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A new simple form of quark mixing matrix

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ABSTRACT

Although different parametrizations of quark mixing matrix are mathematically equivalent, the consequences of experimental analysis may be distinct. Based on the triminimal expansion of Kobayashi–Maskawa matrix around the unit matrix, we propose a new simple parametrization. Compared with the Wolfenstein parametrization, we find that the new form is not only consistent with the original one in the hierarchical structure, but also more convenient for numerical analysis and measurement of the CP-violating phase. By discussing the relation between our new form and the unitarity boomerang, we point out that along with the unitarity boomerang, this new parametrization is useful in hunting for new physics.

The mixing of quarks is one of the fundamental problems in particle physics. However, its origin is still unclear yet and the mixing is currently described phenomenologically by the mixing matrix, i.e., the Cabibbo [1]–Kobayashi–Maskawa [2] (CKM) matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{1}$$

The parametrization proposed by Chau and Keung (CK) [3,4] is the most popular way of parameterizing the matrix. Using three mixing angles and one CP-violating phase, it provides a clear understanding of the mixing. However, some recent works [5,6] reveal that the parameters in the CK parametrization are inconvenient when dealing with the unitarity boomerang (UB). A unitarity boomerang is formed using two unitarity triangles [7] with a common inner angle, thus contains all four independent parameters in the mixing matrix, and is a powerful tool of hunting for new physics beyond the Standard Model [8]. Instead of the CK form, Frampton and He proposed [5] that the original Kobayashi–Maskawa (KM) [2] matrix is kept as the standard parametrization, which is given by

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & s_3 & -c_3 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}. \tag{2}$$

Here $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ $(i = 1, 2, 3)$, and θ_1, θ_2, θ_3 are Euler angles describing the rotation among different generations, δ is the CP-violating phase in the KM parametrization.

Although different parametrizations of quark mixing matrix are mathematically equivalent, the consequences of experimental analysis may be distinct. The magnitudes of the elements V_{ij} are physical quantities which do not depend on parametrization. However, the CP-violating phase does. As a result, the understanding of the origin of CP violation is associated with the parametrization. For example, the prediction based on the maximal CP-violation hypothesis [9] is related with the parametrization or in other words, phase convention. As discussed in Ref. [10], only with the original KM parametrization and the Fritzsch–Xing [11] parametrization, one can get successful predictions on the unitarity triangle [7] from the maximal CP-violation hypothesis. Therefore the original KM matrix is convenient for studying both the maximal CP-violation and unitarity boomerangs, so that a study about it is necessary.

With the data on the magnitudes of the mixing matrix elements [4]

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$$\begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415^{+0.0010}_{-0.0011} \\ 0.00874^{+0.00026}_{-0.00037} & 0.0407 \pm 0.0010 & 0.999133^{+0.000044}_{-0.000043} \end{pmatrix}, \tag{3}$$

one can easily get the ranges of the parameters in the KM parametrization

$$\theta_1 = 0.228 \pm 0.001, \qquad \theta_2 = 0.039^{+0.001}_{-0.002}, \qquad \theta_3 = 0.016 \pm 0.001. \tag{4}$$

When studying mixing, it is useful to parameterize the matrix according to the hierarchical structure of the mixing to reveal more physical information about the underlying theory. A good choice is the idea of triminimal parametrization [12–14] with an approximation as the basis matrix to the lowest order. That is to express a mixing angle in the mixing matrix as the sum of a zeroth order angle θ^0 and a small perturbation angle ϵ with

$$\theta_1 = \theta_1^0 + \epsilon_1, \qquad \theta_2 = \theta_2^0 + \epsilon_2, \qquad \theta_3 = \theta_3^0 + \epsilon_3. \tag{5}$$

With the deviations ϵ_i, one can expand the mixing matrix in powers of ϵ_i while different choices of θ_i^0 lead to different basis. The general expansion of KM matrix is presented in Appendix A. Since Eq. (3) is very close to the unit matrix, it is a good approximation to let

$$\epsilon_1 = \theta_1, \qquad \epsilon_2 = \theta_2, \qquad \epsilon_3 = \theta_3. \tag{6}$$

