JOURNAL OF ALGEBRA **145**, 128–142 (1992)

Almost-Free Groups in Varieties

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Communicated by Peter M. Neumann

Received September 30, 1987

Let $\mathscr{V}$ be a non-trivial variety of groups. For every $0<n<\omega$, $\mathscr{V}$ has a non-free $\aleph_n$-free group of cardinality $\aleph_n$ if and only $\mathscr{V}$ is not a variety of nilpotent groups of prime power exponent. If $\mathscr{V}$ is not a variety of nilpotent groups of prime power exponent then either $\mathscr{V}$ has a non-free κ-free group of power κ in every cardinal κ for which such an abelian group exists or $\mathscr{V}$ has a non-free κ^+-free group of power κ^+ for every cardinal κ.

0. PRELIMINARIES

A natural question to ask about any variety of algebras is: For which infinite cardinals κ are there almost-free algebras of cardinality κ which are not free? This question can be studied from the point of view of universal algebra, where we try to prove results for all varieties or we can concentrate on interesting varieties of algebras. The papers [S1], [S2], [EM1], [MS] are all examples of the first approach. Since it puts the results in the current paper in perspective, it should be noted that Shelah's singular compactness theorem [S1] says that if κ is a singular cardinal then any almost-free algebra of cardinality κ is free. Most of the investigation of specific varieties has concentrated on varieties of modules and varieties of groups. Information on varieties of modules can be found in [EM1] or [EM2]. For the variety of groups, the earliest result was Higman's [H] construction of an almost-free group of cardinality $\aleph_1$ which is not free. For abelian groups it seems that Baer [B] knew that the direct product of countably many copies of $\mathbb{Z}$ contains an almost-free abelian group of cardinality $\aleph_1$ which is not free. (The result appears more clearly in [Sp].) Many papers on the construction of almost-free groups and abelian groups in various cardinalities and with additional properties have been published. We will not try to summarize all these results here but will follow the main line

* Research partially supported by NSERC of Canada under Grant A8948.

0021-8693/92 $3.00

toward the construction of almost-free groups in varieties. In the rest of the paper a variety will be a variety of groups unless we specify that we are considering of algebras.

In the variety of groups or in the variety of abelian groups, a group is *κ-free* if all subsets of cardinality less than κ generate a free group. For other varieties there are several possible definitions of being κ-free. The weakest notion is the one used in [P2], namely, that a group is κ-free if every subset of cardinality less than κ is contained in a free subgroup. Of course, the notion of free is to be interpreted in the variety in which we are working. (When we wish to stress the variety $\mathscr{V}$, we will write $\mathscr{V}$-free, etc.) A stronger notion is to demand that most (in one of several possible senses) subgroups of cardinality less than κ are free. A group of cardinality κ is *almost-free* if it is κ-free. The precise definition of κ-free need not be a great concern in this paper (although it does seem to matter for the singular compactness theorem). Any example of an almost-free group G that we produce will be almost-free according to any of the definitions. On the other hand when we are able to assert that there is no almost-free group which is not free, we are able to use the weakest possible definition. So our results do not depend on which definition has been adopted. (The phrasing of the strongest notion is somewhat awkward but the following notion suffices for a group G of regular cardinality κ; namely that G is the union of a smooth chain of free subgroups of cardinality less than κ.)

Recently, Pope [P2] published a paper which purported to show that any variety containing a finite simple group has non-free almost-free groups of power $\aleph_n$ for every $n>0$. Unfortunately there is an error in the proof. Lemma 1 of that paper is incorrect. However, using some ideas from that paper in combination with results in [EM1] and [S2] it is possible to show that any variety which is not nilpotent of prime power exponent has an almost-free group of power $\aleph_n$ for every $n>0$. We first review the results we will need. No attempt is made to avoid the use of logical machinery even where it is possible. However, we will try to state the consequences which are needed later in this paper in non-logical terms.

In [EM1], the construction principle (CP) for an arbitrary variety was defined as follows:

> There are countable free algebras $H \subseteq K$ so that
>
> (i) H has a set of free generators $\{h_n : n<\omega\}$ such that for every finite $S \subseteq \omega$ the algebra generated by $\{h_n : n \in S\}$ is a free factor of K with a free complementary factor.
>
> (ii) H is not a free factor of $K * \mathrm{F}(\omega)$ with a free complementary factor.

In the above condition $*$ denotes free product, $\mathrm{F}(\kappa)$ denotes the free algebra of rank κ. When we say that A is a free factor of B with a free complementary factor, we mean that there is a free C, so that $A * C = B$. (This definition is not the one which appears in [EM1] but is equivalent to it.) The condition in (i) that the complementary factor of the subalgebra generated by $\{h_n : n \in S\}$ be free is superfluous. It is not hard to see that, using the notation above, if a subalgebra A of K is a free factor of $K * \mathrm{F}(\omega)$ which is free, then it has a complementary factor which is free. Suppose C is a complementary factor and let H_S denote the subalgebra generated by $\{h_n : n \in S\}$. Then, replacing $\mathrm{F}(\omega)$ by $\mathrm{F}(\omega) * \mathrm{F}(\omega)$, we have $K * \mathrm{F}(\omega) \cong A * C * \mathrm{F}(\omega)$. But $C * \mathrm{F}(\omega) \cong C * A * \mathrm{F}(\omega)$, since A is free, and $C * A$ is free. Similar comments apply for (ii).

