statistical convergence, where $A := [a_{jn}]$, $j, n = 1, 2, \ldots$, is any non-negative regular summability matrix. Recall that a matrix A is regular if $\lim_{j\to\infty}(Ax)_j = L$ whenever $\lim_{n\to\infty} x_n = L$, where the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ is called the A-transform of x and defined to be $(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n$ provided that the series is convergent for each $n \in \mathbb{N}$ (see, e.g., [6]). Now, a sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$, $\lim_{j\to\infty} \sum_{n:|x_n-L|\geq\varepsilon} a_{nj} = 0$, which is denoted by $st_A - \lim_n x_n = L$ (see [7]). If $A = C_1 = [c_{jn}]$, the Cesáro matrix of order 1 is defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise; then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [8]. In this case, we use the notation st- lim instead of st_{C_1}- lim (see the last section for this situation). Notice that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A; however, the converse is not always true. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded; however it does not need to be bounded for an A-statistically convergent sequence.

2. Construction of the operators

In this section, we mainly use the ideas as in the papers [9,10]. Let

$$D^2 := D \times D = \left\{(z, w) \in \mathbb{C}^2 : |z| < 1 \text{ and } |w| < 1\right\}$$

* Corresponding author.
E-mail addresses: ganastss@memphis.edu, ganastss@gmail.com (G.A. Anastassiou), oduman@etu.edu.tr (O. Duman).

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and

$$\bar{D}^2 := \bar{D} \times \bar{D} = \left\{(z, w) \in \mathbb{C}^2 : |z| \leq 1 \text{ and } |w| \leq 1\right\}.$$

Assume that $f : \bar{D}^2 \to \mathbb{C}$ is a complex function in two complex variables. If the univariate complex functions $f(\cdot, w)$ and $f(z, \cdot)$ (for each fixed z and $w \in D$, respectively) are analytic on D, then we say that the function $f(\cdot, \cdot)$ is analytic on D^2 (see, e.g., [11,12]). If a function f is analytic on D^2, then f has the following Taylor expansion:

$$f(z, w) = \sum_{k,m=0}^{\infty} a_{k,m}(f) z^k w^m, \quad (z, w) \in D^2, \tag{2.1}$$

having the coefficients $a_{k,m}(f)$ given by

$$a_{k,m}(f) := -\frac{1}{4\pi^2} \int_T \frac{f(p, q)}{p^{k+1} q^{m+1}} \mathrm{d}p\mathrm{d}q, \quad k, m \in \mathbb{N}_0, \tag{2.2}$$

where $T := \left\{(p, q) \in \mathbb{C}^2 : |p| = r \text{ and } |q| = \rho\right\}$ with $0 < r, \rho < 1$.

Now consider the following space:

$$A\left(\bar{D}^2\right) := \left\{f : \bar{D}^2 \to \mathbb{C};\ f \text{ is analytic on } D^2,\ \text{continuous on } \bar{D}^2 \text{ with } f(0, 0) = 0\right\}. \tag{2.3}$$

In this case, $A\left(\bar{D}^2\right)$ is a Banach space with the sup-norm given by

$$\|f\| = \sup\left\{|f(z, w)| : (z, w) \in \bar{D}^2\right\} \quad \text{for } f \in A\left(\bar{D}^2\right).$$

We now define the double-complex Picard-type singular operators as follows:

$$P_n(f; z, w) = \frac{1}{2\pi\xi_n^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) \mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n} \mathrm{d}s\mathrm{d}t, \tag{2.4}$$

where $(z, w) \in \bar{D}^2$, $n \in \mathbb{N}$, $f \in A(\bar{D}^2)$, and also $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers.

It is not hard to see that if f is a constant function on $\bar{D}^2$, say $f(z, w) \equiv C$, then we have, for every $n \in \mathbb{N}$ that $P_n(C; z, w) = C$. Hence, the operators P_n preserve the constant functions.

