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Sonic Hyperbolic Phase Transitions and Chapman–Jouguet Detonations

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We prove that the Cauchy problem for an $n \times n$ system of strictly hyperbolic conservation laws in one space dimension admits a weak global solution also in presence of sonic phase boundaries. Applications to Chapman–Jouguet detonations, liquid–vapor transitions and elastodynamics are considered. © 2002 Elsevier Science (USA)

Key Words: conservation laws; phase boundaries; Chapman–Jouguet detonations.

1. INTRODUCTION

This paper deals with sonic phase boundaries in conservation laws, i.e. in first-order systems of PDEs in divergence form

$$\partial_t u + \partial_x [f(u)] = 0, \tag{1.1}$$

where $t \in [0, +\infty[$, $x \in \mathbf{R}$ and u is the vector of the conserved quantities. The flow function $f : \Omega \subseteq \mathbf{R}^n \mapsto \mathbf{R}^n$ is smooth. We assume that Ω is the disjoint union of two *phases* Ω_0, Ω_1 and that system (1.1) is strictly hyperbolic in Ω.

The present approach allows to handle various physical situations in a unified way: typical examples are the liquid–vapor phase transitions, the

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stretching or shearing of various materials in elastodynamics and chemical reactions in fluids.

We consider two reference states $\underline{u}^\ell \in \Omega_0$ and $\underline{u}^{\mathrm{r}} \in \Omega_1$ and assume that the Riemann problem

$$\begin{cases} \partial_t u + \partial_x[f(u)] = 0, \\ u(0,x) = \begin{cases} \underline{u}^\ell & \text{if } x<0, \\ \underline{u}^{\mathrm{r}} & \text{if } x>0 \end{cases} \end{cases} \tag{1.2}$$

admits the piecewise constant solution

$$\underline{u}(t,x) = \begin{cases} \underline{u}^\ell & \text{if } x<\underline{\Lambda}t, \\ \underline{u}^{\mathrm{r}} & \text{if } x>\underline{\Lambda}t. \end{cases}$$

The line $x = \underline{\Lambda} \cdot t$ is called the phase boundary. We assume that it is *left-sonic*, i.e. $\underline{\Lambda}$ coincides with one of the characteristic speeds of $\underline{u}^\ell$, say $\underline{\Lambda} = \lambda_k(\underline{u}^\ell)$, and moreover $\lambda_k(\underline{u}^{\mathrm{r}}) < \underline{\Lambda} < \lambda_{k+1}(\underline{u}^{\mathrm{r}})$.

Motivated by physical considerations, see for example [12], the Riemann problem for initial data (u^ℓ, u^{r}) close to $(\underline{u}^\ell, \underline{u}^{\mathrm{r}})$ is solved as follows. From left to right, there are k Lax waves in Ω_0, a phase boundary of speed Λ, and $n-k$ Lax waves in Ω_1. The phase boundary can be either left-subsonic or left-sonic. In the former case, the kth wave in Ω_0 is null and the phase boundary behaves exactly as a k-shock. In the latter case, the kth wave is a (possibly null) rarefaction adjacent to the phase boundary.

This kind of solutions are considered in Chapman–Jouguet theory of combustion [12, 13]: in that framework a left-sonic phase boundary is called a Chapman–Jouguet detonation, and a subsonic phase boundary is a strong detonation. We refer to [6, 8, 17] for other related mathematical results on sonic phase transitions. Solutions with features similar to those of sonic phase boundaries appear also in nongenuinely nonlinear systems, see [15]. Physical models are considered in [9, 18, 20].

Under natural stability assumptions on (1.2), we prove by an ad hoc front tracking algorithm that there exists a positive δ such that for all functions $u_0 : \mathbf{R} \mapsto \Omega$ with $\|u_0\|_\infty + \mathrm{TV}(u_0) < \delta$, the Cauchy Problem

$$\begin{cases} \partial_t u + \partial_x[f(u)] = 0, \\ u(0,x) = \begin{cases} \underline{u}^\ell + u_0(x) & \text{if } x<0, \\ \underline{u}^{\mathrm{r}} + u_0(x) & \text{if } x>0 \end{cases} \end{cases} \tag{1.3}$$

admits a global solution. Moreover, the phase boundary remains for all times subsonic on the right and either subsonic or sonic on the left. The total variation of its propagation speed is $\mathcal{O}(\delta)$. Throughout the proof, the phase boundary is considered as a "generalized" Lax wave, whose regularity is $\mathbf{C}^{1,1}$ and in general not $\mathbf{C}^2$, see also [10].

We limit the construction below to perturbations of a single left-sonic phase boundary. The more general situation where the unperturbed Riemann problem is solved in terms of several subsonic or sonic phase boundaries can be recovered by a suitable mixing of the techniques presented below and in [5].

The problem of the continuous dependence of the solutions upon the initial data seems to present the same difficulties of the nongenuinely nonlinear case, which is still open in the $n \times n$ case, even if phase boundaries are absent.

Section 2 contains the precise statement of our result. Proofs are given in Sections 4 and 5. In Section 3, we provide applications to phase transitions both in gas dynamics and elastodynamics, and to Chapman–Jouguet combustion waves.

2. NOTATIONS AND MAIN RESULT

We consider the system of conservation laws (1.1). The function $f : \Omega \mapsto \mathbf{R}^n$ is of class $\mathbf{C}^3$ and $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 and Ω_1 are two disjoint open subsets of $\mathbf{R}^n$. We refer to Ω_0 and Ω_1 as *phases*, see [5].

On system (1.1) we require the following standard conditions:

(1) the $n \times n$ matrix $Df(u)$ is strictly hyperbolic, i.e. it has n real distinct eigenvalues;

(2) each characteristic field is either genuinely nonlinear or linearly degenerate. The k-characteristic family is genuinely nonlinear in Ω_0.

The latter part of assumption (2) is of an essentially technical nature. In the situation where the kth family is linearly degenerate, an existence result similar to the one below is known to hold, see [4, 7].

We denote by $\lambda_i(u)$ and $r_i(u)$, respectively, the ith eigenvalue and the ith right eigenvector of $Df(u)$, for $i = 1, \dots, n$ and for all $u \in \Omega$. The indexes are chosen so that $\lambda_{i-1}(u) < \lambda_i(u)$ for all u in Ω. For simplicity, below we write $A(u)$ for $Df(u)$.

Let $u : [0, +\infty[\times \mathbf{R} \to \Omega$ be a weak solution to (1.1) such that $u(t, \cdot) \in \mathbf{BV}$ for all t. As in [5], we say that a Lipschitz-continuous curve $x = \Lambda(t)$ is a *phase boundary* for u if for a.e. t the traces

$$u(t, \Lambda(t)-) = \lim_{x \to \Lambda(t)-} u(t, x) \quad \text{and} \quad u(t, \Lambda(t)+) = \lim_{x \to \Lambda(t)+} u(t, x)$$

are in different phases. We say that the phase boundary $x = \Lambda(t)$ is *left-sonic* at time t with respect to the kth characteristic field if

$$\lambda_k(u(t,\Lambda(t)+)) < \lambda_k(u(t,\Lambda(t)-)) = \Lambda(t) < \lambda_{k+1}(u(t,\Lambda(t)+)). \tag{2.1}$$

Clearly, the inequality on the right is missing if $k = n$. We say also that the phase boundary $x = \Lambda(t)$ is *left-subsonic*, respectively *left-supersonic*, if $\lambda_k(u(t,\Lambda(t)-)) > \Lambda(t)$, resp. $\lambda_k(u(t,\Lambda(t)-)) < \Lambda(t)$. Since the present results are local in the state space, we may assume that the inequalities in (2.1) hold as well. Analogous definitions may well be given in the right-sonic case.

When no misunderstanding arises, we drop the "left" and refer simply to sonic (subsonic, supersonic) phase boundaries. If a phase boundary $x = \Lambda(t)$ is subsonic, respectively supersonic, then there are $n+1$ (resp. n) characteristics impinging into the phase boundary and $n-1$ (resp. n) outgoing from it. In other words, subsonic phase boundaries behave like k-shock waves, while supersonic ones do not.

We now fix two states $\underline{u}^\ell \in \Omega_0$ and $\underline{u}^{\mathrm{r}} \in \Omega_1$ and assume that the Riemann problem (1.2) has the solution

$$\underline{u}(t,x) = \begin{cases} \underline{u}^\ell & \text{if } x < \underline{\Lambda} \cdot t, \\ \underline{u}^{\mathrm{r}} & \text{if } x > \underline{\Lambda} \cdot t, \end{cases} \tag{2.2}$$

and that the phase boundary $x = \underline{\Lambda} \cdot t$ is *left-sonic* with respect to the kth characteristic field, i.e.,

$$\lambda_k(\underline{u}^{\mathrm{r}}) < \lambda_k(\underline{u}^\ell) = \underline{\Lambda} < \lambda_{k+1}(\underline{u}^{\mathrm{r}}). \tag{2.3}$$

Small perturbations of solution (2.2) lead to consider both subsonic and supersonic phase boundaries. To avoid the under-determinacy caused by supersonic phase boundaries, we now introduce a class of solutions to the Riemann problem for initial data (u^ℓ, u^{r}) close to $(\underline{u}^\ell, \underline{u}^{\mathrm{r}})$ which have only either sonic or subsonic phase boundaries.

