The Value Distribution of Differences of Additive Arithmetic Functions

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Let $a > 0, A > 0, b, B$ be integers which satisfy $aB \neq Ab$, and $\eta(x)$ a real-valued function defined for $x \geq 2$. In this paper I consider the frequency $F_x(z)$, among the positive integers $n$ not exceeding $x$, of those for which the real-valued additive functions $f_j, j = 1, 2$, satisfy $f_1(an + b) - f_2(An + B) - \eta(x) \leq z$.

**Theorem 1.** The following three propositions are equivalent.

(i) There is an $\eta(x)$ so that the frequencies $F_x(z)$ converge weakly to a distribution function as $x \to \infty$.

(ii) There is an $\eta(x)$ so that

$$\lim_{z \to \infty} \limsup_{x \to \infty} (1 - F_x(z) + F_x(-z)) = 0.$$ 

(iii) There are real numbers $\alpha_j$ so that the series

$$\sum_{|f_j(p) - \alpha_j \log p| > 1} \frac{1}{p}, \quad \sum_{|f_j(p) - \alpha_j \log p| \leq 1} \frac{(f_j(p) - \alpha_j \log p)^2}{p}, \quad j = 1, 2,$$

converge.

Suppose that $\beta(x) > 0, \beta(x) \to \infty$ as $x \to \infty$, and for each positive $y, \beta(x'/y)\beta(x) \to 1$ as $x \to \infty$. Combining the methods of this paper with those of Elliott [2, Chap. 21], one may give necessary and sufficient conditions for the frequencies $F_x(z\beta(x))$, with a suitably chosen $\eta(x)$, to

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converge weakly as $x \to \infty$. This gives the appropriate analogue of Theorem (21.1) of that reference without assumptions on the mean and variance of the frequencies. In particular, analogues may be given of all the theorems in Elliott [1, Chap. 16], concerning the simulation of additive functions by values of sums of independent random variables. In place of a single function we may have the difference $f_1(an + b) - f_2(An + B)$.

If $f_1(an + b) - f_2(An + B)$ approaches a constant as $n \to \infty$ through a sequence of asymptotic density 1, then $f_j(n) = a_j \log n, j = 1, 2,$ holds on those integers $n$ which are prime to $aA(aB - Ab)$.

For differences $f(n + 1) - f(n)$ of the same function, with $x(x) = 0$, the equivalence of propositions (i) and (iii) of Theorem 1, and so its ramifications, was obtained by Hildebrand [6]. His argument begins with a novel application of the Large Sieve, suited to the consideration of differences $f(n + k) - f(n)$ for a fixed $k \neq 0$. This seems not to extend to differences $f(an + b) - f(An + B)$ nor to allow translations $\eta(x)$, much less to allow the introduction of differing functions $f_j$. In this paper I abandon this application of the Large Sieve and introduce new ideas (see Section 3). These concern the value distribution of complex multiplicative functions $g$ about which we assume only that $|g(n)| \leq 1$ holds for all $n$. They are of interest for themselves.

The theorem and the lemmas are numbered consecutively.

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It is convenient to recast proposition (ii) of Theorem 1 using the language of Fourier transforms. To this end we note the following result concerning distribution functions $G(z)$ and their corresponding characteristic functions $\phi(t)$, $t$ real.

**Lemma 2.**

\[
\lim_{z \to \infty} \lim_{x \to \infty} \sup \left(1 - G_x(z) + G_x(-z)\right) = 0
\]

for a sequence of distribution functions $G_x(z)$, if and only if

\[
\lim_{\tau \to 0} \lim_{x \to \infty} \sup_{|t| \leq \tau} |\phi_x(t) - 1| = 0.
\]