In order to get some geometric properties of the operators P_n in (2.4) we first need the following concepts.

Let us have $f \in C(\bar{D}^2)$, the space of all continuous functions on $\bar{D}^2$. Then, the first modulus of continuity of f on $\bar{D}^2$, denoted by $\omega_1(f, \delta)_{\bar{D}^2}$, $\delta > 0$, is defined to be

$$\omega_1(f; \delta)_{\bar{D}^2} := \sup\left\{|f(z, w) - f(p, q)| : \sqrt{|z-p|^2 + |w-q|^2} \leq \delta,\ (z, w), (p, q) \in \bar{D}^2\right\}$$

and the second modulus of smoothness of f on $\partial(D^2)$, denoted by $\omega_2(f; \alpha)_{\partial(D^2)}$, $\alpha > 0$, is defined to be

$$\omega_2(f; \alpha)_{\partial(D^2)} := \sup\left\{f\left(\mathrm{e}^{\mathrm{i}(x+s)}, \mathrm{e}^{\mathrm{i}(y+t)}\right) - 2f\left(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}y}\right) + f\left(\mathrm{e}^{\mathrm{i}(x-s)}, \mathrm{e}^{\mathrm{i}(y-t)}\right) : (x, y) \in \mathbb{R}^2 \text{ and } \sqrt{s^2+t^2} \leq \alpha\right\}.$$

Then, by the maximum modulus principle for complex functions of several variables (see, e.g., [11,12]), if $\sqrt{s^2+t^2} \leq \alpha$, we observe that

$$\begin{aligned}
\left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) - 2f(z, w) + f\left(z\mathrm{e}^{-\mathrm{i}s}, w\mathrm{e}^{-\mathrm{i}t}\right)\right| &\leq \sup_{(z,w)\in\bar{D}^2} \left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) - 2f(z, w) + f\left(z\mathrm{e}^{-\mathrm{i}s}, w\mathrm{e}^{-\mathrm{i}t}\right)\right| \\
&= \sup_{(z,w)\in\partial(D^2)} \left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) - 2f(z, w) + f\left(z\mathrm{e}^{-\mathrm{i}s}, w\mathrm{e}^{-\mathrm{i}t}\right)\right| \\
&= \sup_{(x,y)\in\mathbb{R}^2} \left|f\left(\mathrm{e}^{\mathrm{i}(x+s)}, \mathrm{e}^{\mathrm{i}(y+t)}\right) - 2f(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}y}) + f\left(\mathrm{e}^{\mathrm{i}(x-s)}, \mathrm{e}^{\mathrm{i}(y-t)}\right)\right|.
\end{aligned}$$

Thus, we easily get that

$$\left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) - 2f(z, w) + f\left(z\mathrm{e}^{-\mathrm{i}s}, w\mathrm{e}^{-\mathrm{i}t}\right)\right| \leq \omega_2\left(f; \sqrt{s^2+t^2}\right)_{\partial(D^2)}. \tag{2.5}$$

Now let $f \in C\left(\bar{D}^2\right)$ and $\alpha > 0$. Using the function $\varphi_f : \mathbb{R}^2 \to \mathbb{C}$ defined by $\varphi_f(x, y) = f\left(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}y}\right)$, we see that

$$\omega_2(f; \alpha)_{\partial(D^2)} \equiv \omega_2(\varphi_f; \alpha)_{\partial(D^2)}. \tag{2.6}$$

Therefore, the equivalence in (2.6) enables us to write

$$\omega_2(f; c\alpha)_{\partial(D^2)} \leq (1+c)^2 \omega_2(f; \alpha)_{\partial(D^2)}. \tag{2.7}$$

We obtain the following result.