Definition 1. Consider two states $u^\ell \in \Omega_0$, $u^{\mathrm{r}} \in \Omega_1$ close to $\underline{u}^\ell$, $\underline{u}^{\mathrm{r}}$, respectively. An *admissible solution* to the Riemann problem

$$\begin{cases} \partial_t u + \partial_x[f(u)] = 0, \\ u(0,x) = \begin{cases} u^\ell & \text{if } x < 0, \\ u^{\mathrm{r}} & \text{if } x > 0 \end{cases} \end{cases} \tag{2.4}$$

is a function consisting (from the left to the right) of $k-1$ Lax waves, a phase boundary (with possibly a k-rarefaction attached to its left), and $(n-k)$ Lax waves.

More precisely, according to the definition above, the phase boundary is either left-subsonic or left-sonic with respect to the kth characteristic field, see Fig. 1. In the former case, no k-wave is present in the solution. In the latter case, a k-rarefaction and the phase boundary constitute a so-called *compound* (or *mixed*) wave, similarly to what happens in nongenuinely nonlinear systems of conservation laws, see [15].

A well-known example of phase boundaries in the sense of Definition 1 is found in the Chapman–Jouguet detonations model of combustion theory, [12, 13]. The class of admissible solutions introduced above is a natural generalization to $n \times n$ systems of the solutions considered there. We refer to Section 3 for more details and other examples.

In order to obtain the unique solvability of the Riemann problem (2.4) in the sense of Definition 1, in addition to hypotheses (1) and (2) we need a stability assumption on the solution $\underline{u}$:

(3) The solution $\underline{u}$ defined in (2.2) satisfies

$$\det[r_1(\underline{u}^\ell), \ldots, r_{k-1}(\underline{u}^\ell), \underline{u}^{\mathrm{r}} - \underline{u}^\ell, r_{k+1}(\underline{u}^{\mathrm{r}}), \ldots, r_n(\underline{u}^{\mathrm{r}})] \neq 0.$$

The condition above is also the stability condition for a (large) k-shock wave, see [16].

PROPOSITION 2.1. *Let assumptions* (1) *and* (2) *hold. If the Riemann problem* (1.2) *admits solution* (2.2) *satisfying to* (3), *then for all* u^ℓ, u^{r} *in suitable neighborhoods of* $\underline{u}^\ell$ *and* $\underline{u}^{\mathrm{r}}$ *the Riemann problem* (2.4) *admits a unique admissible solution in the sense of Definition* 1.

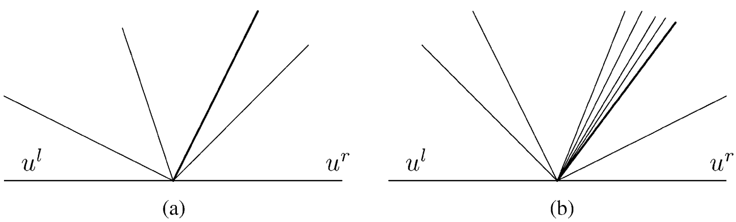

FIG. 1. Solutions to the Riemann problem in the case $n = 4$. (a) with a subsonic phase boundary and (b) with a left-sonic phase boundary joined to a rarefaction attached on its left. The phase boundary is represented by a thick line, the small waves by thin lines.

Consider now the Cauchy problem (1.3). In the definition below we refer to the classical Lax entropy conditions as stated in [14].

DEFINITION 2. By *admissible solution* to the Cauchy problem (1.3) we mean a pair (u, Λ) with $u : [0, +\infty[\times \mathbf{R} \mapsto \Omega_0 \cup \Omega_1$ and $\Lambda : [0, +\infty[\mapsto \mathbf{R}$ and

1. in the region $x < \Lambda(t)$, u is a weak entropic solution to (1.3) and $u(t, x) \in \Omega_0$;
2. in the region $x > \Lambda(t)$, u is a weak entropic solution to (1.3) and $u(t, x) \in \Omega_1$;
3. along $x = \Lambda(t)$ the Rankine–Hugoniot relations

$$\begin{aligned} &f(u(t, \Lambda(t)+)) - f(u(t, \Lambda(t)-)) \\ &\quad = \dot{\Lambda}(t) \cdot (u(t, \Lambda(t)+) - u(t, \Lambda(t)-)) \end{aligned} \tag{2.5}$$

hold for a.e. t.

We are now ready to state the main result of this paper.

THEOREM 2.1. *Let $f : \Omega \mapsto \mathbf{R}^n$ be a smooth function satisfying* (1) *and* (2). *Assume that the Riemann problem* (1.2) *admits the weak solution* (2.2) *consisting of a left-sonic phase boundary and satisfying condition* (3). *Let u_0 be such that*

$$\|u_0\|_{\mathbf{L}^\infty} + \mathrm{TV}(u_0) \leqslant \delta$$

for some positive δ.

Then, if δ is sufficiently small, there exists a global solution

$$u \in \mathbf{BV}_{\mathrm{loc}}(]0, +\infty[\times \mathbf{R})$$

to the Cauchy problem (1.3) *in the sense of Definition* 2. *The solution u has a Lipschitz-continuous phase boundary $x = \Lambda(t)$ which is for a.e. t either left-sonic or left-subsonic. Moreover,*

$$\mathrm{TV}(u(t, \cdot);]-\infty, \Lambda(t)[) + \mathrm{TV}(\dot{\Lambda}) + \mathrm{TV}(u(t, \cdot);]\Lambda(t), +\infty[) = \mathcal{O}(1) \cdot \delta. \tag{2.6}$$

Above, $\mathrm{TV}(u; I)$ denotes the total variation of the function u over the (space) interval I, while $\mathrm{TV}(\dot{\Lambda})$ is the total variation of $\dot{\Lambda}$ over the whole (time) interval $[0, +\infty[$.

This result is proved by means of a wave-front tracking scheme as in [2, 3], suitably adapted to the present situation, see condition (K) in Section 5.

3. APPLICATIONS

A first example of phase transition is given by the p-system

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p(\tau) = 0. \end{cases} \tag{3.1}$$

Here, τ is the specific volume, v the speed and $p\colon \Omega_0 \cup \Omega_1 \mapsto \mathbf{R}$ is the pressure. Ω_0 represents the liquid phase while Ω_1 stand for the vapor one. Under standard assumption on p, namely

$$p'(\tau) < 0 \quad \text{and} \quad p''(\tau) > 0 \qquad \forall \tau \in \Omega_0 \cup \Omega_1, \tag{3.2}$$

in [6] it is shown that if also $p(\sup \Omega_0) \geqslant p(\inf(\Omega_1)$ holds, in general a global and continuous Riemann solver leads to sonic phase boundaries. Theorem 2.1 applies since the above assumptions are well known to imply (1) and (2). Also (3) holds, as it follows from simple computations.

More generally, the full system of Euler equations of gas dynamics in Lagrangian coordinates is

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t (e + \frac{1}{2} v^2) + \partial_x (p \cdot v) = 0, \end{cases} \tag{3.3}$$

where e is the internal energy density. The pressure law $p = p(\tau, e)$ satisfies the conditions

$$\partial_\tau p(\tau, e) < 0 \quad \text{and} \quad \partial_e p(\tau, e) > 0. \tag{3.4}$$

It is well known that due to the former inequality above, (3.3) satisfies (1) with eigenvalues 0, $\pm \lambda$, where $\lambda = \sqrt{p p_e - p_\tau}$. Moreover, if

$$p \partial_e \lambda - \partial_\tau \lambda \neq 0 \tag{3.5}$$

then also (2) holds. Assumptions (3.4)–(3.5) hold in the polytropic case, where $p = (\gamma - 1)e/\tau$, $\gamma > 1$.

Let u denote the triple of the conserved quantities $(\tau, v, e + v^2/2)$.

LEMMA 3.1. *Let* $\underline{u}^\ell$, $\underline{u}^{\mathrm{r}}$ *be such that* (2.2) *is a weak solution of* (3.3) *with* $\underline{\Lambda}$ *left-sonic with respect to the third characteristic family. If* (3.4) *holds, then the stability condition* (3) *is satisfied, provided*

$$\underline{u}^{\mathrm{r}} \neq \underline{u}^\ell. \tag{3.6}$$

Proof. Let $\underline{u}^\ell = (\tau, v, e + v^2/2)$ and $\underline{u}^{\mathrm{r}} = (\tau^{\mathrm{r}}, v^{\mathrm{r}}, e^{\mathrm{r}} + (v^{\mathrm{r}})^2/2)$. First, note that the flow in (3.3) has the Jacobian

$$\begin{bmatrix} 0 & -1 & 0 \\ \partial_\tau p & -v\partial_e p & \partial_e p \\ v\partial_\tau p & p - v^2\partial_e p & v\partial_e p \end{bmatrix}.$$

Then (3) is equivalent to

$$(\tau^{\mathrm{r}} - \tau)\partial_\tau p - (v^{\mathrm{r}} - v)(v\partial_e p - \lambda) + \left(e^{\mathrm{r}} + \frac{(v^{\mathrm{r}})^2}{2} - e - \frac{v^2}{2}\right)\partial_e p \neq 0.$$

Using the Rankine–Hugoniot conditions (2.5) and (3.6) the above relation amounts to

$$(p^{\mathrm{r}} - p)(-\partial_\tau p + p^{\mathrm{r}}\partial_e p + \lambda^2) \neq 0.$$

By (3.6) the first factor does not vanish, and by (3.4) the second one is strictly positive, completing the proof. ∎

Remark that the above lemma holds thanks only to the form (3.3) of the equations, the thermodynamic assumptions (3.4) and the left-sonicity of the phase boundary.