To make the lowest order be the unit matrix, we still need to adjust the phases of quarks with

$$c \to c e^{i\pi}, \qquad s \to s e^{i\pi}, \qquad b \to b e^{i(\pi+\delta)}. \tag{7}$$

According to Eq. (4), we have $\epsilon_1^2 \sim \epsilon_2 \sim \epsilon_3$. Therefore, in order to keep the magnitude consistency of the expansion, we display all terms of $\mathcal{O}(\epsilon_1^3)$ in our parametrization with

$$V = \begin{pmatrix} 1 - \frac{\epsilon_1^2}{2} & \epsilon_1 - \frac{\epsilon_1^3}{6} & e^{-i\delta}\epsilon_1\epsilon_3 \\ \frac{\epsilon_1^3}{6} - \epsilon_1 & 1 - \frac{\epsilon_1^2}{2} & \epsilon_2 + e^{-i\delta}\epsilon_3 \\ \epsilon_1\epsilon_2 & -\epsilon_2 - e^{i\delta}\epsilon_3 & 1 \end{pmatrix} + \mathcal{O}(\epsilon_1^4). \tag{8}$$

Comparing with the Wolfenstein parametrization [15]

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{9}$$

in which $\lambda = s_1$, $A\lambda^2(\rho^2 + \eta^2)^{\frac{1}{2}} = s_3$ and $A\lambda^2[(1-\rho)^2 + \eta^2]^{\frac{1}{2}} = s_2$, Eq. (8) has the same hierarchical structure with the Wolfenstein parametrization. We can check the magnitude consistency by substituting these relations into Eq. (8) and only focus on the modulus of each element in terms of all four Wolfenstein parameters, which gives

$$\begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho^2 + \eta^2)^{\frac{1}{2}} \\ \lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2(1 - 2\rho + 2\rho^2 + 2\eta^2)^{\frac{1}{2}} \\ A\lambda^3((1-\rho)^2 + \eta^2)^{\frac{1}{2}} & A\lambda^2(1 - 2\rho + 2\rho^2 + 2\eta^2)^{\frac{1}{2}} & 1 \end{pmatrix}. \tag{10}$$

Here we take $\delta \approx 90^\circ$, which implies the maximal CP violation. The only difference comes from $|V_{cb}|$ and $|V_{ts|}$ with an extra coefficient. However, numerical calculation gives $(1 - 2\rho + 2\rho^2 + 2\eta^2)^{\frac{1}{2}} = 1.0089 \approx 1$, so that the hierarchical structure of the quark mixing is well preserved in Eq. (8).

A natural idea is to find the relation between these two forms. However, it is complicated in adjusting the phases by rephasing the quark fields, as shown in Ref. [6]. This is because the phase convention adopted by Eq. (8) is different from Eq. (9). Actually, the Wolfenstein parametrization takes the same phase convention with the standard CK form [3,4], which implies another choice of the phase δ. Therefore one has difficulty to arrive at the Wolfenstein parametrization from triminimal parametrization of KM matrix. This is different from the situation of triminimal parametrization of CK matrix, as shown in Ref. [14], where the Wolfenstein parametrization can be understood as a simple form "derived" from the CK matrix.