THEOREM 1. *Let $\mathscr{V}$ be a variety of algebras in a countable similarity type.*

(a) [EM1] *The construction principle holds if and only if there is an $\mathrm{L}_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ which is not free.*

(b) [MS] *If the construction principle fails to hold then either every $\mathrm{L}_{\infty\omega}$-free algebra is free or free every infinite cardinal κ there is an algebra of cardinality κ^+ which is κ^+-free but not free.*

There is an easily understood algebraic characterization of $\mathrm{L}_{\infty\omega}$-free algebras of cardinality $\aleph_1$. By a theorem of Kueker [Ku], an algebra of cardinality $\aleph_1$ is $\mathrm{L}_{\infty\omega}$-free if and only if it can be written as the union of a smooth chain of $\aleph_1$ countable free algebras. In [EM1] it is shown that in the presence of certain set-theoretic principles, the construction principle can be used to build κ-free non-free algebras of cardinality κ for any regular non-weakly compact cardinal, in particular for each $\aleph_n$ $(0 < n < \omega)$. The set-theoretic principles used, while consistent with the axioms for set theory, are independent assuming the consistency of some large cardinal axioms. For an arbitrary variety of algebras there is no way to deduce the existence of an almost-free algebra of cardinality $\aleph_2$ from the existence of an almost-free algebra of cardinality $\aleph_1$, without appealing to additional set-theoretic assumptions. (It is possible to go in the other direction.) For varieties of groups rather more is known. Building on previous work of Eklof [E] and Mekler [M], Pope showed:

THEOREM 2 [P1]. *If $\mathscr{V}$ is a variety of groups of infinite exponent (i.e., containing $\mathbb{Z}$), then for all $0 < n < \omega$, there is an $\aleph_n$-free group of cardinality $\aleph_n$ which is not free.*

Actually this theorem could have been deduced easily from earlier results in [M]. First to fix some notation, if $\mathscr{V}$ is a variety and G is a group let $\mathscr{V}G$ denote G modulo the verbal subgroup defined by laws determining $\mathscr{V}$. For $g \in G$ denote its image in $\mathscr{V}G$ by $g/\mathscr{V}$. In [M] for any $0 < n < \omega$, an

$L_{\infty\omega_n}$-free group G of cardinality $\aleph_n$ is constructed so that $G/[G, G]$ is not free abelian. So for any variety $\mathscr{V}$ containing $\mathbb{Z}$, $\mathscr{V}G/[\mathscr{V}G, \mathscr{V}G] \cong G/[G, G]$. Hence $\mathscr{V}G$ is not free, but $\mathscr{V}G$ is $L_{\infty\omega_n}$-free since the verbal subgroup determined by $\mathscr{V}$ is definable in $L_{\omega_1\omega}$. For $\aleph_1$, Pope's theorem can be derived using the group which Higman constructed in [H], since that group is $L_{\infty\omega}$-free.

This result can be approached differently by using a stronger version of the construction principle. The strong construction principle (CP+) is defined as follows:

> For every $0 < n < \omega$ there are countable free algebras $H \subseteq K$ and a partition of ω into n infinite blocks $s^1, ..., s^n$ so that
>
> (i) H has a set of free generators $\{h_m : m < \omega\}$, and for every subset $S \subseteq \omega$ if for some k, $S \cap s^k$ is finite then the algebra generated by $\{h_m : m \in S\}$ is a free factor of K with a free complementary factor.
>
> (ii) H is not a free factor of $K * \mathrm{F}(\omega)$ with a free complementary factor.

The following theorem is implicit in the methods of [S2] and [EM1] and is explicitly stated in [EM2].

THEOREM 3. *Suppose $\mathscr{V}$ is a variety of algebras which satisfies* (CP+). *For any uncountable cardinal κ, if there is an almost-free abelian group of cardinality κ which is not free then there is an almost-free algebra of cardinality κ in $\mathscr{V}$ which is not free.*

Although we will not try to summarize Shelah's construction, it may help the reader to understand the definition of (CP+) if we say a few words about the proof of Theorem 3. From the existence of an almost-free non-free abelian group of cardinality κ the existence of a family $\{s_i : i \in I\}$ of countable sets is deduced, where I is a set of cardinality κ. For some fixed n, each s_i is the disjoint union of infinite sets $s_i^1, ..., s_i^n$. For any $J \subseteq I$, if the cardinality of J is less than κ, then there is a well ordering $<_J$ of J so that for all $i \in J$ there is some k so that $s_i^k \cap \bigcup_{j <_J i} s_j$ is finite. (For $n > 0$ and finite and $\kappa = \aleph_n$ the construction of such a family is fairly straightforward.) To construct the algebra take K_i a copy of K for each $i \in I$. The algebra is the free product of the K_i's, where we identify the generators of the copy of H in K_i with the elements of s_i. It should not be difficult to believe that the algebra constructed is almost-free. Other properties of the family are used to guarantee that the algebra is not free.