Theorem 2.1 applies also to liquid–vapor sonic phase transitions in (3.3). Assume that the fluid satisfies, for instance, van der Waals equation of state. Then, it is well known that (3.4) and (3.5) are locally satisfied a.e. in the hyperbolic region. The present result ensures that the phase boundary is stable with respect to small **BV** perturbation for all times in the sonic case.

3.1. Chapman–Jouguet Detonations

We briefly recall here the following standard combustion model, consisting of two Euler systems coupled by a free boundary where the reaction takes place.

At time $t = 0$, burnt gas covers the half-line $x < 0$, while unburnt gas fills $x > 0$. The two ideal gases satisfy Euler equations, with eventually different equations of state. The two gases differ in the expression for the internal energy, since the one of the unburnt gas contains also a term accounting for the energy to be released through combustion.

Using Lagrangian coordinates:

$$
\begin{array}{ll}
\text{Burnt Gas} & \text{Unburnt Gas} \\
\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t (e + \frac{1}{2} v^2) + \partial_x (pv) = 0, \end{cases} & \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t (e + \frac{1}{2} v^2) + \partial_x (pv) = 0, \end{cases} \\
p\tau = RT/\mu_B, & p\tau = RT/\mu_U, \\
e = c_v^B T, & e = c_v^U T + Q.
\end{array}
\tag{3.7}
$$

As usual, c_v^B (resp. c_v^U) is the specific heat of the burnt (resp. unburnt) gas, μ_B and μ_U are the molar weights, and R is the universal gas constant. The constant Q is the energy density that is released from the unburnt gas through combustion. Let $\varepsilon = e + v^2/2$ be the total energy density.

Introducing the parameters $\alpha_B = c_v^B/(n_B R)$ and $\alpha_U = c_v^U/(n_U R)$ with α_B, $\alpha_U \in [1, +\infty[$, the relations $e_B = \alpha_B p\tau$ and $e_U = \alpha_U p\tau + Q$ allow to close system (3.7) with the aid of the only two parameters α_B and α_U.

The (free) boundary between the two gases is the reaction front. Its propagation speed is chosen according to the Rankine–Hugoniot conditions.

We refer to [21] for the study of the global Riemann problem for (3.7). Locally, i.e. in the spirit of the present paper, on the left we fix a burnt state $\underline{u}_B = (\underline{\tau}_B, \underline{v}_B, \underline{\varepsilon}_B)$ and an unburnt state $\underline{u}_U = (\underline{\tau}_U, \underline{v}_U, \underline{\varepsilon}_U)$ so that $\underline{u}_B$ is the Chapman–Jouguet detonation point related to $\underline{u}_U$, see for instance [12,13]. Clearly, (3.6) holds. Let Ω_B be a neighborhood of $\underline{u}_B$ and Ω_U a neighborhood of $\underline{u}_U$, with $\Omega_B \cap \Omega_U = \emptyset$. Let $f : \Omega_0 \cup \Omega_1 \mapsto \mathbf{R}^3$ defined by

$$
(\tau, v, \varepsilon) \mapsto \begin{cases} \left(v, \dfrac{\varepsilon - \frac{1}{2} v^2}{\alpha_B \tau}, \dfrac{v\left(\varepsilon - \frac{1}{2} v^2\right)}{\alpha_B \tau} \right) & \text{if} \quad (\tau, v, \varepsilon) \in \Omega_B, \\ \left(v, \dfrac{\varepsilon - \frac{1}{2} v^2 - Q}{\alpha_U \tau}, \dfrac{v\left(\varepsilon - \frac{1}{2} v^2 - Q\right)}{\alpha_U \tau} \right) & \text{if} \quad (\tau, v, \varepsilon) \in \Omega_U. \end{cases}
$$

With the f above, problem (3.7) is equivalent to (1.1) and is well known to satisfy (1) and (2). Moreover, it enjoys also (3), as follows from Lemma 3.1. Theorem 2.1 then ensures the global existence of solutions to (3.7) for small **BV** perturbations of initial data near to the Chapman–Jouguet detonation point. Note that bound (2.6) on the total variation of the speed of the phase boundary imply that the reaction continues for all times.

3.2. Phase Transitions in Elastodynamics

In the more general case of adiabatic thermoelastic materials, system (3.3) reads as

$$\begin{cases} \partial_t w - \partial_x v = 0, \\ \partial_t v - \partial_x \sigma = 0, \\ \partial_t (e + \frac{1}{2} v^2) - \partial_x (\sigma v) = 0. \end{cases} \tag{3.8}$$

Above, w is the strain, v the velocity, $\sigma = \sigma(w, S)$ the stress and $e = e(w, S)$ the internal energy, where S is the entropy. We denote by T the temperature. See [11] for further reference.

In particular materials, see [18, 20], suitable stresses produce changes in the crystalline structure propagating at a speed comparable to that of sound. These *phase transitions* fall in the same framework provided by Theorem 2.1, under suitable assumptions on the stress function σ, namely (3.4) and (3.5) are replaced by

$$\partial_w \sigma(w, S) > 0, \ \ \partial_S \sigma(w, S) < 0 \quad \text{and} \quad \partial^2_{ww} \sigma(w, S) \neq 0.$$

Moreover, (3) now reads as

$$\sigma^{\mathrm{r}} - \sigma \neq -2T \frac{\partial_w \sigma}{\partial_S \sigma},$$

where σ^{r} (resp. σ) is the stress on the right (left) of the unperturbed left-sonic phase transition and the terms on the r.h.s. are computed on the left state.

4. THE RIEMANN PROBLEM

This section is devoted to the study of the Riemann problem (2.4). For $u \in \Omega$ and $i = 1, \ldots, n$ we introduce the following curves exiting u: the ith shock curve $\sigma \mapsto S_i(u, \sigma)$, the ith rarefaction curve $\sigma \mapsto R_i(u, \sigma)$ and their gluing, the ith Lax curve

$$\Phi_i(u, \sigma) = \begin{cases} R_i(u, \sigma) & \text{if } \sigma \geqslant 0, \\ S_i(u, \sigma) & \text{if } \sigma \leqslant 0. \end{cases}$$

If the ith family is linearly degenerate, then the curves above are parametrized by means of the arc length. If the ith family is genuinely nonlinear, following [3] we choose the parameter σ so that

$$\lambda_i(R_i(u, \sigma)) = \lambda_i(u) + \sigma = \lambda_i(S_i(u, \sigma)). \tag{4.1}$$

It is well known [3, 19] that all the curves above are $\mathbf{C}^2$. Below, Id denotes the $n \times n$ identity matrix.

LEMMA 4.1. *The Rankine–Hugoniot condition* $\Lambda \cdot (u^{\mathrm{r}} - u^{\ell}) = f(u^{\mathrm{r}}) - f(u^{\ell})$ *implicitly defines a unique smooth function* $u^{\mathrm{r}} = H(u^{\ell}, \Lambda)$ *in a neighborhood of the unperturbed solution* $\underline{u}^{\mathrm{r}} = H(\underline{u}^{\ell}, \underline{\Lambda})$. *Moreover,*

$$D_{\Lambda}H(u, \Lambda) = [A(H(u, \Lambda)) - \Lambda \cdot \mathrm{Id}]^{-1} \cdot (H(u, \Lambda) - u),$$

$$D_{u}H(u, \Lambda) = [A(H(u, \Lambda)) - \Lambda \cdot \mathrm{Id}]^{-1} \cdot (A(u) - \Lambda \cdot \mathrm{Id}).$$

Proof. Write $RH(u^{\ell}, u^{\mathrm{r}}, \Lambda) = f(u^{\mathrm{r}}) - f(u^{\ell}) - \Lambda \cdot (u^{\mathrm{r}} - u^{\ell})$. Note that

$$D_{u^{\mathrm{r}}}RH(\underline{u}^{\ell}, \underline{u}^{\mathrm{r}}, \underline{\Lambda}) = A(\underline{u}^{\mathrm{r}}) - \underline{\Lambda} \cdot \mathrm{Id}$$

which is a nonsingular matrix, due to (2.3). The Implicit Function Theorem allows to complete the proof. ∎

A property of the Hugoniot function H of key importance in the sequel is that for all u

$$\begin{aligned} &D_{u}H(u, \lambda_k(u))r_k(u) \\ &\quad = [A(H(u, \lambda_k(u))) - \lambda_k(u) \cdot \mathrm{Id}]^{-1} \cdot (A(u) - \lambda_k(u) \cdot \mathrm{Id}) \cdot r_k(u) = 0 \qquad (4.2) \end{aligned}$$

since r_k is a right k-eigenvector.

It turns out very useful to consider the phase boundary as part of a *generalized* k-wave. We thus introduce the *generalized kth Lax curve* as

$$\tilde{\Phi}_k(u, \sigma) = \begin{cases} H(R_k(u, \sigma), \lambda_k(R_k(u, \sigma))) & \text{if } \sigma \geqslant 0, \\ H(u, \lambda_k(S_k(u, \sigma))) & \text{if } \sigma \leqslant 0. \end{cases} \qquad (4.3)$$

The above parameterization of the generalized Lax curve is of key importance in the sequel. In fact, it amounts to assign a size σ to the phase transition exiting from u. This size is a (signed) measure of the distance from the left-sonic phase boundary exiting from u. More precisely, if $\sigma \leqslant 0$ then the states u and $\tilde{\Phi}_k(u, \sigma)$ are connected by a subsonic phase boundary; if instead $\sigma \geqslant 0$ then the state u is connected first to the state $R_k(u, \sigma)$ (by a k-rarefaction) and subsequently to $\tilde{\Phi}_k(u, \sigma)$ by a sonic phase boundary (attached to the right part of the rarefaction).