**Proof.** For $\tau \neq 0$

\[
\int_{x}^{\infty} \frac{\sin \tau z}{\tau z} dG(z) = \frac{1}{2\tau} \int_{-\tau}^{\tau} \phi(t) dt.
\]
Hence
\[(1 - r) \left( 1 - G \left( \frac{1}{\tau^2} \right) + G \left( -\frac{1}{\tau^2} \right) \right) \leq \int_{|z| > 1/\tau} \left( 1 - \frac{\sin \tau z}{\tau z} \right) dG(z) \leq \int_{-\infty}^{\infty} \left( 1 - \frac{\sin \tau z}{\tau z} \right) dG(z) = \frac{1}{2\tau} \int_{-\tau}^{\tau} (1 - \phi(t)) \, dt,\]
so that (1) follows from (2). In the other direction
\[
\phi(t) - 1 = \int_{|z| \leq 1/\tau} (e^{it\tau} - 1) \, dG(z) + \int_{|z| > 1/\sqrt{\tau}} (e^{it\tau} - 1) \, dG(z).
\]
For $|t| \leq \tau$ the first integrand is $\leq |t| \leq \sqrt{\tau}$ in absolute value; the second is $\leq 2$. Condition (2) follows from (1).

That for a suitable $\eta(x)$ the weak convergence of the frequencies $F_x(z)$ follows from the convergence of the series (iii) may be obtained by truncating the functions $f_j$, as in the proof of the Erdős–Wintner theorem given in Elliott [1, Chap. 5]. See, also, Elliott [2, Chap. 21]. It is easy to see that if the $F_x(z)$ converge weakly, then the condition of (ii) is satisfied. Accordingly, I shall concentrate on the assertion that (ii) implies (iii).

In our present circumstances the role of $\phi_x(t)$ is played by
\[
[x^{-1}] \exp(-i\eta(x)) \sum_{n \in x} g_1(\alpha n + b) \overline{g_2(\beta n + b)},
\]
where $g_j(n) = \exp(i f_j(n))$, $j = 1, 2$, and we examine the consequences of the hypothesis (2).

Remark. The quantity $|\phi_x(t) - 1|$ in the condition (2) of Lemma 2 may be replaced by
\[
\psi(t) = \int_{-\infty}^{\infty} |e^{it\tau} - 1| \, dG(z).
\]
Clearly $|\phi_x(t) - 1| \leq \psi(t)$, and by the Cauchy–Schwarz inequality
\[
\psi(t)^2 \leq \int_{-\infty}^{\infty} |e^{it\tau} - 1|^2 \, dG(z) = 2(1 - \text{Re} \phi_x(t)) \leq 2 |1 - \phi_x(t)|.
\]
This brings a practical advantage.
In this section $g$ will be a complex-valued multiplicative function which satisfies $|g(n)| \leq 1$ for all positive integers $n$. It is the results of this section that allow the introduction of the generality in Theorem 1.

**Lemma 3.** The estimate
\[
\sum_{n \leq y \atop n \equiv r (\text{mod } D)} g(n) - \frac{1}{\phi(D)} \sum_{n \leq y} g(n) \leq y \left( \frac{\log \log x}{\log x} \right)^{1/8} \frac{\log x}{\log y}
\]
holds uniformly for $y \leq x$, all $(r, D) = 1$, for all moduli $D$, except possibly for those moduli $D$ which are multiples of a certain $D_0 > 1$.

**Proof.** See Elliott [3]. The choice of a non-principal Dirichlet character (mod 3) for $g$ shows that the exceptional moduli may all occur.

**Lemma 4.** There is a positive constant $c_0$, so that if $0 < \beta < \frac{1}{2}$ and $D > 0, k$ are integers, then
\[
\sum' p \max_{y \leq x} \max_{(r, p) = 1} \left| \sum_{n \leq y \atop n \equiv k (\text{mod } D)} g(n) - \frac{1}{p-1} \sum_{n \leq y \atop n \equiv k (\text{mod } p)} g(n) \right|^2 \ll \frac{x^2}{(\log x)^{c_0}}.
\]
Here $'$ denotes that the moduli $p$ run over all the primes up to $x^\beta$, with the possible exception of at most $\phi(D)$.