The strong version of the construction principle holds in any variety of groups of infinite exponent. Indeed the usual example of the construction principle, where K is the group freely generated by $\{t_n : n < \omega\}$ and H is the subgroup freely generated by $\{t_n t_{n+1}^{-2} : n < \omega\}$ is an example of (CP+). So Theorem 2 is a consequence of Theorem 3 and the fact that (CP+) holds in every variety of infinite exponent. In [P1] there is an inductive argument which shows that if a variety $\mathscr{V}$ of groups satisfies (CP), then for every $0<n<\omega$, $\mathscr{V}$ has a non-free $L_{\infty\omega_n}$-free group of cardinality $\aleph_n$. The group-theoretic content of the inductive argument in [P1] (and of [E] and [M]) is contained in the following theorem.

THEOREM 4. *If $\mathscr{V}$ is a variety of groups which satisfies* (CP) *then $\mathscr{V}$ satisfies* (CP+).

Proof. In this proof we use the following simple fact.

FACT. *Suppose X and Y are disjoint sets whose union freely generates a $\mathscr{V}$-free group G and for each $x \in X$, u_x is a word in Y. Then $Y \cup \{xu_x : x \in X\}$ freely generates G.*

Proof (*of fact*). Consider two endomorphisms, φ, ψ of G defined by $\varphi(y)=y$, for $y \in Y$, and $\varphi(x)=xu_x$ for $x \in X$ and $\psi(y)=y$, for $y \in Y$, and $\psi(x)=xu_x^{-1}$ for $x \in X$. Both endomorphisms act as the identity on Y and so also on each u_x. Hence $\varphi(\psi(x))=x$; i.e., $\psi=\varphi^{-1}$. So φ is an automorphism of G and $Y \cup \{xu_x : x \in X\}$ is the image of a set of free generators of G.

We now prove the theorem by induction on n. (We now work in $\mathscr{V}$.) Let $H \subseteq K$ satisfy the requirements for n with respect to a set of free generators $\{h_m : m<\omega\}$ and some partition of ω into n pieces. Let G be a free group freely generated by $\{g_m : m<\omega\}$. Let $K_1 = G * K$ and $H_1 = G * H$. By the fact, H_1 is freely generated by $\{g_m : m<\omega\} \cup \{g_m h_m : m<\omega\}$. To prove that part (i) of (CP+) holds for $n+1$ in place of n it is enough to show:

(a) for all $r<\omega$, the subgroup generated by $\{g_m : m<r\} \cup \{g_m h_m : m<\omega\}$ is a free factor of K_1;

(b) if S is a subset of ω so that $\{h_m : m \in S\}$ generates a free factor of K, then $\{g_m : m<\omega\} \cup \{g_m h_m : m \in S\}$ generates a free factor of K_1.

First we consider (a). Choose a free group C so that $K = C * \langle h_m : m<r\rangle$. (Here $\langle X\rangle$ denotes the subgroup generated by X.) Then

$$K_1 = C * \langle h_m : m<r\rangle * \langle g_m : m<\omega\rangle,$$

which by the fact gives

$$K_1 = C * \langle h_m : m<r\rangle * \langle g_m : m<r\rangle * \langle g_m h_m : r \leqslant m\rangle,$$

and applying the fact again gives

$$K_1 = C * \langle g_m h_m : m<r\rangle * \langle g_m : m<r\rangle * \langle g_m h_m : r \leqslant m\rangle,$$

which is the desired expression.

Now consider (b). Choose C so that $K = C * \langle h_m : m \in S \rangle$. Then $K_1 = C * \langle h_m : m \in S \rangle * \langle g_m : m < \omega \rangle$. Applying the fact to this equation yields $K_1 = C * \langle g_m h_m : m \in S \rangle * \langle g_m : m < \omega \rangle$, which is as required.

To prove part (ii) of (CP+) holds, suppose that

$$K_1 * \mathrm{F}(\omega) = H_1 * C.$$

First note that H is a free factor of the right-hand side. Since $K_1 = K * G$, $K_1 * \mathrm{F}(\omega)$ is isomorphic over K to $K * \mathrm{F}(\omega)$. This contradicts the fact that H and K satisfy (ii). ▌