Note that due to choice (4.1), definition (4.3) can be rewritten as

$$\tilde{\Phi}_k(u,\sigma) = \begin{cases} H(R_k(u,\sigma), \lambda_k(u)+\sigma) & \text{if } \sigma \geqslant 0, \\ H(u, \lambda_k(u)+\sigma) & \text{if } \sigma \leqslant 0. \end{cases} \tag{4.4}$$

LEMMA 4.2. *The function* $u \mapsto \tilde{\Phi}_k(u,\sigma)$ *is of class* $\mathbf{C}^2$ *for every fixed* σ. *The function* $(u,\sigma) \mapsto \tilde{\Phi}_k(u,\sigma)$ *is* $\mathbf{C}^{1,1}$. *Moreover,*

$$D_\sigma \tilde{\Phi}_k(u,0) = D_\Lambda H(u, \lambda_k(u)), \tag{4.5}$$

$$D_u \tilde{\Phi}_k(u,0) = D_u H(u, \lambda_k(u)) + D_\Lambda H(u, \lambda_k(u)) \cdot \nabla \lambda_k(u). \tag{4.6}$$

Proof. The regularity of $u \mapsto \tilde{\Phi}_k(u,\sigma)$ for fixed σ is well known [3, 19]. Consider now (4.5). If $\sigma > 0$, from (4.2) it follows that

$$\begin{aligned} D_\sigma \tilde{\Phi}_k(u,\sigma) &= D_u H(R_k(u,\sigma), \lambda_k(u)+\sigma)) \cdot r_k(R_k(u,\sigma)) \\ &\quad + D_\Lambda H(R_k(u,\sigma), \lambda_k(u)+\sigma) \\ &= D_\Lambda H(R_k(u,\sigma), \lambda_k(u)+\sigma), \end{aligned}$$

hence $D_\sigma \tilde{\Phi}_k(u,0^+) = D_\Lambda H(u, \lambda_k(u))$. On the other hand, if $\sigma < 0$ then

$$D_\sigma \tilde{\Phi}_k(u,\sigma) = D_\Lambda H(u, \lambda_k(u)+\sigma)),$$

hence $D_\sigma \tilde{\Phi}_k(u,0^-) = D_\Lambda H(u, \lambda_k(u))$. Comparing the expressions above, (4.5) is proved.

To prove (4.6), compute

$$\begin{aligned} \sigma > 0: \quad D_u \tilde{\Phi}(u,\sigma) &= D_u H(R_k(u,\sigma), \lambda_k(u)+\sigma) \cdot D_u R_k(u,\sigma) \\ &\quad + D_\Lambda H(R_k(u,\sigma), \lambda_k(u)+\sigma) \cdot \nabla \lambda_k(u,\sigma), \\ D_u \tilde{\Phi}(u,0^+) &= D_u H(u, \lambda_k(u)) + D_\Lambda H(u, \lambda_k(u)) \cdot \nabla \lambda_k(u), \\ \sigma < 0: \quad D_u \tilde{\Phi}_k(u,\sigma) &= D_u H(u, \lambda_k(u)+\sigma) \\ &\quad + D_\Lambda H(u, \lambda_k(u)+\sigma) \cdot \nabla \lambda_k(u), \\ D_u \tilde{\Phi}_k(u,0^-) &= D_u H(u, \lambda_k(u)) + D_\Lambda H(u, \lambda_k(u)) \cdot \nabla \lambda_k(u). \end{aligned}$$

This proves (4.6).

The functions $D_\sigma \tilde{\Phi}_k$ and $D_u \tilde{\Phi}_k$ are thus continuous in both the variables (u,σ), hence $\tilde{\Phi}_k \in \mathbf{C}^1$. Furthermore, $\tilde{\Phi}_k$ is $\mathbf{C}^2$ for $\sigma \geqslant 0$ and, separately, for $\sigma \leqslant 0$. By an easy argument we find that $\tilde{\Phi}_k(u,\sigma) \in \mathbf{C}^{1,1}$, completing the proof. ∎

The fact that $\sigma \mapsto \tilde{\Phi}_k(u,\sigma)$ is $\mathbf{C}^{1,1}$ is optimal, as shown by the next lemma in the case of the p-system.

LEMMA 4.3. *In the case of the* p*-system* (3.1) *with a pressure law satisfying* (3.2), *the function* $\sigma \mapsto \tilde{\Phi}_2(u,\sigma)$ (*here*, $k = 2$) *is not twice differentiable.*

Proof. Fix a state u. From the proof of Lemma 4.2 we see that

$$D_\sigma \tilde{\Phi}_k(u,\sigma) = \begin{cases} D_\Lambda H(R_k(u,\sigma), \lambda_k(u) + \sigma) & \text{if } \sigma \geqslant 0, \\ D_\Lambda H(u, \lambda_k(u) + \sigma) & \text{if } \sigma \leqslant 0. \end{cases}$$

By computation of $D^2_\sigma \tilde{\Phi}_k(u,\sigma)$ we see that the map $\sigma \mapsto \tilde{\Phi}_2(u,\sigma)$ is twice differentiable if and only if

$$(D_u(D_\Lambda H))(u, \lambda_k(u)) \cdot r_k(u) = 0. \tag{4.7}$$

In the case of the p-system (3.1), $u = (v, \tau)$, $c(\tau) = \sqrt{-p'(\tau)}$ and for a suitable function $\tau_H = \tau_H(\tau, \Lambda)$,

$$H(u,\Lambda) = \begin{bmatrix} v - \Lambda \cdot (\tau_H(\tau,\Lambda) - \tau) \\ \tau_H(\tau,\Lambda) \end{bmatrix}.$$

In contrast with (4.7), we prove that

$$(D_u(D_\Lambda H))(u,\Lambda) \cdot r_k(u) \neq 0$$

for every u and Λ close to $\underline{u}^\ell$, $\underline{\Lambda}$, respectively. In fact, from the formula

$$D_\Lambda H(u,\Lambda) = \frac{\tau_H(\tau,\Lambda) - \tau}{\Lambda^2 - c^2(\tau_H(\tau,\Lambda))} \begin{pmatrix} \Lambda^2 + c^2(\tau_H(\tau,\Lambda)) \\ -2\Lambda \end{pmatrix} = \begin{pmatrix} I(\tau,\Lambda) \\ J(\tau,\Lambda) \end{pmatrix}$$

we find that

$$(D_u(D_\Lambda H))(u,\Lambda) = \begin{pmatrix} 0 & D_\tau I(\tau,\Lambda) \\ 0 & D_\tau J(\tau,\Lambda) \end{pmatrix}.$$

Let us write τ_H for $\tau_H(\tau,\Lambda)$. A simple computation shows that $D_\tau J(\tau,\Lambda) = 0$ if and only if

$$D_\tau \tau_H = \frac{\Lambda^2 - c^2(\tau_H)}{\Lambda^2 - c^2(\tau_H) + 2 \cdot c(\tau_H) \cdot c'(\tau_H) \cdot (\tau_H - \tau)}$$

provided

$$\Lambda^2 - c^2(\tau_H) + 2 \cdot c(\tau_H) \cdot c'(\tau_H) \cdot (\tau_H - \tau) \neq 0.$$

If this happens then

$$D_\tau I(\tau, \Lambda) = 2 \cdot \frac{\tau_H - \tau}{\Lambda^2 - c^2(\tau_H)} \cdot c(\tau_H) \cdot c'(\tau_H) \cdot D_\tau \tau_H$$

which does not vanish. ∎

In the following, we use the notation

$$\tilde{\Phi}(u^\ell; \sigma_1, \ldots, \sigma_n) = \Phi_n(\ldots \tilde{\Phi}_k(\ldots \Phi_1(u^\ell, \sigma_1), \ldots, \sigma_k), \ldots, \sigma_n),$$

$$\mathscr{F}(u^\ell, u^{\mathrm{r}}; \sigma_1, \ldots, \sigma_n) = \tilde{\Phi}(u^\ell; \sigma_1, \ldots, \sigma_n) - u^{\mathrm{r}}. \tag{4.8}$$

Note that solving the Riemann problem (2.4) according to Definition 1 amounts to solve

$$u^{\mathrm{r}} = \tilde{\Phi}(u^\ell; \sigma_1, \ldots, \sigma_n)$$

for the wave sizes $\sigma_1, \ldots, \sigma_n$ in terms of the states u^ℓ, u^{r}.