By keeping the original Wolfenstein parameter $\lambda = \sin\theta_1 \approx \epsilon_1 - \frac{\epsilon_1^3}{6}$ and the CP-violating phase δ, and introducing two new parameters with

$$f\lambda^2 = \sin\theta_2 \approx \epsilon_2, \qquad h\lambda^2 = \sin\theta_3 \approx \epsilon_3, \tag{11}$$

we obtain a new Wolfenstein-like parametrization through substitution of them into Eq. (8), that is

$$V = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & e^{-i\delta}h\lambda^3 \\ -\lambda & 1 - \frac{\lambda^2}{2} & (f + e^{-i\delta}h)\lambda^2 \\ f\lambda^3 & -(f + e^{i\delta}h)\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{12}$$

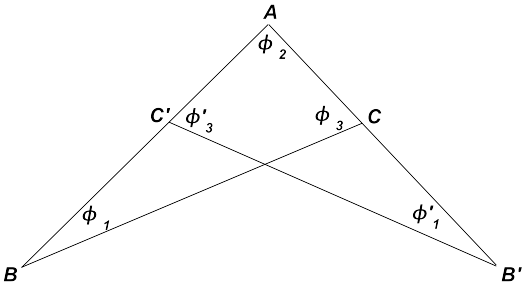

Fig. 1. The unitarity boomerang of quark mixing with the common angle ϕ_2. The sides are $AC = |V_{ud}V_{ub}^*|$, $AC' = |V_{ub}V_{tb}^*|$, $AB = |V_{td}V_{tb}^*|$, $AB' = |V_{ud}V_{td}^*|$, $BC = |V_{cd}V_{cb}^*|$, $B'C' = |V_{us}V_{ts}^*|$.

This new simple form obviously preserves the unitarity of the matrix to the third order of λ and the hierarchical structure of the quark mixing as we discussed above. The choice of two new parameters is quite natural since $h\lambda^3 = |V_{ub}|$ and $f\lambda^3 = |V_{td}|$, thus can directly be determined with $\lambda = 0.2257^{+0.0009}_{-0.0010}$ [4] and Eq. (3), which gives

$$h = 0.312^{+0.018}_{-0.014}, \qquad f = 0.760^{+0.023}_{-0.032}. \tag{13}$$

Different from the original Wolfenstein form, in which the CP violation is determined by two parameters, i.e., ρ and η, there is only one phase δ independent of other parameters in Eq. (12). Another advantage of this new form is that V_{cb} and V_{ts}, with magnitudes of $\mathcal{O}(10^{-2})$, contribute to the constraint of CP-violating phase δ, while in the original Wolfenstein form we need to consider V_{ub} and V_{td}, whose magnitudes being one order smaller than those of V_{cb} and V_{ts} but with all four parameters involved, making it inconvenient when doing experimental analysis. Therefore, from this point of view, Eq. (12) is more convenient than the original Wolfenstein parametrization. Simple numerical calculation of equation $|(f + e^{-i\delta}h)\lambda^2| = |V_{cb}|$ gives

$$\delta \approx 91.4^\circ, \tag{14}$$

which means approximate maximal CP violation as we mentioned before.

A useful and natural application of this new simple parametrization is to study the unitarity boomerangs with it. The commonly used unitarity boomerang is consisted by two unitarity triangles with the same order of the three sides, say, λ^3, arising from

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \qquad V_{ud}V_{td}^* + V_{us}V_{ts}^* + V_{ub}V_{tb}^* = 0. \tag{15}$$

Since the common angle of the two chosen unitarity boomerangs could be determined by the CP-violating measurement J [16,17], the CP-violating phase could then be constrained. The Jarlskog parameter satisfies

$$\begin{aligned} J &= 2\left|V_{td}V_{tb}^*\right|\left|V_{ud}V_{ub}^*\right|\sin\phi_2 \\ &= 2\left|V_{ud}V_{td}^*\right|\left|V_{ub}V_{tb}^*\right|\sin\phi_2' \end{aligned}$$

with $\phi_2 = \phi_2'$ as the common angle of the unitarity boomerang as illustrated in Fig. 1. Using Eq. (12), we easily parameterize the sides and angles of the unitarity boomerang with

$$AB = AB' = f\lambda^3; \qquad AC = AC' = h\lambda^3; \qquad BC = B'C' = \lambda^3\left(f^2 + 2fh\cos\delta + h^2\right)^{\frac{1}{2}};$$
$$\phi_1 = \phi_1' = \arctan\frac{h\sin\delta}{f + h\cos\delta}; \qquad \phi_3 = \phi_3' = \arctan\frac{f\sin\delta}{h + f\cos\delta}; \qquad \phi_2 = \phi_2' = \pi - \delta,$$