**Proof.** This result is proved by Elliott [4].

The next two lemmas show that when considering the mean-value of $g(n)$, extra conditions which do not severely modify the range of integers $n$ or their prime divisors may be factored out. As will be seen, there is a tacit restriction upon the values of $g(p)$. With an appropriate reformulation of the lemmas, this restriction may be removed. Since that would require more extensive proofs and would not be of advantage in the present situation, I leave it to another occasion.

**Lemma 5.** Let
\[
L = \sum_{p \leq x} p^{-1} |1 - g(p)|.
\]
Then
\[
\sum_{n \leq x/w} g(n) = \frac{1}{w} \sum_{n \leq x} g(n) + O\left( \frac{x}{w} e^{L/8} \left( \frac{\log 2w}{\log x} \right)^{1/8} \right),
\]
uniformly for $1 \leq w \leq x$. 
Proof. Without loss of generality we may assume \(w > 1\). For complex \(s = \sigma + it, \sigma = \text{Re}(s)\), let \(G(s) = \sum g(n) n^{-s}\), with the summation over all positive integers \(n\). If necessary we extend the definition of \(g\) by setting it to be zero on the prime-powers greater than \(x\). For \(1 \leq y \leq x, x \geq 2\), there is a representation

\[
S(y) = \frac{1}{y} \sum_{n \leq y} g(n) \log n \log \frac{y}{n} = -\frac{1}{2\pi i} \int_{(\sigma_0)} \frac{G'(s) y^{s-1}}{s^2} \, ds,
\]

(4)

taken over the line \(\text{Re}(s) = \sigma_0 = 1 + (\log x)^{-1}\). By applying the method of Halász, as described in Elliott [1, Chap. 6], the contribution to this integral arising from the range \(|\tau| > (\sigma_0 - 1)^{-4}\) is seen to be \(\ll (\log x)^{-1}\).

On the intervals \(K(\sigma_0 - 1) < |\tau| \leq (\sigma_0 - 1)^{-4}\) we may employ the Euler product representations of \(G(s)\) and the Riemann zeta function \(\zeta(s)\) to obtain the bound

\[
G(s) \zeta(s)^{-1} \ll \exp \left( \sum_{p \leq x} \frac{1}{p^\sigma_0} \left| 1 - g(p) \right| \right) \ll e^L.
\]

Since \(\zeta(s)\) has a simple pole at \(s = 1\) and is \(\ll \log(2 + |\tau|)\) in the region \(\text{Re}(s) \geq 1, |\tau| > 1\) (Prachar [8, Satz 4.3, p. 64]), we have

\[
G(s) \ll e^L \left( \frac{1}{K(\sigma_0 - 1)} + \log \log x \right)
\]

there. As in Elliott [1, Chap. 6], the contribution towards \(S(y)\) coming from these intervals of \(\tau\)-values is seen to be

\[
\ll e^{L/3} \left( \frac{1}{K} + \frac{\log \log x}{\log x} \right)^{1/3} \log x.
\]

We may clearly suppose that \(1 < w < x/4\). We replace \(y\) in (4) by \(x/w\) and over the range \(\text{Re}(s) = \sigma_0, |\tau| \leq K(\sigma_0 - 1)\), estimate \(w^{-\tau - 1}\) by \(1 + O((1 + K) \log w/\log x)\). Since

\[
\int_{(\sigma_0)} \frac{\left| G'(s) \right|}{|s|^2} \frac{d\tau}{|\tau|^2} \ll \log x
\]

(cf. Elliott [1, Chap. 6, pp. 228 and 234–238]), we have

\[
S\left( \frac{x}{w} \right) = -\frac{1}{2\pi i} \int_{(\sigma_0)} \frac{G'(s) x^{s-1}}{s^2} \, ds
\]

\[
+ O \left( \log x \left( \frac{(1 + K) \log w}{\log x} + e^{L/3} \left( \frac{1}{K} + \frac{\log \log x}{\log x} \right)^{1/3} \right) \right).
\]
At the expense of a further error of the same type we may remove the condition $|\tau| \leq K(\sigma_0 - 1)$ in the integral. With the choice $K = (e^{L/3} \log x / \log w)^{3/4}$ this leads to the estimate

$$S \left( \frac{x}{w} \right) = S(x) + O \left( e^{L/4} \left( \frac{\log 2w}{\log x} \right)^{1/4} \log x \right).$$