Proof of Proposition 2.1. According to the previous notations, solving (2.4) as specified in Definition 1 is equivalent to solving

$$\mathscr{F}(u^\ell, u^{\mathrm{r}}; \sigma_1, \ldots, \sigma_n) = 0$$

in terms of $\sigma_1, \ldots, \sigma_n$ for given u^ℓ and u^{r}. Applying the Implicit Function Theorem in $\mathbf{C}^{1,1}$ amounts to require that the determinant of the matrix

$$\mathscr{A} = (\mathscr{A}_1, \ldots, \mathscr{A}_n) = D_{\sigma_1, \ldots, \sigma_n} \mathscr{F}(\underline{u}^\ell, \underline{u}^{\mathrm{r}}; 0, \ldots, 0) \tag{4.9}$$

does not vanish, where $\mathscr{A}$ is the $n \times n$ matrix having columns $\mathscr{A}_h$ defined by

$$\mathscr{A}_i = [\mathscr{A}(\underline{u}^{\mathrm{r}}) - \lambda_k(\underline{u}^\ell) \cdot \mathrm{Id}]^{-1}[(\lambda_i(\underline{u}^\ell) - \lambda_k(\underline{u}^\ell))\mathrm{Id} + [\underline{u}]\nabla\lambda_k(\underline{u}^\ell)]r_i, (\underline{u}^\ell)$$

$$1 \leqslant i \leqslant k-1,$$

$$\mathscr{A}_k = [A(\underline{u}^{\mathrm{r}}) - \lambda_k(\underline{u}^\ell) \cdot \mathrm{Id}]^{-1}[\underline{u}],$$

$$\mathscr{A}_j = r_j(\underline{u}^{\mathrm{r}}), \qquad k+1 \leqslant j \leqslant n.$$

By multiplying $\mathscr{A}$ by $A(\underline{u}^{\mathrm{r}}) - \lambda_k(\underline{u}^{\ell}) \cdot \mathrm{Id}$ we see that $\det \mathscr{A} \neq 0$ iff $\det \mathscr{B} \neq 0$, where

$$\begin{aligned}
\mathscr{B}_i &= [(\lambda_i(\underline{u}^{\ell}) - \lambda_k(\underline{u}^{\ell}))\mathrm{Id} + [\underline{u}]\nabla\lambda_k(\underline{u}^{\ell})]r_i(\underline{u}^{\ell}), \quad 1 \leqslant i \leqslant k-1, \\
\mathscr{B}_k &= [\underline{u}], \\
\mathscr{B}_j &= [\lambda_j(\underline{u}^{\mathrm{r}}) - \lambda_k(\underline{u}^{\ell})]r_j(\underline{u}^{\mathrm{r}}), \quad k+1 \leqslant j \leqslant n.
\end{aligned} \tag{4.10}$$

Since $\lambda_i(\underline{u}^{\ell}) - \lambda_k(\underline{u}^{\ell}) \neq 0$ and $\lambda_j(\underline{u}^{\mathrm{r}}) - \lambda_k(\underline{u}^{\ell}) \neq 0$, from the properties of the determinant we deduce that $\det \mathscr{B} \neq 0$ iff $\det \mathscr{C} \neq 0$, where

$$\begin{aligned}
\mathscr{C}_i &= r_i(\underline{u}^{\ell}) + \frac{\nabla\lambda_k(\underline{u}^{\ell})r_i(\underline{u}^{\ell})}{\lambda_i(\underline{u}^{\ell}) - \lambda_k(\underline{u}^{\ell})}[\underline{u}], \qquad 1 \leqslant i \leqslant k-1, \\
\mathscr{C}_k &= [\underline{u}], \\
\mathscr{C}_j &= r_j(\underline{u}^{\mathrm{r}}), \qquad k+1 \leqslant j \leqslant n.
\end{aligned}$$

In the first $k-1$ columns a scalar multiple of the kth column $[\underline{u}]$ appears as a summand. Therefore, the determinant does not change if we skip those columns, concluding the proof. ∎

For future reference let us denote $\beta_j = \lambda_j(u^{\mathrm{r}}) - \lambda_k(u^{\ell})$, $\Gamma = [u]\nabla\lambda_k(u^{\ell})$, $\gamma_i = \lambda_i(u^{\ell}) - \lambda_k(u^{\ell}) + \Gamma$, for $u^{\mathrm{r}} = \tilde{\Phi}_k(u^{\ell}, 0)$ and $i, j = 1, \ldots, n$. Remark that β_j are numbers while Γ, γ_i are $n \times n$ matrices.

Under these notations we remark that we have, for $u^{\mathrm{r}} = \tilde{\Phi}_k(u^{\ell}, 0)$,

$$D_{\sigma_1,\ldots,\sigma_n}\mathscr{F}(u^{\ell}, u^{\mathrm{r}}; 0, \ldots, 0) = (A(u^{\mathrm{r}}) - \lambda_k(u^{\ell}) \cdot \mathrm{Id})^{-1} B(u^{\ell}, u^{\mathrm{r}}),$$

where the matrix B is defined as

$$B(u^{\ell}, u^{\mathrm{r}}) = [\gamma_1 r_1(u^{\ell}), \ldots, \gamma_{k-1} r_{k-1}(u^{\ell}), [u], \beta_{k+1} r_{k+1}(u^{\mathrm{r}}), \ldots, \beta_n r_n(u^{\mathrm{r}})]. \tag{4.11}$$

5. THE CAUCHY PROBLEM

The existence Theorem 2.1 is achieved through the construction of piecewise constant approximate solutions having uniformly bounded total variation. We use the standard wave front tracking algorithm for $n \times n$ systems as defined in [3, 11]. Hence, we underline here only those modifications necessary in the present construction and refer to the cited books for further reference.

The whole technique relies on the *Approximate Riemann Solver* and on the *Simplified Riemann Solver*. While keeping the later essentially unchanged, we modify the former so that

(K) Each rarefaction wavelet of the kth family is assigned the k-characteristic speed of the state to its left.

In the spirit of (4.3), the phase boundary is considered as a k-wave, whenever a Riemann problem involving data in different phases is tackled. Note that, as a consequence of (K), approximate sonic phase boundaries are assigned null size and propagate with the exact speed.

Furthermore, we prescribe that all the Riemann problems along the phase boundary be solved by the Accurate Solver, unless the Riemann problem arises from the interaction of a nonphysical wave. In this case, the simplified solver is used, that is the phase boundary is prolonged with the same speed while the nonphysical wave changes slightly its size.

The above modifications do not alter the key properties of the algorithm, provided suitable interaction estimates are proved, which is the scope of the next paragraph. Later, in Subsection 5.2 we shall show that any limit of the sequence of approximate solutions satisfy (2.5).

5.1. *Interaction Estimates*

We consider now the interaction estimates. We limit the present study only to those simple interactions to which the phase boundary takes part, the other cases being covered as in [3].

As it is standard in this context, the Landau symbol $\mathcal{O}(1)$ denotes a function whose modulus is uniformly bounded as u and σ range over a compact set.

LEMMA 5.1. *Assume that a* (*possibly nonphysical*) *j-wave σ_j^-, with $j \neq k$, hits the phase boundary $\tilde{\sigma}_k^-$, see* Fig. 2a. *Then, the sizes $\tilde{\sigma}_k^+$ and σ_i^+, $i = 1, \ldots, k-1, k+1, \ldots, n$ of the outcoming waves satisfy*

$$|\tilde{\sigma}_k^+ - \tilde{\sigma}_k^-| + |\sigma_j^+| + \sum_{\substack{i=1,\ldots,n \\ i \neq j, i \neq k}} |\sigma_i^+| = \mathcal{O}(1) \cdot |\sigma_j^-|, \tag{5.1}$$

$$|\tilde{\Lambda}_k^+ - \tilde{\Lambda}_k^-| = \mathcal{O}(1) \cdot |\sigma_j^-|, \tag{5.2}$$

$$\|u_\ell^+ - u_\ell^-\| + \|u_r^+ - u_r^-\| = \mathcal{O}(1) \cdot |\sigma_j^-|, \tag{5.3}$$

where $\tilde{\Lambda}_k^+$ (*resp. $\tilde{\Lambda}_k^-$*) *is the speed of the outgoing* (*resp. incoming*) *phase boundary and $u_\ell^\pm$* (*resp. $u_r^\pm$*) *is the state to its left* (*resp. right*).

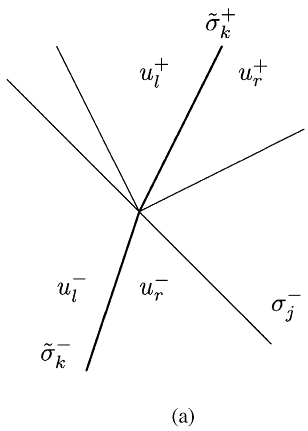

FIG. 2. Interactions of the phase boundary $\tilde{\sigma}_k^-$ with a wave σ_j^- from the right, (a), and with a wave σ_k^- from the left, (b). The small outgoing waves are denoted in Lemmas 5.1 and 5.2 by $\sigma_i^+,\ i\neq k$.

Proof. Assume that $j > k$, the other case being entirely analogous. Note that the value $j = n+1$ is not excluded. Let $\mathscr{F}$ be as in (4.8). By the Implicit Function Theorem in $\mathbf{C}^{1,1}$ the equality

$$\mathscr{F}(u^\ell, \tilde{\Phi}_k(\Phi_j(u^\ell, \sigma_j^-), \tilde{\sigma}_k^-); \sigma_1^+, \ldots, \tilde{\sigma}_k^+, \ldots, \sigma_n^+) = 0, \tag{5.4}$$

implicitly defines a $\mathbf{C}^{1,1}$ function $(\sigma_1^+, \ldots, \tilde{\sigma}_k^+, \ldots, \sigma_n^+) = \Sigma^j(\sigma_j^-, \tilde{\sigma}_k^-; u^\ell)$ with the property

$$\Sigma_i^j(0, \tilde{\sigma}_k^-; u^\ell) = \begin{cases} 0 & \text{if } i \neq k, \\ \tilde{\sigma}_k^- & \text{if } i = k. \end{cases}$$

The Lipschitzeanity of Σ^j then implies (5.1).

To prove (5.2), simply use the Implicit Function Theorem and the linear independency of the right eigenvectors. ∎

LEMMA 5.2. *Assume that a k-wave σ_k^- hits the phase boundary $\tilde{\sigma}_k^-$, and denote by $\sigma_i^+,\ i = 1, \ldots, n,\ i \neq k$ and $\tilde{\sigma}_k^+$ the sizes of the outcoming waves, see Fig.* 2b.