In considering all infinite cardinals, we are led to consider the *incompactness spectrum* of a variety, that is, the class of uncountable cardinals κ for which there is a non-free κ-free algebra of cardinality κ. By the singular compactness theorem, the incompactness spectrum of any variety consists of regular cardinals. The incompactness spectrum may be influenced by the underlying set theory. In [EM1] it is shown that if $V = L$ is true then the incompactness spectrum of any variety of algebras which satisfies the construction principle contains all uncountable regular non-weakly compact cardinals. In [MS], it is shown that the incompactness spectrum of any variety of algebras which does not satisfy the construction principle is either empty or is exactly the class of successor cardinals. Since there are models of set theory in which the incompactness spectrum of abelian groups, the class of uncountable regular cardinals, and the class of successor cardinals all coincide (e.g., a model of $V = L$ and there are no inaccessible cardinals), it is consistent that there are exactly two possible incompactness spectra. However, it is consistent, assuming the consistency of some large cardinal, that the incompactness spectrum of abelian groups is very different from the class of regular cardinals. For example, given the consistency of a supercompact cardinal, it is consistent that the incompactness spectrum of abelian groups contains some weakly inacessible cardinals but does not contain any cardinals greater than or equal to $2^{\aleph_0}$ [BD]. One can hope to obtain absolute information by comparing the incompactness spectrum of a variety with that of abelian groups. Theorem 3 can be restated as saying that the incompactness spectrum of any variety which satisfies (CP+) contains that of abelian groups. Much is known about the incompactness spectrum of abelian groups; for example, it is known that it contains $\aleph_n$ for each $n > 0$ and that if a cardinal κ is in the spectrum then κ^+ is also in the spectrum [E]. Theorem 2 can be strengthened to the following.

THEOREM 5 [S2]. *Suppose $\mathcal{V}$ is a variety of groups of infinite exponent. The incompactness spectrum of $\mathcal{V}$ contains that of abelian groups.*

If a variety does not satisfy the construction principle then its incompactness spectrum is either empty or consists of the class of successor cardinals. A variety of nilpotent groups has the ame incompactness spectrum ao the variety of abelian groups it contains. The following theorem contains what is known for these varieties of finite exponent.

THEOREM 6 [EM1]. *Suppose $\mathscr{V}$ is a variety of nilpotent groups of finite exponent.*

(a) *$\mathscr{V}$ does not satisfy* (CP).

(b) *If $\mathscr{V}$ is of prime power exponent then every $\aleph_1$-free group* (*of any cardinality*) *is free.*

(c) *If $\mathscr{V}$ is not of prime power exponent, then for every infinite cardinal κ there is an almost-free group of cardinality κ^+ which is not free.*

More recently, Pope [P2] with the help of a lemma due to Kovács has claimed an extension of these results to varieties which contain a finite simple group. We will show that Kovács' lemma can be extended to arbitrary varieties which are not solvable. Unfortunately there is an error in another part of [P2].

Lemma 1 of [P2] says:

> Let $\mathscr{V}$ be a variety of groups. Suppose there are countable $\mathscr{V}$-free groups H_n $(n < \omega)$, H, L so that $H_n \mid H_{n+1} \mid L$, $H = \bigcup H_n$, and $H \subseteq L$. Let N be the smallest normal subgroup of L containing H. If L/N is not free, then the construction principle holds in $\mathscr{V}$.

This statement is not true. For example, let $\mathscr{V}$ be the class of abelian groups of exponent 6. The last statement would be true if the hypothesis "L/N is not free" were replaced by "$L/N * F$ is not free, for any free group F."

Our goal (in which we will be only partly successful) is to determine which varieties satisfy (CP+) and which varieties have non-free almost-free algebras of cardinality $\aleph_n$, for $0 < n < \omega$. We consider a series of overlapping cases. The nilpotent and torsion-free varieties have already been discussed. It should be stressed again that the ideas in [P2] form an important part of our approach. By Theorem 4, to show that a variety of groups satisfies (CP+) it is enough to show that it satisfies (CP) (although in the examples we will give, it is not much harder to verify (CP+) directly.)

Our basic strategy in proving that (CP) holds in a variety $\mathscr{V}$ will be to find H and K in the variety of all groups which satisfy (i) and then show that in $\mathscr{V}$ they satisfy (ii). The following lemma makes the strategy explicit.

LEMMA 7. *Let $H \subseteq K$ be countable free groups in the variety of all groups such that H and K satisfy part* (i) *of the definition of* (CP). *Let A denote the quotient of K by the smallest normal subgroup containing H. If $\mathscr{U}$ is a variety of groups so that $\mathscr{U}A *_{\mathscr{U}} \mathrm{F}_{\mathscr{U}}(\omega)$ is not free, then any variety $\mathscr{V} \supseteq \mathscr{U}$ satisfies* (CP) (*or equivalently* (CP+)).

Proof. The groups $\mathscr{V}H \subseteq \mathscr{V}K$ satisfy (i) of (CP). Suppose now that $\mathscr{V}H \mid \mathscr{V}K *_{\mathscr{V}} \mathrm{F}_{\mathscr{V}}(\omega)$. Then $\mathscr{V}A *_{\mathscr{V}} \mathrm{F}_{\mathscr{V}}(\omega)$ is free, but $\mathscr{U}(\mathscr{V}A *_{\mathscr{V}} \mathrm{F}_{\mathscr{V}}(\omega)) = \mathscr{U}A *_{\mathscr{U}} \mathrm{F}_{\mathscr{U}}(\omega)$, which is not free. This is a contradiction, so $\mathscr{V}H \subseteq \mathscr{V}K$ also satisfies (ii). ∎

1. NON-SOLVABLE VARIETIES

First we review Pope's attempt to find an example of the construction principle. Following the strategy outlined in Lemma 7, we first work in the variety of all groups.