showing that to the third order of λ, the two chosen unitarity triangles are identical. Using the last one of these equations, we can check the maximal CP-violation hypothesis [10] easily, and the experimental analysis consistently gives $\phi_2 = (88^{+6}_{-5})^\circ$ [18]. High order corrections to the boomerang bring about difference between these two triangles (see Appendix B). To the lowest order, the Jarlskog parameter is given by

$$J = fh\lambda^6\sin\delta.$$

We get simple relations between these two parametrizations, i.e., diagrammatical and matrix forms. This implies that the parametrization (12) is natural in discussing the unitarity boomerangs of quark mixing. In Ref. [5] and Ref. [8], Frampton and He pointed out that the unitarity boomerang is very helpful in searching new physics since it contains all the information about the mixing matrix and reflects the precision attained by high-energy experiments. Thus deviations from the expected unitarity boomerang may imply possibility for new physics beyond the Standard Model. Therefore, if new physics information show up in the unitarity boomerang analysis, we could get corresponding signals in the parameters and consequently the mixing matrix through the relations above. Then by studying how the new physics modify the original matrix, we may get hints of understanding the underlying theory.

Finally, we present a conclusion of this Letter. The new form of quark mixing matrix (12) is our main result. It exhibits the hierarchical structure of the mixing, and is convenient for numerical analysis, especially for constraint of the CP-violating phase. Combined with the unitarity boomerang, it is also helpful to study the presence of new physics. Therefore, we humbly suggest it as a simple form corresponding to the KM matrix in theoretical and experimental studies.

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Appendix A. The general triminimal expansion of the KM matrix

We present here the general triminimal expansion of KM matrix. To second order of ϵ_i, the KM matrix is given by