Theorem 2.1. *For each fixed $n \in \mathbb{N}$, $P_n\left(A\left(\bar{D}^2\right)\right) \subset A\left(\bar{D}^2\right)$.*

Proof. Let $n \in \mathbb{N}$ and $f \in A\left(\bar{D}^2\right)$ be fixed. Since $f(0,0)=0$, we easily see that

$$P_n(f;0,0) = \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(0,0)\,\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t = 0.$$

Now we show that $P_n(f)$ is continuous on $\bar{D}^2$. To see this assume that $(p,q),(z_m,w_m)\in\bar{D}^2$ and that $\lim_m(z_m,w_m)=(p,q)$. Hence, we get from the definition of ω_1 that

$$\begin{aligned}|P_n(f;z_m,w_m)-P_n(f;p,q)| &\le \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f\left(z_m\mathrm{e}^{\mathrm{i}s},w_m\mathrm{e}^{\mathrm{i}t}\right)-f\left(p\mathrm{e}^{\mathrm{i}s},q\mathrm{e}^{\mathrm{i}t}\right)\right|\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\\ &\le \frac{\omega_1\left(f,\sqrt{|z_m-p|^2+|w_m-q|^2}\right)_{\bar{D}^2}}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\\ &= \omega_1\left(f,\sqrt{|z_m-p|^2+|w_m-q|^2}\right)_{\bar{D}^2}.\end{aligned}$$

Since $\lim_m(z_m,w_m)=(p,q)$, we may write that

$$\lim_m\sqrt{|z_m-p|^2+|w_m-q|^2}=0,$$

which implies that

$$\lim_m\omega_1\left(f,\sqrt{|z_m-p|^2+|w_m-q|^2}\right)_{\bar{D}^2}=0$$

due to the right continuity of $\omega_1(f,\cdot)$ at zero. Hence, we get

$$\lim_m P_n(f;z_m,w_m)=P_n(f;p,q),$$

which gives the continuity of $P_n(f)$ at the point $(p,q)\in\bar{D}^2$.

Finally, since $f\in A\left(\bar{D}^2\right)$, the function f has the Taylor expansion in (2.1) with the coefficients $a_{k,m}(f)$ in (2.2). Then, for $(z,w)\in D^2$, we get

$$f(z\mathrm{e}^{\mathrm{i}s},w\mathrm{e}^{\mathrm{i}t})=\sum_{k,m=0}^{\infty}a_{k,m}(f)z^kw^m\mathrm{e}^{\mathrm{i}(sk+tm)}. \tag{2.8}$$

Since $\left|a_{k,m}(f)\mathrm{e}^{\mathrm{i}(sk+tm)}\right|=\left|a_{k,m}(f)\right|$ for every $(s,t)\in\mathbb{R}^2$, the series in (2.8) is uniformly convergent with respect to $(s,t)\in\mathbb{R}^2$. Hence, we conclude that

$$\begin{aligned}P_n(f;z,w) &= \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{k,m=0}^{\infty}a_{k,m}(f)z^kw^m\mathrm{e}^{\mathrm{i}(sk+tm)}\right)\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\\ &= \frac{1}{2\pi\xi_n^2}\sum_{k,m=0}^{\infty}a_{k,m}(f)z^kw^m\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(sk+tm)}\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\right)\\ &= \frac{1}{2\pi\xi_n^2}\sum_{k,m=0}^{\infty}a_{k,m}(f)z^kw^m\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(sk+tm)\,\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\right)\\ &= \frac{2}{\pi\xi_n^2}\sum_{k,m=0}^{\infty}a_{k,m}(f)z^kw^m\left(\int_0^{\infty}\int_0^{\infty}\cos(sk+tm)\,\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\right)\\ &= \sum_{k,m=0}^{\infty}a_{k,m}(f)\ell_n(k,m)z^kw^m,\end{aligned}$$

where, for $k,m\in\mathbb{N}_0$,

$$\begin{aligned}\ell_n(k,m) &:= \frac{2}{\pi\xi_n^2}\int_0^{\infty}\int_0^{\infty}\cos(sk+tm)\,\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t\\ &= \frac{2}{\pi\xi_n^2}\int_0^{\pi/2}\int_0^{\infty}\cos\left[\rho\left(k\cos\theta+m\sin\theta\right)\right]\mathrm{e}^{-\rho/\xi_n}\rho\,\mathrm{d}\rho\mathrm{d}\theta\\ &= \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\infty}\cos\left[u\xi_n\left(k\cos\theta+m\sin\theta\right)\right]\mathrm{e}^{-u}\mathrm{d}u\mathrm{d}\theta.\end{aligned} \tag{2.9}$$