EXAMPLE 8. Let K be the group freely generated by $\{t_n : n < \omega\}$ and H the subgroup generated by $\{t_m^{-1}[t_{2m+1}, t_{2m+2}] : m < \omega\}$. It is not hard to see that for all m, the subgroup, H_m, generated by $\{t_k^{-1}[t_{2k+1}, t_{2k+2}] : k \leqslant m\}$ is a free factor of K. (This example is close to the one in Section 4 of [M] and [BS], where it was used to construct groups which were parafree as well as being almost free.) In fact if $S \subseteq \omega$ is such that $\{m : m+1, m+2, ..., 2m+2 \notin S\}$ is unbounded then $\{t_m^{-1}[t_{2m+1}, t_{2m+2}] : m \in S\}$ freely generates a free factor of K. The group generated by $\{t_m : m \notin S\}$ is a complementary factor. To see this it suffices to show that $\{t_m^{-1}[t_{2m+1}, t_{2m+2}] : m \in S\} \cup \{t_m : m \notin S\}$ freely generates K. Choose m so that $m+1, ..., 2m+2 \notin S$. Then $\{t_k^{-1}[t_{2k+1}, t_{2k+2}] : k \in S,\ k \leqslant m\} \cup \{t_k : k \notin S,\ k \leqslant 2m+2\}$ generates the same subgroup as $\{t_k : k \leqslant 2m+2\}$. For any finite k, if k elements generate a free group of rank k, then they generate it freely (cf. Corollary 2.13.1 of [MKS]). It is also easy in this specific case to prove directly or using the fact in the proof of Lemma 4 that these are free generators.

We could now argue directly that part (i) of (CP+) is satisfied. But to apply Lemma 7, it is enough to note that $H \subseteq K$ satisfy part (i) of (CP). Let N be the smallest normal subgroup of K containing H. Let $A = K/N$.

Our generalization of Kovács' lemma is the following lemma. (Kovács' version requires that $\mathscr{V}$ contain a finite simple group.)

LEMMA 9. *Let A be as in Example* 8 *and $\mathscr{V}$ a variety of groups which is not solvable* (*i.e., contains a non-solvable group*). *Then $\mathscr{V}A$ is non-trivial.*

Proof. Rephrasing the statement of the lemma, we are claiming that the first order theory consisting of the laws of the variety $\mathscr{V}$, the equations $t_n = [t_{2n+1}, t_{2n+2}]$, and the inequation $t_0 \neq e$ has a model. By the compactness theorem, it suffices to show that any finite subset of the axioms is consistent. Let G be the $\mathscr{V}$-free group on countably many generators $\{g_n : n < \omega\}$. Consider a finite subset of axioms. Let n be the largest number so that the equation $t_n = [t_{2n+1}, t_{2n+2}]$ is in the subset. Expand G to a structure for the language of the theory by interpreting t_{n+m} as g_m and the other constants according to the equations in the subset. The interpretation of t_0 is not e, since G is not solvable. So G can be expanded to a model of any finite subset of the axioms. ∎

THEOREM 10. *Suppose $\mathscr{V}$ is a variety of groups which is not solvable. Then either $\mathscr{V}$ satisfies* (CP+) *or for every infinite cardinal κ there is an almost-free algebra of cardinality κ^+ which is not free.*

Proof. Let A be as above. There are two possibilities: either $\mathscr{V}A * \mathrm{F}(\omega)$ is free, where $\mathrm{F}(\mu)$ denotes the $\mathscr{V}$-free group on μ generators, or it is not. If this group is not free then, by Lemma 7, $\mathscr{V}$ satisfies (CP+). Suppose $\mathscr{V}A * \mathrm{F}(\omega)$ is free. Let $G = \mathrm{F}(\kappa) * *_{\kappa^+} \mathscr{V}A$, where $*$ denotes the $\mathscr{V}$-free product. G is almost-free, since $\mathrm{F}(\kappa) * *_{\kappa} \mathscr{V}A$ is free.

Suppose now that G is free. Since $\mathscr{V}A$ is non-trivial, G has cardinality κ^+. So the abelianization of G is the direct sim of κ^+ copies of $\mathscr{V}\mathbb{Z}$ and so has cardinality κ^+. However, the abelianization of $*_{\kappa^+} \mathscr{V}A$ is trivial since $A = [A, A]$. So the abelianization of G has cardinality κ. This is a contradiction. (The argument that G is not free is based on Pope's argument for Lemma 3 of [P2].) ∎

2. VARIETIES CONTAINING NON-NILPOTENT LOCALLY NILPOTENT VARIETIES

We now turn to considering solvable varieties which are not nilpotent. We will show that these varieties satisfy the strong construction principle. First we consider the easier case of the varieties which are locally nilpotent.

EXAMPLE 11. Let K be the group (in the variety of all groups) freely generated by $\{t_m : m < \omega\}$ and let H be the subgroup generated by $\{t_{2m}^{-1}[t_{2m+1}, t_{2m+2}] : m < \omega\}$. As in Example 7, we can show that H and K satisfy (i) of (CP). Now let N be the smallest normal subgroup containing H and $A = K/H$.