$$
\begin{aligned}
V = {} & \begin{pmatrix} c_1^0 & -s_1^0c_3^0 & -s_1^0s_3^0 \\ s_1^0c_2^0 & c_1^0c_2^0c_3^0 - s_2^0s_3^0e^{i\delta} & c_1^0c_2^0s_3^0 + s_2^0c_3^0e^{i\delta} \\ s_1^0s_2^0 & c_1^0s_2^0c_3^0 + c_2^0s_3^0e^{i\delta} & c_1^0s_2^0s_3^0 - c_2^0c_3^0e^{i\delta} \end{pmatrix} + \epsilon_1 \begin{pmatrix} -s_1^0 & -c_1^0c_3^0 & -c_1^0s_3^0 \\ c_1^0c_2^0 & -c_2^0c_3^0s_1^0 & -c_2^0s_1^0s_3^0 \\ c_1^0s_2^0 & -c_3^0s_1^0s_2^0 & -s_1^0s_2^0s_3^0 \end{pmatrix} \\
& + \epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -s_1^0s_2^0 & -c_1^0c_3^0s_2^0 - c_2^0s_3^0e^{i\delta} & -c_1^0s_2^0s_3^0 + c_2^0c_3^0e^{i\delta} \\ c_2^0s_1^0 & c_1^0c_2^0c_3^0 - s_2^0s_3^0e^{i\delta} & c_1^0c_2^0s_3^0 + c_3^0s_2^0e^{i\delta} \end{pmatrix} \\
& + \epsilon_3 \begin{pmatrix} 0 & s_1^0s_3^0 & -c_3^0s_1^0 \\ 0 & -c_1^0c_2^0s_3^0 - c_3^0s_2^0e^{i\delta} & c_1^0c_2^0c_3^0 - s_2^0s_3^0e^{i\delta} \\ 0 & -c_1^0s_2^0s_3^0 + c_2^0c_3^0e^{i\delta} & c_1^0c_3^0s_2^0 + c_2^0s_3^0e^{i\delta} \end{pmatrix} + \frac{1}{2}\epsilon_1^2 \begin{pmatrix} -c_1^0 & c_3^0s_1^0 & s_1^0s_3^0 \\ -c_2^0s_1^0 & -c_1^0c_2^0c_3^0 & -c_1^0c_2^0s_3^0 \\ -s_1^0s_2^0 & -c_1^0c_3^0s_2^0 & -c_1^0s_2^0s_3^0 \end{pmatrix} \\
& + \frac{1}{2}\epsilon_2^2 \begin{pmatrix} 0 & 0 & 0 \\ -c_2^0s_1^0 & -c_1^0c_2^0c_3^0 + s_2^0s_3^0e^{i\delta} & -c_1^0c_2^0s_3^0 - c_3^0s_2^0e^{i\delta} \\ -s_1^0s_2^0 & -c_1^0c_3^0s_2^0 - c_2^0s_3^0e^{i\delta} & -c_1^0s_2^0s_3^0 + c_2^0c_3^0e^{i\delta} \end{pmatrix} \\
& + \frac{1}{2}\epsilon_3^2 \begin{pmatrix} 0 & c_3^0s_1^0 & s_1^0s_3^0 \\ 0 & -c_1^0c_2^0c_3^0 + s_2^0s_3^0e^{i\delta} & -c_1^0c_2^0s_3^0 - c_3^0s_2^0e^{i\delta} \\ 0 & -c_1^0c_3^0s_2^0 - c_2^0s_3^0e^{i\delta} & -c_1^0s_2^0s_3^0 + c_2^0c_3^0e^{i\delta} \end{pmatrix} + \epsilon_1\epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -c_1^0s_2^0 & c_3^0s_1^0s_2^0 & s_1^0s_2^0s_3^0 \\ c_1^0c_2^0 & -c_2^0c_3^0s_1^0 & -c_2^0s_1^0s_3^0 \end{pmatrix} \\
& + \epsilon_2\epsilon_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1^0s_2^0s_3^0 - c_2^0c_3^0e^{i\delta} & -c_1^0c_3^0s_2^0 - c_2^0s_3^0e^{i\delta} \\ 0 & -c_1^0c_2^0s_3^0 - c_3^0s_2^0e^{i\delta} & c_1^0c_2^0c_3^0 - s_2^0s_3^0e^{i\delta} \end{pmatrix} + \epsilon_1\epsilon_3 \begin{pmatrix} 0 & c_1^0s_3^0 & -c_1^0c_3^0 \\ 0 & c_2^0s_1^0s_3^0 & -c_2^0c_3^0s_1^0 \\ 0 & s_1^0s_2^0s_3^0 & -c_3^0s_1^0s_2^0 \end{pmatrix} \\
& + \mathcal{O}\big(\epsilon_i^3\big),
\end{aligned}
$$

where $s_i^0 = \sin\theta_i^0$ and $c_i^0 = \cos\theta_i^0$.

The Jarlskog parameter given by

$$J = \mathrm{Im}\big(V_{11}V_{22}V_{12}^*V_{21}^*\big) = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin\delta$$

is independent of phase convention, making it important when discussing CP violation. Expanding J with ϵ_i to the second order gives

$$
\begin{aligned}
J = J_0 \Big(& 1 + \epsilon_1\big(3\cot 2\theta_1^0 + \csc 2\theta_1^0\big) + 2\epsilon_2 \cot 2\theta_2^0 + 2\epsilon_3 \cot 2\theta_3^0 + \frac{1}{4}\epsilon_1^2\big(9\cos 2\theta_1^0 - 5\big)\csc^2\theta_1^0 \\
& - 2\epsilon_2^2 - 2\epsilon_3^2 + 2\epsilon_1\epsilon_2\big(3\cos 2\theta_1^0 + 1\big)\cot 2\theta_2^0 \csc 2\theta_1^0 + 4\epsilon_2\epsilon_3 \cot 2\theta_2^0 \cot 2\theta_3^0 \\
& + 2\epsilon_1\epsilon_3\big(3\cos 2\theta_1^0 + 1\big)\cot 2\theta_3^0 \csc 2\theta_1^0 \Big) + \mathcal{O}\big(\epsilon_i^3\big),
\end{aligned}
$$

in which $J_0 = (s_1^0)^2 s_2^0 s_3^0 c_1^0 c_2^0 c_3^0 \sin\delta$.