We should remark that

$$|\ell_n(k,m)| \leq 1 \quad \text{for every } n \in \mathbb{N} \text{ and } k, m \in \mathbb{N}_0.$$

Therefore, for each $n \in \mathbb{N}$ and $f \in A\left(\bar{D}^2\right)$, the function $P_n(f)$ has a Taylor series expansion whose Taylor coefficients are given by

$$a_{k,m}(P_n(f)) := a_{k,m}(f)\ell_n(k,m), \quad k, m \in \mathbb{N}_0. \tag{2.10}$$

Combining the above facts we obtain the desired result. □

Now consider the following space:

$$B\left(\bar{D}^2\right) := \left\{f : \bar{D}^2 \to \mathbb{C}; f \text{ is analytic on } D^2,\ f(0,0) = 1 \text{ and } \mathrm{Re}\,[f(z,w)] > 0 \text{ for every } (z,w) \in D^2\right\}.$$

Then we have the next result.

Theorem 2.2. *For each fixed $n \in \mathbb{N}$, $P_n\left(B\left(\bar{D}^2\right)\right) \subset B\left(\bar{D}^2\right)$.*

Proof. Let $n \in \mathbb{N}$ and $f \in B\left(\bar{D}^2\right)$ be fixed. As in the proof of Theorem 2.1, we see that $P_n(f)$ is analytic on D^2. Since $f(0,0) = 1$, we easily get that

$$P_n(f;0,0) = \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(0,0)\,\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t = 1.$$

Finally, we may write that, for every $(z,w) \in D^2$,

$$\mathrm{Re}\,[P_n(f;z,w)] = \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Re}\left[f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right)\right]\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t > 0$$

since $\mathrm{Re}\,[f(z,w)] > 0$. Thus, the proof is completed. □

Using the definition of $\omega_1(f;\delta)_{\bar{D}^2}$ for $f \in C\left(\bar{D}^2\right)$ and $\delta > 0$, we obtain the following theorem.

Theorem 2.3. *For each fixed $n \in \mathbb{N}$ and $f \in C\left(\bar{D}^2\right)$, we have*

$$\omega_1(P_n(f);\delta)_{\bar{D}^2} \leq \omega_1(f;\delta)_{\bar{D}^2}.$$

Proof. Let $\delta > 0$, $n \in \mathbb{N}$ and $f \in C\left(\bar{D}^2\right)$ be given. Assume that $(z,w), (p,q) \in \bar{D}^2$ and $\sqrt{|z-p|^2 + |w-q|^2} \leq \delta$. Then, we have

$$\begin{aligned}|P_n(f;z,w) - P_n(f;p,q)| &\leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f\left(z\mathrm{e}^{\mathrm{i}s}, w\mathrm{e}^{\mathrm{i}t}\right) - f\left(p\mathrm{e}^{\mathrm{i}s}, q\mathrm{e}^{\mathrm{i}t}\right)\right|\mathrm{e}^{-\sqrt{s^2+t^2}/\xi_n}\mathrm{d}s\mathrm{d}t \\ &\leq \omega_1\left(f;\sqrt{|z-p|^2+|w-q|^2}\right)_{\bar{D}^2} \\ &\leq \omega_1(f;\delta)_{\bar{D}^2}.\end{aligned}$$

Then, taking the supremum over $\sqrt{|z-p|^2+|w-q|^2} \leq \delta$, we conclude that

$$\omega_1(P_n(f);\delta)_{\bar{D}^2} \leq \omega_1(f;\delta)_{\bar{D}^2},$$

whence the result. □

3. Statistical approximation by the operators P_n

We first obtain the following estimate for the operators P_n defined by (2.4).