LEMMA 12. *Suppose $\mathscr{V}$ is a variety of groups which is not nilpotent but is locally nilpotent* (*i.e., every finitely generated group is nilpotent*). *Let A be* *the group constructed in Example* 11. *Then $\mathscr{V}A$ is not contained in a $\mathscr{V}$-free group.*

Proof. First note that $\mathscr{V}A$ is the $\mathscr{V}$-group presented by $(t_n (n<\omega);$ $t_{2n}^{-1}[t_{2n+1}, t_{2n+2}]$ $(n<\omega))$. As in the proof of Lemma 9, we can see that t_0 is not e. Suppose $\mathscr{V}A$ is contained in the $\mathscr{V}$-group F freely generated by $\{f_n : n<\omega\}$. Choose n so that $t_0 \in \langle f_0, ..., f_n \rangle$. (Recall $\langle\ \rangle$ denotes "subgroup generated by.") Let φ be the homomorphism from G onto $\langle f_0, ..., f_n\rangle$ defined by

$$\varphi(f_k) = \begin{cases} f_k, & k \leqslant n \\ e, & \text{otherwise.} \end{cases}$$

Then φ is the identity on $\langle f_0, ..., f_n\rangle$. In particular, $\varphi(t_0) \neq e$. Hence $\varphi(t_0)$ is a non-identity element of a nilpotent group and in that group can be expressed as a commutator of arbitrarily high weight. Thus we have a contradiction. ∎

Lemma 12 can be used to show that any variety which contains a non-nilpotent locally nilpotent variety satisfies the strong construction principle. A group will generate a non-nilpotent locally nilpotent variety if and only if it is not nilpotent but is *uniformly locally nilpotent*. A group is uniformly locally nilpotent if for every n there is m such that every subgroup generated by n elements is nilpotent of class m.

THEOREM 13. *Suppose $\mathscr{V}$ is a variety which contains a non-nilpotent uniformly locally nilpotent group (or equivalently contains a non-nilpotent locally nilpotent variety); then $\mathscr{V}$ satisfies* (CP+).

Proof. By Lemma 7, this is a direct consequence of Example 11 and Lemma 12. ∎

3. VARIETIES CONTAINING A FINITE NON-NILPOTENT GROUP

Recall that the *lower central series* of a group G is defined by $G_1 = G$ and $G_{n+1} = [G_n, G]$. We let $G^{(1)}$ denote the commutator subgroup.

LEMMA 14. *Suppose G is a finite group which is not nilpotent and $\mathscr{V}$ is a variety containing G. Then $\mathscr{V}$ contains a finite group H so that $H^{(1)}$ is an elementary abelian p-group for some prime p, and $H = \langle z, H^{(1)}\rangle$, where the order of z is a prime $q \neq p$ and $z^{-1}az \neq a$ for any non-identity a in $H^{(1)}$.*

Proof. In this proof we use the following straightforward group-theoretic fact. If $G = QP$, where Q is an abelian q-group, P an abelian

p-group which is normal in G, then the centre of $G/Z(G)$ is trivial. Here $Z(G)$ denotes the centre of G. In particular $G/Z(G)$ is nilpotent if and only if G is abelian. Choose H an element of $\mathscr{V}$ which is not nilpotent so that the cardinality of H is minimal. Every proper subgroup of H is nilpotent. It is known (cf. [Sc, 6.5.7, p. 148]) that $H = QP$, where Q is a cyclic q-group, P is a p-group, and P is normal. Fix z a generator of Q. Choose the greatest n so that z does not commute with every element of P_n. By the observation at the beginning of this proof, QP_n/P_{n+1} is not nilpotent. By the minimality of H, P is abelian and the centre of H is trivial. By further applying the minimality of H, we can deduce that the order of z is q and P is elementary abelian.

Finally we can see that $H^{(1)} = P$, by noting that $[z, -]$ is a one–one function on P. ▮

Let H be as in the conclusion of the last lemma. We, now, make some comments on H and subgroups of powers of H. Let $\bar{H}$ denote $H/H^{(1)}$. We denote elements of $\bar{H}$ by small Greek letters. For any $a \in H$, the function $[a, -]$ on $H^{(1)}$ depends only on the image of a in $\bar{H}$. So there is no ambiguity in denoting these functions as $[\sigma, -]$, where $\sigma \in \bar{H}$. To emphasize that $\bar{H}$ is abelian we use additive notation for the elements of $\bar{H}$. Similarly we use additive notation for $H^{(1)}$, but multiplicative notation if we work in all of H. For any non-zero σ, $[\sigma, -]$ is a non-singular linear operator on $H^{(1)}$ and so is invertible. Let m denote the order of $\mathrm{GL}(H^{(1)})$. So for any non-zero σ and any $a \in H^{(1)}$, $[\sigma, -]^m(a) = a$.

Consider H^I, where I is some index set. Extending our notation we will let small greek letters stand for elements of $\bar{H}^I$. By the *support* of an element a (or σ), we mean $\{i : a(i) \neq e\}$ ($\{i : \sigma(i) \neq 0\}$). There are two basic facts which we will use.