Appendix B. High order calculation of the boomerang

The leading order of the sides of the unitarity boomerang in Fig. 1 are of $\mathcal{O}(\lambda^3)$ and the two unitarity triangles are identical with each other. When high order corrections are included, difference between the two triangles comes out. We need to parameterize the CKM matrix to high order of λ, here we expand it to $\mathcal{O}(\lambda^5)$

$$
V = \begin{pmatrix} 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8} & \lambda - \frac{h^2\lambda^5}{2} & e^{-i\delta}h\lambda^3 \\ -\lambda + \frac{f^2\lambda^5}{2} & 1 - \frac{\lambda^2}{2} - \frac{1}{8}(1 + 4h^2 + 8e^{i\delta}fh + 4f^2)\lambda^4 & (f + e^{-i\delta}h)\lambda^2 - \frac{1}{2}e^{-i\delta}h\lambda^4 \\ f\lambda^3 & -(f + e^{i\delta}h)\lambda^2 + \frac{1}{2}f\lambda^4 & 1 - \frac{1}{2}(f^2 + 2e^{-i\delta}fh + h^2)\lambda^4 \end{pmatrix}.
$$

With this expression we can get the sides and the angles in Fig. 1 as

$$AB = f\lambda^3; \qquad AB' = f\lambda^3 - \frac{1}{2}f\lambda^5; \qquad AC = h\lambda^3 - \frac{1}{2}h\lambda^5; \qquad AC' = h\lambda^3;$$

$$BC = \lambda^3\kappa^{\frac{1}{2}} - \frac{1}{2}\lambda^5\big(h^2 + fh\big)\kappa^{-\frac{1}{2}}; \qquad B'C' = \lambda^3\kappa^{\frac{1}{2}} - \frac{1}{2}\lambda^5\big(f^2 + fh\big)\kappa^{-\frac{1}{2}};$$

$$\phi_1 = \arctan\frac{h\sin\delta}{f + h\sin\delta} - \frac{fh\lambda^2\sin\delta}{2\kappa} + fh\lambda^4\sin\delta\left(1 - \frac{fh\cos\delta + h^2}{4\kappa^2}\right);$$

$$\phi_1' = \arctan\frac{h\sin\delta}{f + h\sin\delta} + \frac{fh\lambda^2\sin\delta}{2\kappa} + \frac{fh\lambda^4\sin\delta(f^2 + fh\cos\delta)}{4\kappa^2};$$

$$\phi_3 = \arctan\frac{f\sin\delta}{h + f\sin\delta} + \frac{fh\lambda^2\sin\delta}{2\kappa} + \frac{fh\lambda^4\sin\delta(h^2 + fh\cos\delta)}{4\kappa^2};$$

$$\phi_3' = \arctan\frac{f\sin\delta}{h + f\sin\delta} - \frac{fh\lambda^2\sin\delta}{2\kappa} + fh\lambda^4\sin\delta\left(1 - \frac{fh\cos\delta + f^2}{4\kappa^2}\right);$$

$$\phi_2 = \phi_2' = \pi - \delta - fh\lambda^4\sin\delta,$$

in which $\kappa = f^2 + 2fh\cos\delta + h^2$. In the expressions for angles (except the common inner angle ϕ_2), the terms proportional to λ^2 come from the fraction of the high order terms of the elements since the definition of the angles is, for example $\phi_1 = \mathrm{Arg}[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}]$, thus we do not have this kind of corrections when we only consider the leading order.

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