If σ_k^- interacts from the right then

$$|\tilde{\sigma}_k^+ - \tilde{\sigma}_k^-| + \sum_{\substack{i=1,\ldots,n \\ i\neq k}} |\sigma_i^+| = \mathcal{O}(1) \cdot |\sigma_k^-|, \tag{5.5}$$

while if σ_k^- interacts from the left we have

$$\tilde{\sigma}_k^+ = \sigma_k^- + \tilde{\sigma}_k^- + \mathcal{O}(1) \cdot (|\sigma_k^-| + |\tilde{\sigma}_k^-|) \cdot |\sigma_k^-|, \tag{5.6}$$

$$\sigma_i^+ = \mathcal{O}(1) \cdot (|\sigma_k^-| + |\tilde{\sigma}_k^-|) \cdot |\sigma_k^-|, \qquad i = 1, \dots, n, \quad i \neq k. \tag{5.7}$$

In both cases

$$|\tilde{\Lambda}_k^+ - \tilde{\Lambda}_k^-| = \mathcal{O}(1) \cdot |\sigma_k^-|,$$

$$\|u_\ell^+ - u_\ell^-\| + \|u_{\mathrm{r}}^+ - u_{\mathrm{r}}^-\| = \mathcal{O}(1) \cdot |\sigma_k^-|.$$

Proof. We prove first (5.5) and assume that σ_k^- comes from the right. Replace the l.h.s. in (5.4) with

$$\begin{aligned} &\mathcal{G}(u^\ell; \sigma_k^-, \tilde{\sigma}_k^-; \sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+) \\ &\quad = \mathcal{F}(u^\ell, \Phi_k(\tilde{\Phi}_k(u^\ell, \tilde{\sigma}_k^-), \sigma_k^-); \sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+). \end{aligned} \tag{5.8}$$

The function implicitly defined by $\mathcal{G}(u^\ell; \sigma_k^-, \tilde{\sigma}_k^-; \sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+) = 0$ is

$$(\sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+) = \Sigma^k(\sigma_k^-, \tilde{\sigma}_k^-; u^\ell) \tag{5.9}$$

and enjoys the property

$$\Sigma_i^k(0, \tilde{\sigma}_k^-; u^\ell) = \begin{cases} 0 & \text{if } i \neq k, \\ \tilde{\sigma}_k^- & \text{if } i = k. \end{cases} \tag{5.10}$$

Then (5.5) follows as in the previous proof by Lipschitzeanity of Σ^k.

We prove now (5.6) and (5.7).

Define

$$\begin{aligned} &\mathcal{G}(u^\ell; \sigma_k^-, \tilde{\sigma}_k^-; \sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+) \\ &\quad = \mathcal{F}(u^\ell, \tilde{\Phi}_k(\Phi_k(u^\ell, \sigma_k^-), \tilde{\sigma}_k^-); \sigma_1^+, \dots, \tilde{\sigma}_k^+, \dots, \sigma_n^+). \end{aligned}$$

Then

$$D_{\sigma_k^-}\mathcal{G}(u^\ell; 0, 0; 0, \dots, 0) = -[A(u^{\mathrm{r}}) - \lambda_k(u^\ell) \cdot \mathrm{Id}]^{-1} \cdot [u]$$

for $u^{\mathrm{r}} = \tilde{\Phi}_k(u^\ell, 0)$. Hence, by the Implicit Function Theorem and (4.11)

$$D_{\sigma_k^-}\Sigma^k(0, 0; u^\ell) = B^{-1}(u^\ell, u^{\mathrm{r}})[u] = e_k, \tag{5.11}$$

where e_k is the kth vector of the canonical basis of $\mathbf{R}^n$. Therefore,

$$D_{\sigma_k^-}\Sigma_i^k(0,0;u^\ell) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k \end{cases}$$

and moreover, from (5.10),

$$D_{\tilde{\sigma}_k^-}\Sigma_i^k(0,\tilde{\sigma}_k^-;u^\ell) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

This proves (5.6) and (5.7). The latter estimates on the change in the propagation speed and on the variation of the side states are entirely analogous to those in Lemma 5.1. ∎

We point out that estimates (5.6) and (5.7) do not hold if the wave σ_k^- interacts from the right. In fact, if we still use the notation Σ^k for the set of outgoing waves produced by the interaction, then in this case we find

$$D_{\sigma_k^-}\Sigma^k(0,0;u^\ell) = (\lambda_k(u^{\mathrm{r}}) - \lambda_k(u^\ell))B^{-1}(u^\ell,u^{\mathrm{r}})r_k(u^{\mathrm{r}}),$$

instead of (5.11).

5.2. Convergence

Assume now that an ε-approximate solution u^ε to (1.1) has been defined according to the algorithm in [3]. Then, at every fixed time t,

$$u^\varepsilon(t,x) = \underline{u}^\ell \cdot \chi_{]-\infty,x_0]}(x) + \sum_{\alpha=1}^{N} u_\alpha \cdot \chi_{]x_{\alpha-1},x_\alpha]}(x) + \underline{u}^{\mathrm{r}} \cdot \chi_{]x_N,+\infty[^{(x)}}. \tag{5.12}$$

Here, x_α is a (time-dependent) point of jump of u^ε. Moreover, the Riemann problem with data $u_{\alpha-1},\ u_\alpha$ is solved by waves of (total) sizes $\sigma_{1,\alpha},\dots,\sigma_{n,\alpha}$. Let $\tilde{\alpha}$ be such that

$$u_{\tilde{\alpha}-1} \in \Omega_0 \quad \text{and} \quad u_{\tilde{\alpha}} \in \Omega_1, \tag{5.13}$$

so that $\sigma_{k,\tilde{\alpha}}$ is the size of the phase boundary, in the sense of parameterization (4.3) and (4.4).

Define the following Glimm functionals:

$$V^\varepsilon(u^\varepsilon) = \sum_{\alpha<\tilde{\alpha}} \left(\sum_{i=1}^{k-1} |\sigma_{i,\alpha}| + W \cdot \sum_{i=k+1}^{n+1} |\sigma_{i,\alpha}| \right) + \sum_{\alpha<\tilde{\alpha}} (W^- \cdot [\![\sigma_{k,\alpha}]\!]_- + W^+ \cdot [\![\sigma_{k,\alpha}]\!]_+) + W^+ \cdot [\![\sigma_{k,\tilde{\alpha}}]\!]_+$$

$$+\sum_{\alpha>\tilde\alpha}\left(W\cdot\sum_{i=1}^{k}|\sigma_{i,\alpha}|+\sum_{i=k+1}^{n+1}|\sigma_{i,\alpha}|\right),$$

$$Q^{\varepsilon}(u^{\varepsilon})=\sum_{[(i,\alpha),(j,\beta)]\in\mathscr{A}}|\sigma_{i,\alpha}\sigma_{j,\beta}|,$$

$$\Upsilon^{\varepsilon}(u^{\varepsilon})=V^{\varepsilon}(u^{\varepsilon})+Q^{\varepsilon}(u^{\varepsilon}).$$

The term $W^{+}\cdot[\![\sigma_{k,\tilde\alpha}]\!]_{+}$ deserves some explanation. It is needed to assure the decrease of the functional Υ^{ε} at times t_0 when a sonic phase boundary arises by interaction. At t_0 the strength of the phase boundary is positive, though for later times the rarefaction attached on the left is split and separates from it; at those times the phase boundary has strength 0.

Above, we followed the standard notation of considering nonphysical waves as waves belonging to the $(n+1)$th family. Moreover, $\mathscr{A}$ denotes the set of approaching waves, see [3, 19]. In the set $\mathscr{A}$ is included also the wave associated to phase boundary, differently from [5].

The role of the functionals above is to provide a bound for the total variation. This is achieved by showing first that Υ^{ε} is "equivalent" to $\mathrm{TV}(u^{\varepsilon})$ and, secondly, showing that the map $t\mapsto\Upsilon^{\varepsilon}(u^{\varepsilon}(t))$ is a nonincreasing function of time.

PROPOSITION 5.1. *Let u be a function of the form* (5.12), *satisfying* (5.13). *Then there exists positive constants c and C such that*

$$c\cdot\Upsilon(u)\leqslant\mathrm{TV}(u)\leqslant C\cdot(\Upsilon(u)+\|\underline{u}^{\mathrm{r}}-\underline{u}^{\ell}\|). \tag{5.14}$$

Proof. The former inequality is a standard consequence of strict hyperbolicity. To obtain the latter inequality, observe that

$$\|u_{\tilde\alpha}-u_{\tilde\alpha-1}\|\leqslant\|u_{\tilde\alpha}-\underline{u}^{\mathrm{r}}\|+\|\underline{u}^{\mathrm{r}}-\underline{u}^{\ell}\|+\|\underline{u}^{\ell}-u_{\tilde\alpha-1}\|$$

$$\leqslant C\cdot\Upsilon(u)+\|\underline{u}^{\mathrm{r}}-\underline{u}^{\ell}\|. \qquad ∎$$

PROPOSITION 5.2. *The map $t\mapsto\Upsilon^{\varepsilon}(u^{\varepsilon}(t))$ is nonincreasing.*

Proof. Several cases are in order, depending on the types of waves that interact. We consider in detail however, only the interaction of the small waves with the phase boundary. In fact, the interactions of small waves is analogous to [3], because the weights that we assigned to the functional Υ^{ε} do not play any role far from the phase boundary. We emphasize that the interaction of a physical wave with the phase boundary never produces a nonphysical wave, as a consequence of the choice of the Riemann solvers, see Section 4.