Theorem 3.1. *For every $f \in A\left(\bar{D}^2\right)$, we have*

$$\|P_n(f) - f\| \leq M\omega_2(f,\xi_n)_{\partial(D^2)}$$

for some (finite) positive constant M.

Proof. Let $(z, w) \in \bar{D}^2$ and $f \in A\left(\bar{D}^2\right)$ be fixed. We first observe that

$$\begin{aligned} P_n(f; z, w) - f(z, w) &= \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{f\left(ze^{is}, we^{it}\right) - f(z, w)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &= \frac{1}{2\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\left\{f\left(ze^{is}, we^{it}\right) - f(z, w)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &\quad + \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{0}\int_{-\infty}^{0}\left\{f\left(ze^{is}, we^{it}\right) - f(z, w)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &\quad + \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{0}\int_{0}^{\infty}\left\{f\left(ze^{is}, we^{it}\right) - f(z, w)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &\quad + \frac{1}{2\pi\xi_n^2}\int_{0}^{\infty}\int_{-\infty}^{0}\left\{f\left(ze^{is}, we^{it}\right) - f(z, w)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt. \end{aligned}$$

After some simple calculations, we have

$$\begin{aligned} P_n(f; z, w) - f(z, w) &= \frac{1}{2\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\left\{f\left(ze^{is}, we^{it}\right) - 2f(z, w) + f\left(ze^{-is}, we^{-it}\right)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &\quad + \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{0}\int_{0}^{\infty}\left\{f\left(ze^{is}, we^{it}\right) - 2f(z, w) + f\left(ze^{-is}, we^{-it}\right)\right\} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt. \end{aligned}$$

It follows from the property (2.5) that, for all $(z, w) \in \bar{D}^2$,

$$\begin{aligned} |P_n(f; z, w) - f(z, w)| &\le \frac{1}{2\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\omega_2\left(f, \sqrt{s^2+t^2}\right)_{\partial(D^2)} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &\quad + \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{0}\int_{0}^{\infty}\omega_2\left(f, \sqrt{s^2+t^2}\right)_{\partial(D^2)} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &= \frac{1}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\omega_2\left(f, \sqrt{s^2+t^2}\right)_{\partial(D^2)} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &= \frac{1}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\omega_2\left(f, \frac{\sqrt{s^2+t^2}}{\xi_n}\xi_n\right)_{\partial(D^2)} e^{-\sqrt{s^2+t^2}/\xi_n} dsdt. \end{aligned}$$

If we also consider the property (2.7), then we see that

$$\begin{aligned} |P_n(f; z, w) - f(z, w)| &\le \frac{\omega_2(f, \xi_n)_{\partial(D^2)}}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\left(1 + \frac{\sqrt{s^2+t^2}}{\xi_n}\right)^2 e^{-\sqrt{s^2+t^2}/\xi_n} dsdt \\ &= \frac{\omega_2(f, \xi_n)_{\partial(D^2)}}{\pi\xi_n^2}\int_{0}^{\pi/2}\int_{0}^{\infty}\left(1 + \frac{\rho}{\xi_n}\right)^2 \rho e^{-\rho/\xi_n} d\rho d\theta \\ &= \frac{\omega_2(f, \xi_n)_{\partial(D^2)}}{2}\int_{0}^{\infty}(1+u)^2 ue^{-u} du \\ &= M\omega_2(f, \xi_n)_{\partial(D^2)}, \end{aligned}$$

where

$$M = \frac{1}{2}\int_{0}^{\infty}(1+u)^2 ue^{-u} du < \infty.$$

Taking the supremum over $(z, w) \in \bar{D}^2$ for the last inequality, the proof is completed. □

In order to get a statistical approximation by the operators P_n we need the following lemma.