To remove the weights $\log x/n$, $\log x/(wn)$ we consider

$$\frac{x}{e} (S(x) - (1 - \varepsilon) S(x(1 - \varepsilon))) = \sum_{n \leq x} g(n) \log n + O(\varepsilon x \log x + \log x)$$

uniformly for $0 < \varepsilon \leq \frac{1}{2}$. A similar representation holds with $x/w$ in place of $x$, since $x(1 - \varepsilon)/w \geq 2$. Choosing $\varepsilon$ suitable shows that

$$\frac{w}{x} \sum_{n \leq x/w} g(n) \log n = \frac{1}{x} \sum_{n \leq x} g(n) \log n + O \left( e^{L/8} \left( \frac{\log 2w}{\log x} \right)^{1/8} \log x \right)$$

provided $e^{L/8}(\log 2w/\log x)^{1/8} < \frac{1}{2}$. But if this restriction fails, then the inequality (5) is trivially valid.

To remove the weights $\log n$ we estimate these last sums crudely over the range $n < x/(w \log x)$, and over the range $x/(w \log x) < n \leq x$ replace $\log n$ by $\log x + O(\log(w \log x))$. This introduces an extra error which is $\ll \log(w \log x)$, and so within an error similar to that at (5).

This completes the proof of Lemma 5.

**Lemma 6.** Let

$$L = \sum_{p \leq x} p^{-1} |1 - g(p)|.$$

Then

$$\sum_{n \leq x} g(n) = \eta(D) \sum_{n \leq x} g(n) + O \left( \frac{xe^{L/3}(\log \log 3D)^2}{(\log x)^{1/8}} \right)$$

with

$$\eta(D) = \prod_{p \mid D} \left( 1 + \sum_{k \leq \log x / \log p} p^{-k} g(p^k) \right)^{-1}$$

holds uniformly for $x \geq 2$ and odd integers $D$. It holds for even integers also, provided

$$\left| 1 + \sum_{k \leq \log x / \log 2} 2^{-k} g(2^k) \right| \geq c_1 > 0.$$  

In this case the error term depends on $c_1$. 

Proof. For the proof of this lemma we note that the Dirichlet series which corresponds to the multiplicative function which is \( g(n) \) when \((n, D) = 1\), and \( = 0 \) otherwise, has the form \( \theta(s)^{-1} G(s) \), where

\[
\theta(s) = \prod_{p \mid D} \left( 1 + \sum_{k < \frac{\log x}{\log p}} p^{-k} g(p^k) \right).
\]

We need not continue the summation over \( k \) any further, because it will not affect the representations such as (4) when \( 1 \leq y \leq x \). In our present circumstances

\[
\left( \theta(s)^{-1} G(s) \right)' = \theta(s)^{-1} G'(s) - \theta(s)^{-2} \theta'(s) G(s).
\]

As in the proof of (3) we treat the integral involving \( \left( \theta(s)^{-1} G(s) \right)' \) by first reducing it to the range \(|\tau| \leq K(\sigma_0 - 1)\). On that range we split the integral into two pieces, corresponding to the above decomposition of the integrand.