1. A k-rarefaction σ_k^- hits the phase boundary $\tilde{\sigma}_k^-$ from the left. Due to the particular choice (K) of the wave speed made above, the phase boundary $\tilde{\sigma}_k^-$ needs to be subsonic before the interaction, i.e. $\tilde{\sigma}_k^- < 0$, for otherwise no interaction may occur. Moreover, for the same reason, $\sigma_k^- + \tilde{\sigma}_k^- < 0$. To estimate ΔQ^ε we remark that the set of waves approaching $\tilde{\sigma}_k^-$ contains the set of waves approaching $\tilde{\sigma}_k^+$. Due to (5.6)

$$\begin{aligned}
\Delta V^\varepsilon &\leqslant \sum_{j\neq k} |\sigma_j^+| + W^+ [\![\tilde{\sigma}_k^+]\!]_+ - W^+|\sigma_k^-| \\
&= \mathcal{O}(1)(1+W^+)(|\sigma_k^-| + |\tilde{\sigma}_k^-|)|\sigma_k^-| - W^+|\sigma_k^-| \\
&= (\mathcal{O}(1)\cdot(1+W^+)\delta - W^+)\cdot|\sigma_k^-|, \\
\Delta Q^\varepsilon &\leqslant \sum_{j\neq k} |\sigma_j^+|\delta + |\tilde{\sigma}_k^+ - \tilde{\sigma}_k^-|\delta - |\sigma_k^- \tilde{\sigma}_k^-| \\
&= \mathcal{O}(1)\cdot\delta\cdot|\sigma_k^-|, \\
\Delta \Upsilon^\varepsilon &= (\mathcal{O}(1)\cdot(W^+ + 1)\delta - W^+)\cdot|\sigma_k^-|.
\end{aligned}$$

The latter quantity is negative, provided W^+ is sufficiently large (independently from W and W^-!) and δ sufficiently small.

2. A k-shock hits the phase boundary coming from the left. Then, $\tilde{\sigma}_k^+$ may have either signs. In both cases, due to (5.6),

$$[\![\tilde{\sigma}_k^+]\!]_+ - [\![\tilde{\sigma}_k^-]\!]_+ \leqslant \mathcal{O}(1)\cdot(|\sigma_k^-| + |\tilde{\sigma}_k^-|)|\sigma_k^-|,$$

$$\begin{aligned}
\Delta V^\varepsilon &\leqslant \sum_{j\neq k} |\sigma_j^+| + W^+([\![\tilde{\sigma}_k^+]\!]_+ - [\![\tilde{\sigma}_k^-]\!]_+) - W^-|\sigma_k^-| \\
&= (\mathcal{O}(1)\cdot(1+W^+)\delta - W^-)|\sigma_k^-|.
\end{aligned}$$

To estimate ΔQ^ε we remark that if $\tilde{\sigma}_k^- < 0$ then the set of waves approaching to it contains the set of waves approaching $\tilde{\sigma}_k^+$; these sets are equal if both $\tilde{\sigma}_k^- > 0$ and $\tilde{\sigma}_k^+ > 0$. In the remaining case $\tilde{\sigma}_k^- > 0$, $\tilde{\sigma}_k^+ < 0$ we remark that $|\tilde{\sigma}_k^+| \leqslant |\tilde{\sigma}_k^+ - \tilde{\sigma}_k^-|$. In any case we have the estimates

$$\begin{aligned}
\Delta Q^\varepsilon &\leqslant \sum_{j\neq k} |\sigma_j^+|\delta + |\tilde{\sigma}_k^+ - \tilde{\sigma}_k^-|\delta - |\sigma_k^- \tilde{\sigma}_k^-| \\
&= \mathcal{O}(1)\cdot\delta\cdot|\sigma_k^-|, \\
\Delta \Upsilon^\varepsilon &= (\mathcal{O}(1)\cdot(2+W^+)\delta - W^-)|\sigma_k^-|,
\end{aligned}$$

which is negative provided W^- is sufficiently large and δ sufficiently small.

3. Let $i = 1, 2, \ldots, n$, but $i \neq k$. Assume that an i-wave σ_i^- hits the phase boundary $\tilde{\sigma}_k^-$, resulting in various outcoming waves σ_j^+ and in the phase boundary $\tilde{\sigma}_k^+$. Due to (5.1) and arguing as in the previous case

$$\begin{aligned}
\Delta V^\varepsilon &\leqslant \sum_{j \neq k} |\sigma_j^+| + W^+ [\![\tilde{\sigma}_k^+]\!]_+ - W^+ [\![\tilde{\sigma}_k^-]\!]_+ - W|\sigma_i^-| \\
&= (\mathcal{O}(1) \cdot (1 + W^+) - W) \cdot |\sigma_i^-|, \\
\Delta Q^\varepsilon &= \mathcal{O}(1) \cdot \delta \cdot |\sigma_i^-|, \\
\Delta \Upsilon^\varepsilon &= (\mathcal{O}(1) \cdot (1 + W^+) - W) \cdot |\sigma_i^-|.
\end{aligned}$$

If $i = k$ and the wave σ_k is coming from the right then we use (5.5) and obtain analogous estimates.

4. We consider now the interaction of a nonphysical wave σ^- with the phase boundary $\tilde{\sigma}_k$; the outcome is a phase boundary with the same strength $\tilde{\sigma}_k$ and a nonphysical wave σ^+. Since $\sigma^+ = \mathcal{O}(1)\sigma^-$ from (5.1), we have

$$\begin{aligned}
\Delta V^\varepsilon &= (\mathcal{O}(1) - W)|\sigma^-|, \\
\Delta Q^\varepsilon &= -|\sigma^- \tilde{\sigma}_k^-|, \\
\Delta \Upsilon^\varepsilon &= (\mathcal{O}(1) - |\tilde{\sigma}_k^-| - W)|\sigma^-|
\end{aligned}$$

and so again $\Delta \Upsilon^\varepsilon < 0$ if W is sufficiently large.

5. The phase boundary does not take part to the interaction. Then, the standard estimates in [3] ensure that if the interacting waves have sizes σ and σ', then

$$\Delta \Upsilon^\varepsilon < -\tfrac{1}{2}|\sigma\sigma'|, \tag{5.15}$$

provided that δ is sufficiently small with respect to the weights that have been fixed in the previous steps. ∎

To complete the construction of the solution to (1.3) it is now necessary to choose sequences ε_n and ρ_n converging to 0 as in [3]. The corresponding sequences u_n and Λ_n satisfy Helly's Compactness Theorem and, up to a subsequence, converge to u and Λ.

We are thus left with the task of showing that the solution so obtained in the limit does satisfy the Rankine–Hugoniot conditions as stated in Definition 2. To this aim, let

$$\mathcal{RH}(u, w, \lambda) = f(u) - f(w) - \lambda \cdot (u - w)$$

and denote by $\Upsilon_n(t) = \Upsilon^{\varepsilon_n}(u_n(t))$. The limit $\tilde{\Upsilon}(t) = \lim_{n \to +\infty} \Upsilon_n(t)$ is a bounded nonincreasing function.

LEMMA 5.3. *For a.e. positive t and for every positive ε there exists a positive δ and an increasing (positive) sequence* $\{\delta_n : n \in \mathbf{N}\}$ *such that* $\lim_{n\to+\infty} \delta_n = \delta$ *and*

$$\tilde{\Upsilon}(t - \delta_n-) - \tilde{\Upsilon}(t + \delta_n+) < \varepsilon$$

for all large n.

Proof. Fix $\varepsilon > 0$. For every t but a finite set there exists a positive δ such that

$$\tilde{\Upsilon}(t - \delta-) - \tilde{\Upsilon}(t - \delta+) < \frac{\varepsilon}{2}.$$

By the point-wise convergence and the monotonicity of the Υ_n the thesis follows.

LEMMA 5.4. *For a.e. positive t, there exist two positive sequences* $\{l_n\}$ *and* $\{r_n\}$ *such that* $\lim_{n\to+\infty} l_n = 0$, $\lim_{n\to+\infty} r_n = 0$ *and*

$$u(t, \Lambda(t)-) = \lim_{n\to+\infty} u_n(t, \Lambda_n(t) - l_n),$$
$$u(t, \Lambda(t)+) = \lim_{n\to+\infty} u_n(t, \Lambda_n(t) + r_n).$$

For the proof, see Lemma 6.1 in [1].

LEMMA 5.5. *For a.e. positive t, the following estimates hold:*

$$\lim_{n\to+\infty} \sum_{x_\alpha \in [\Lambda_n(t), \Lambda_n(t)+r_n]} \sum_{i=1}^{n+1} |\sigma_{i,\alpha}| = 0, \tag{5.16}$$

$$\lim_{n\to+\infty} \sum_{x_\alpha \in [\Lambda_n(t)-l_n, \Lambda_n(t)]} \left(\sum_{i=1}^{k-1} |\sigma_{i,\alpha}| + \sum_{i=k+1}^{n+1} |\sigma_{i,\alpha}| \right) = 0. \tag{5.17}$$

Proof. Consider first (5.16). Fix $\varepsilon > 0$ and a time $\tau > 0$. Define $\lambda' = \inf_{\Omega_1} \lambda_{k+1}$, $\lambda'' = \sup_{\Omega_1} \lambda_k$ and

$$\mathcal{T}_n^{\mathrm{r}}(\tau) = \left\{ (t,x) \in [0, +\infty[\times \mathbf{R} \ : \begin{cases} x > \Lambda_n(t) \\ x < \lambda'' \cdot (t - \tau) + \Lambda_n(\tau) + r_n \\ x < \lambda' \cdot (t - \tau) + \Lambda_n(\tau) + r_n \end{cases} \right\},$$

see Fig. 3a. Due to (2.3), $\mathcal{T}_n^{\mathrm{r}}(\tau)$ is nonempty and bounded, for $\mathrm{TV}(u_0)$ sufficiently small. Without loss of generality, we may assume that

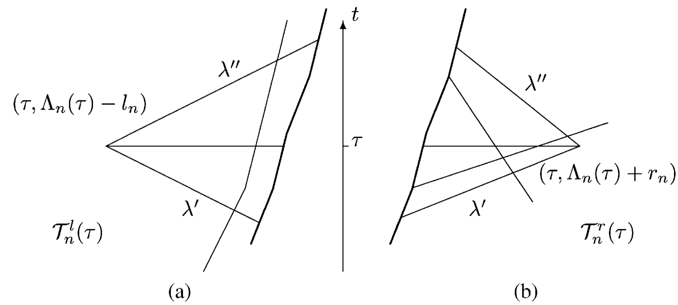

FIG. 3. A k wave "almost parallel" to the phase boundary (a), waves incoming and outgoing from the phase boundary (b).