PROPOSITION 15. *Let H be as above, and suppose that $a \in (H^{(1)})^I$ and $\sigma \in \bar{H}^I$.*

(a) *The support of $[\sigma, a]$ is contained in the support of σ and in the support of a.*

(b) *$[\sigma, -]^m(a) = a$ if and only if the support of a is contained in the support of σ.*

Proof. Clear. ▮

LEMMA 16. *Suppose H is as in the conclusion of Lemma 14 and $\mathscr{V}$ is the variety generated by H. Then the $\mathscr{V}$-group B presented by $(t_n (n < \omega)$; $t_{2n}^{-1}[t_{2n+1}, t_{2n+2}]$ $(n < \omega))$ is not contained in a $\mathscr{V}$-free group.*

Proof. We first show in B that the elements, $\{t_{2n} : n < \omega\}$, are pairwise distinct non-zero elements of $B^{(1)}$. Fix a a non-zero element of $H^{(1)}$ and let

z be as in the definition of H. Define elements in H^ω as follows. For all n, $t_{2n+1}(k)=z$, if $k \leqslant n$ and let $t_{2n+1}(k)=e$, otherwise. Let $t_0(0)=a$ and $t_0(n)=0$, for $n>0$. Let $t_{2n+2}=[t_{2n+1}, -]^{m-1}(t_{2n})+g$, where $g(n+1)=a$, and $g(k)=0$ if $k \neq n+1$. Since these elements of H^ω satisfy the relations in the presentation, we are done.

We now turn to studying the $\mathscr{V}$-free group on countably many generators. Consider the $\mathscr{V}$-free group F freely generated by $\{f_n : n<\omega\}$. As is well known, F can be considered a subgroup of H^I for some index set I, where for every map φ from ω to H there is a coordinate i so that for all n, $f_n(i)=\varphi(n)$. Note that $F^{(1)} \subseteq (H^{(1)})^I$ and $\bar{F}=F(H^{(1)})^I/(H^{(1)})^I$ is an elementary abelian q-group freely generated by $\{\bar{f}_n : n<\omega\}$. Here $\bar{x}$ denotes $x/(H^{(1)})^I$. We now make the following claim.

Claim. *Suppose* $g \in \langle f_0, ..., f_n \rangle \cap (H^{(1)})^I$, $g \neq 0$ *and* $x \in F$. *If* $[x, -]^m(g)=g$, *then* $\bar{x} \in \langle \bar{f}_0, ..., \bar{f}_n \rangle$.

Proof (of Claim). Suppose not. Then there is some $r>n$ such that $\bar{x}=\sum_{s \leqslant r} k_s \bar{f}_s$, where $k_r \not\equiv 0 \pmod{q}$. Choose i so that $g(i) \neq 0$. Since $\bar{H}$ is a group of order q, there are elements $\sigma_{n+1}, ..., \sigma_r \in \bar{H}$ so that

$$\sum_{s \leqslant n} k_s \bar{f}_s(i) + \sum_{n<s \leqslant r} k_s \sigma_s = 0.$$

By the construction of the $\mathscr{V}$-free group, there is j so that for all $s \leqslant n$, $f_s(j)=f_s(i)$ and for $n<s \leqslant r$, $\bar{f}_s(j)=\sigma_s$. So j is not in the support of $[\bar{x}, g]$, but j is in the support of g. This is a contradiction.

Suppose now that A is a subgroup of F. Choose n so that $t_0 \in \langle f_0, ..., f_n \rangle$. For all n, $t_{2n}=[t_{2n+1}, t_{2n+2}]$. So for all n, $t_{2n} \in (H^{(1)})^I$ and, by Proposition 15(a), the support of t_{2n} is contained in both the support of $\bar{t}_{2n+1}$ and the support of t_{2n+2}. In particular the support of t_0 is contained in the support of $\bar{t}_{2n+1}$ for all n. So by Proposition 15(b), for all $n[t_{2n+1}, -]^m(t_0)=t_0$. Hence for all n, $\bar{t}_{2n+1} \in \langle \bar{f}_0, ..., \bar{f}_n \rangle$. Choose k greater than the cardinality of any group in $\mathscr{V}$ with $n+2$ generators. Then for all $r<k$, $t_{2r} \in \langle f_0, ..., f_n, t_{2k} \rangle$. Since for $r \neq s$, $t_{2s} \neq t_{2r}$, we have a contradiction. ∎

Now we can settle the last case.

Theorem 17. *Suppose $\mathscr{V}$ is a variety of groups which either contains a finitely generated solvable group which is not nilpotent or contains a finite group which is not nilpotent; then $\mathscr{V}$ satisfies* (CP+).

Proof. If $\mathscr{V}$ contains an infinite finitely generated solvable group, then $\mathscr{V}$ is of infinite exponent and so satisfies (CP+). Hence we can assume that $\mathscr{V}$ contains a finite group which is not nilpotent. But now Example 11 and Lemmas 7 and 16 show that $\mathscr{V}$ satisfies (CP+). ∎

4. Conclusion

Putting all the information together we have the following theorem.