Lemma 3.2. *Let $A := [a_{jn}], j, n = 1, 2, \ldots$, be a non-negative regular summability matrix. If a bounded sequence $(\xi_n)_{n\in\mathbb{N}}$ in (2.4) satisfies the condition*

$$st_A - \lim \xi_n = 0, \tag{3.1}$$

then we have, for all $f \in C\left(\bar{D}^2\right)$, that

$$st_A - \lim_n \omega_2(f; \xi_n)_{\partial(D^2)} = 0.$$

Proof. Let $f \in C\left(\bar{D}^2\right)$. Then, the proof immediately follows from (3.1) and the right continuity of $\omega_2(f; \cdot)_{\partial(D^2)}$ at zero. □

Now we are ready to give our statistical approximation result.

Theorem 3.3. *Let $A := [a_{jn}]$, $j, n = 1, 2, \ldots$, be a non-negative regular summability matrix. Assume that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is the same as in Lemma 3.2. Then, for every $f \in A\left(\bar{D}^2\right)$, we have*

$$st_A - \lim_n \|P_n(f) - f\| = 0.$$

Proof. Let $f \in A\left(\bar{D}^2\right)$. Then, for a given $\varepsilon > 0$, we may write from Theorem 3.1 that

$$U := \{n \in \mathbb{N} : \|P_n(f) - f\| \geq \varepsilon\} \subseteq \left\{n \in \mathbb{N} : \omega_2(f;\xi_n)_{\partial(D^2)} \geq \frac{\varepsilon}{M}\right\} =: V,$$

where M is the positive constant as in Theorem 3.1. Thus, for every $j \in \mathbb{N}$, we get

$$\sum_{n\in U} a_{jn} \leq \sum_{n\in V} a_{jn}.$$

Now taking the limit as $j \to \infty$ in both sides of the last inequality and also considering Lemma 3.2 we obtain that

$$\lim_j \sum_{n\in U} a_{jn} = 0,$$

which gives

$$st_A - \lim_n \|P_n(f) - f\| = 0.$$

The proof is completed. □

Taking $A = C_1$, the Cesáro matrix of order 1, in Theorem 3.3 we immediately get the following result.

Corollary 3.4. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$st\text{-}\lim_n \xi_n = 0$$

holds. Then, for every $f \in A\left(\bar{D}^2\right)$, we have

$$st\text{-}\lim_n \|P_n(f) - f\| = 0.$$

Of course, if we choose $A = I$, the identity matrix, in Theorem 3.3, then we get the following uniform approximation result.

Corollary 3.5. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a null sequence of positive real numbers. Then, for every $f \in A\left(\bar{D}^2\right)$, the sequence $\{P_n(f)\}_{n\in\mathbb{N}}$ is uniformly convergent to f on $\bar{D}^2$.*

Finally, define the sequence $(\xi_n)_{n\in\mathbb{N}}$ as follows:

$$\xi_n := \begin{cases} 1, & \text{if } n = k^2,\ k = 1, 2, \ldots \\ \dfrac{1}{n}, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then, observe that $st\text{-}\lim_n \xi_n = 0$. In this case, by Corollary 3.4 (i.e., Theorem 3.3 for $A = C_1$) we obtain that

$$st\text{-}\lim_n \|P_n(f) - f\| = 0$$

for every $f \in A\left(\bar{D}^2\right)$. However, since the sequence $(\xi_n)_{n\in\mathbb{N}}$ given by (3.2) is non-convergent, uniform approximation to a function f by the operators $P_n(f)$ is impossible.

Notice that our statistical results are still valid when $\lim \xi_n = 0$ because every convergent sequence is A-statistically convergent, and so statistically convergent. But, as in the above example, our theorems still work although $(\xi_n)_{n\in\mathbb{N}}$ is non-convergent. Therefore, we can say that our approach presented in this work is more applicable than the classical approach.

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