$\mathscr{T}^{\mathrm{r}}_n(\tau)$ is contained in the strip $[\tau - \delta_n, \tau + \delta_n] \times \mathbf{R}$. Consider a wave σ of the ith family crossing the segment $\mathscr{S} = \{t\} \times [\Lambda_n(t), \Lambda_n(t) + r_n]$. If σ vanishes at some time within $[\tau - \delta_n, \tau + \delta_n]$, prolong it in the future if $i \leqslant k$ and in the past if $i > k$ with null size. With this provision, all the waves crossing $\mathscr{S}$ either hit the phase boundary (if $i \leqslant k$) or arise out of it (if $i > k$). Thus, by the interaction estimates in Proposition 5.2 their total size is small, since

$$\sum_{x_\alpha \in [\Lambda_n(t), \Lambda_n(t) + r_n]} \sum_{i=1}^{n+1} |\sigma_{i,\alpha}| = \mathcal{O}(1) \cdot (\tilde{\Upsilon}(t - \delta_n -) - \tilde{\Upsilon}(t + \delta_n +)) < \mathcal{O}(1) \cdot \varepsilon.$$

This proves (5.16). Concerning (5.17), replace $\mathscr{T}^{\mathrm{r}}_n(\tau)$ with

$$\mathscr{T}^{\ell}_n(\tau) = \left\{ (t,x) \in [0, +\infty[\times \mathbf{R} \ : \begin{cases} x < \Lambda_n(t) \\ x > \lambda'' \cdot (t - \tau) + \Lambda_n(\tau) - l_n \\ x > \lambda' \cdot (t - \tau) + \Lambda_n(\tau) - l_n \end{cases} \right\},$$

see Fig. 3b, where now $\lambda' = \sup_{\Omega_0} \lambda_{k-1}$, $\lambda'' = \inf_{\Omega_0} \lambda_{k+1}$ and follow an entirely similar procedure. ∎

The following proposition concludes the proof of Theorem 2.1 proving that the Rankine–Hugoniot conditions (2.5) are satisfied in the limit.

PROPOSITION 5.3. *For a.e. positive t,*

$$\lim_{n \to +\infty} \mathscr{RH}(u_n(t, \Lambda_n(t) - l_n), u_n(t, \Lambda_n(t) + r_n), \Lambda_n(t)) = 0.$$

Proof. Write

$$\mathscr{RH}(u_n(t,\Lambda_n(t)-l_n),u_n(t,\Lambda_n(t)+r_n),\Lambda_n(t))$$
$$=\mathscr{RH}(u_n(t,\Lambda_n(t)-l_n),u_n(t,\Lambda_n(t)-),\Lambda_n(t)) \tag{5.18}$$
$$+\mathscr{RH}(u_n(t,\Lambda_n(t)-),u_n(t,\Lambda_n(t)+),\Lambda_n(t)) \tag{5.19}$$
$$+\mathscr{RH}(u_n(t,\Lambda_n(t)+),u_n(t,\Lambda_n(t)+r_n),\Lambda_n(t)). \tag{5.20}$$

The second summand (5.19) vanishes since the Rankine-Hugoniot conditions are exactly satisfied along the approximate phase boundary. The third summand (5.20) is estimated as

$$\mathscr{RH}(u_n(t,\Lambda_n(t)+),u_n(t,\Lambda_n(t)+r_n),\Lambda_n(t))$$
$$=\mathscr{O}(1)\cdot\sum_{x_\alpha\in[\Lambda_n(t),\Lambda_n(t)+r_n]}\sum_{i=1}^{n+1}|\sigma_{i,\alpha}|,$$

hence it vanishes when $n\to+\infty$ due to (5.16).

The first summand (5.18) is the key point. Let $w_0=u_n(t,\Lambda_n(t)-l_n+)$ and call $w_1,w_2,\dots,w_N$ the values attained by u_n at time t for $x\in[\Lambda_n(t)-l_n,\Lambda_n(t)]$, with $w_N=u_n(t,\Lambda_n(t)-)$. We have, with an obvious notation,

$$\mathscr{RH}(u_n(t,\Lambda_n(t)-l_n),u_n(t,\Lambda_n(t)-),\Lambda_n(t))$$
$$=\sum_{j=1}^{N}\mathscr{RH}(w_{j-1},w_j,\Lambda_n(t))$$
$$=\sum_{k\text{-jump}}\mathscr{RH}(w_{j-1},w_j,\Lambda_n(t))+\sum_{\text{non }k\text{-jump}}\mathscr{RH}(w_{j-1},w_j,\Lambda_n(t)) \tag{5.21}$$
$$\leqslant\sum_{k\text{-shock}}\mathscr{RH}(w_{j-1},w_j,\Lambda_n(t)) \tag{5.22}$$
$$+\sum_{k\text{-rar}}\mathscr{RH}(w_{j-1},w_j,\Lambda_n(t)) \tag{5.23}$$
$$+\mathscr{O}(1)\cdot\sum_{x_\alpha\in[\Lambda_n(t)-l_n,\Lambda_n(t)]}\left(\sum_{i=1}^{k-1}|\sigma_{i,\alpha}|+\sum_{i=k+1}^{n+1}|\sigma_{i,\alpha}|\right). \tag{5.24}$$

Consider the last three summands separately.

In (5.22), note that the approximate shocks satisfy Rankine-Hugoniot conditions. Secondly, split the sum in three parts, the former relative to those k-shocks that arise in the time interval $[t-\delta_n,t+\delta_n]$, the second relative to those k-shocks that hit the phase boundary within $[t-\delta_n,t+\delta_n]$

and the latter relative to those shocks that do not interact with the phase boundary:

$$\begin{aligned}
&\sum_{k\text{-shock}} \mathcal{RH}(w_{j-1}, w_j, \Lambda_n(t)) \\
&\quad = \sum_{k\text{-shock}} |\lambda_k(w_{j-1}, w_j) - \Lambda_n(t)| \cdot \|w_{j-1} - w_j\| \\
&\quad = \sum{}' |\lambda_k(w_{j-1}, w_j) - \Lambda_n(t)| \cdot \|w_{j-1} - w_j\| \\
&\qquad + \sum{}'' |\lambda_k(w_{j-1}, w_j) - \Lambda_n(t)| \cdot \|w_{j-1} - w_j\| \\
&\qquad + \sum{}''' |\lambda_k(w_{j-1}, w_j) - \Lambda_n(t)| \cdot \|w_{j-1} - w_j\|.
\end{aligned}$$

The first and the second sums involve only *small* waves. Indeed, the strengths of those k-shocks that either arise or hit the phase boundary are bounded by $\mathcal{O}(1)\Delta\Upsilon_n$, see Proposition 5.2. In the last summand, it is the difference in the wave speeds that needs to be small, since the latter sum refer to those k-shocks that are "almost parallel" to the phase boundary. Thus

$$\begin{aligned}
&\sum_{k\text{-shock}} \mathcal{RH}(w_{j-1}, w_j, \Lambda_n(t)) \\
&\quad \leqslant \mathcal{O}(1) \cdot |\Delta\Upsilon_n| + \mathcal{O}(1) \cdot \left(|\Delta\Upsilon_n| + \sum{}''' \frac{l_n}{\delta} \cdot \|w_{j-1} - w_j\| \right)
\end{aligned}$$

which is small due to Lemmas 5.3 and 5.4.

The rarefactions in (5.23) are treated similarly to shocks. In fact, introduce the three sums

$$\begin{aligned}
&\sum_{k\text{-rar}} \mathcal{RH}(w_{j-1}, w_j, \Lambda_n(t)) \\
&\quad = \left(\sum{}' + \sum{}'' + \sum{}''' \right) \mathcal{RH}(w_{j-1}, w_j, \Lambda_n(t)) \\
&\quad = \mathcal{O}(1) \cdot |\Delta\tilde{\Upsilon}| + \sum{}''' \mathcal{RH}(w_{j-1}, w_j, \Lambda_n(t)) \\
&\quad = \mathcal{O}(1) \cdot |\Delta\tilde{\Upsilon}| + \mathcal{O}(1) \cdot \sum{}''' |\sigma_{k,\alpha}| \\
&\quad = \mathcal{O}(1) \cdot |\Delta\tilde{\Upsilon}| + \mathcal{O}(1) \cdot \left(|\Delta\Upsilon_n| + \sum{}''' \frac{l_n}{\delta} \cdot \|w_{j-1} - w_j\| \right).
\end{aligned}$$

The last term obtained can be made arbitrarily small.

Finally, the last summand (5.24) is small due to (5.17). ∎

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