THEOREM 18. *Let $\mathscr{V}$ be a non-trivial variety of groups. Then at least one of the following possibilities holds in $\mathscr{V}$.*

(a) *Every almost-free group is free.*

(b) *For every in infinite cardinal κ, there is an almost-free group (in $\mathscr{V}$) of cardinality κ^+ which is not free.*

(c) *For every uncountable cardinal κ, if there is an almost-free abelian group of cardinality κ which is not free then there is an almost-free group (in $\mathscr{V}$) of cardinality κ which is not free.*

Furthermore possibility (a) *holds if and only if $\mathscr{V}$ is a variety of nilpotent groups of prime power exponent.*

In particular, for all $0<n<\omega$ there is an $\aleph_n$-free group of cardinality $\aleph_n$ which is not free if and only if $\mathscr{V}$ is not a variety of nilpotent groups of prime power exponent.

Stated in terms of the incompactness spectrum, the last theorem says the incompactness spectrum of any variety is empty or else contains either the incompactness spectrum of abelian group or the class of successor cardinals. The last two possibilities are not mutually exclusive. There are varieties of groups whose incompactness spectrum contains the successor cardinals and the incompactness spectrum of abelian group. Consider the variety $\mathscr{V}$ generated by $S_3 \oplus C(5)$, where $C(5)$ is the cyclic group of order 5. At cardinals κ, for which there is an almost-free non-free abelian group of cardinality κ we can construct a $\mathscr{V}$-free group using the (CP+). For successor cardinals the construction is easier. Let G be the free group of cardinality κ^+ in the variety generated by S_3. Then $G \oplus \bigoplus_\kappa C(5)$ is κ^+-free but not free in $\mathscr{V}$.

There are several unanswered questions which concern which varieties satisfy (CP+). For example, for $0<n<\omega$ we can ask which varieties of groups have $L_{\infty\omega_n}$-free groups of cardinality $\aleph_n$ which are not free. These are exactly those varieties in which the construction principle holds. As well we can ask how many $\aleph_n$-free groups there are of cardinality $\aleph_n$. In a variety of groups, for any $0<n<\omega$, there are $2^{\aleph_n}$ $\aleph_n$-free groups of cardinality $\aleph_n$ if and only if the construction principle holds. Otherwise there are only finitely many such groups for any n. (The finiteness result appears in [MS].)

The only varieties about which we are in doubt are varieties of finite exponent which are not solvable but with the property that any solvable group in the variety is nilpotent and with the property that any uniformly locally nilpotent group is nilpotent. I am indebted to the referee for pointing out that there are such varieties. Ol'shanskii [O] announces the existence of a nonabelian variety of finite exponent whose solvable groups are all abelian. (A variety of infinite exponent in which all finite groups are abelian is constructed. The last sentence of the paper says: "Apparently, not very substantial modifications in the proof would suffice for an analogous example with an identity of the form $x^{n_0 n_1} \equiv 1$, where n_0 and n_1 are sufficiently large odd numbers.") There remains the question: "If $\mathscr{V}$ such a variety and A is the group constructed in Example 8, is $\mathscr{V}A * \mathrm{F}(\omega)$ free?" Of course the existence of a variety, which we do not know how to prove satisfies (CP+), would not in itself give an example of a non-nilpotent variety which does not satisfy the construction principle. We hope that someone who is more familiar with varieties of finite exponent will be able to prove that all varieties which are not nilpotent satisfy (CP+).

A question which may be harder to resolve is: What are the possible incompactness spectra of varieties of groups? The obvious conjecture is that if $\mathscr{V}$ is any variety of groups its incompactness spectrum is either empty, the class of successor cardinals, the incompactness spectrum of abelian groups, or the union of the class of successor cardinals and the incompactness spectrum of abelian groups. Since it is consistent with the axioms of set theory that the incompactness spectrum of abelian groups, the class of uncountable regular cardinals, and the class of successor cardinals coincide, this conjecture is consistent. The point of the conjecture is to prove it as a theorem of ZFC.

The reader may have observed, as one of the referees did, that we have not allowed $\aleph_0$ to appear in an incompactness spectrum. It is natural to ask what happens for $\aleph_0$. For varieties of infinite exponent, $\mathbb{Q}$ is an example of an almost-free non-free group of cardinality $\aleph_0$. On the other hand, for nilpotent varieties of finite exponent any almost-free group of cardinality $\aleph_0$ is free.

One aspect of the construction whose significance is unclear is the fact that every construction we know of a almost-free non-free group in a variety $\mathscr{V}$ which uses (CP+) can be done as follows: first construct an almost-free group (in the variety of all groups); then consider its image modulo a verbal subgroup. Is this observation an accident brought about by our lack of knowledge or does it follow from more general principles? Note that we are not dealing with a (crude) universal algebraic principle, since in the variety of algebras in the language of groups which satisfies no laws every almost-free algebra is free.

ACKNOWLEDGMENTS

We thank the referee for finding a potentially embarassing mistake in the first version of this paper and for suggesting a proof for varieties which are locally nilpotent but not nilpotent. We are also grateful to J. L. Berggren, John Lawrence, and Narain Gupta for their helpful comments.

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