In the integral of \( \theta(s)^{-1} G'(s) \) we replace \( \theta(s) \) by \( \theta(1) \). The elementary bounds

\[
\sum_{p \mid D} \frac{\log p}{p} \ll \log \log 3D, \quad \sum_{p \mid D} \frac{1}{p} \ll \log \log \log 3D + O(1)
\]

are useful in this step. For even \( D \) we apply the extra hypothesis (7) concerning the behavior of \( g \) on the powers of 2 to get an upper bound for \( \theta(s)^{-1} \) on the line segment \( \Re(s) = \sigma_0 \), \(|\tau| \leq K(\sigma_0 - 1)\) which is uniform in \( x \). Provided \( K(\sigma_0 - 1) \) does not exceed a certain constant dependent on \( c_1 \), we have

\[
\theta(s)^{-1} \ll \exp \left( \sum_{p \mid D} \frac{1}{p} \right) \ll \log \log 3D.
\]

Moreover, if

\[
K(\sigma_0 - 1) \sum_{p \mid D} \frac{\log p}{p} \ll 1, \quad (8)
\]

then the replacement of \( \theta(s) \) by \( \theta(1) \) introduces an error of

\[
\ll |\theta(1)|^{-1} K \log \log 3D \ll K(\log \log 3D)^2.
\]

We may now continue as in the proof of (3), save that a factor \( \theta(1)^{-1} = \eta(D) \) appears in the leading term and contributes an extra factor of \(|\theta(1)|^{-1}\) to the error terms.

There is a new error term introduced by the integral

\[
-\frac{1}{2\pi i} \int_{|\tau| = K(\sigma_0 - 1)} \frac{\theta'(s) G(s) x^{s-1}}{(\theta(s)s)^2} ds.
\]
For this range of $\tau$

$$\frac{\theta'(s)}{\theta(s)} \ll 1 + \sum_{p|D} \frac{\log p}{p} \ll \log \log 3D,$$

so that the factors of the integrand involving $\theta(s)$ are $\ll (\log \log 3D)^2$. An application of the Cauchy–Schwarz inequality and then of Plancherel's theorem (cf. Elliott [1, Chap. 6, p. 228]) shows that

$$\int_{(\sigma_0)} \frac{|G(s)|}{|s|^2} \frac{d\tau}{\tau} \ll (\log x)^{1/2},$$

and the extra error term is manageable.

We remove the weights $\log x/n$ and $\log n$ as for (3), there being no $\omega$. The division of cases is now between $(\log x)^{-1} \log \log 3D \ll 1$ so that we may choose $K = (\log x)^{3/4}$ and satisfy (8); and $(\log x)^{-1/4} \log \log 3D \gg 1$ when the assertion of (6) is trivially valid.

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Before proceeding I note that we may assume the additive functions $f_j$ in Theorem 1 to satisfy $f_j(uv) = f_j(u) + f_j(v)$ whether the integers $u, v$ are coprime or not, classically labelled completely additive.

For $j = 1, 2$, let $h_j$ be the additive function defined by $h_j(p^m) = m f_j(p)$, and let $H_x(z)$ be the distribution function corresponding to $F_x(z)$. For $\varepsilon > 0$ we can choose a positive $r$ so that the asymptotic density of those integers divisible by the square of a prime exceeding $r$, or a power $p^k$, with $k > r$, of a prime $p$ not exceeding $r$, is at most

$$\sum_{p \leq r} \frac{1}{p^r} \left(1 - \frac{1}{p^r}\right) + \sum_{p > r} \frac{1}{p^r} < \varepsilon.$$ 

Let

$$M = \sum_{j=1}^2 \sum_{p \leq r} \sum_{k \leq r} |f_j(p^k)|.$$ 

After removing a sequence of density at most $\varepsilon$, we see that on the remaining integers $|f_j(n) - h_j(n)| \leq M$, $j = 1, 2$. In particular

$$\limsup_{x \to \infty} (1 - H_x(z + 2M) + H_x(-z - 2M))$$

$$\leq \limsup_{x \to \infty} (1 - F_x(z) + F_x(-z)) + 2\varepsilon.$$
for each positive $z$. Letting $z \to \infty$ and $\varepsilon \to 0$ shows that if the $F_x(z)$ satisfy proposition (i) of Theorem 1, then so do the $H_x(z)$. Since the propositions (iii) for the $f_j$ and $h_j$ coincide, our assumption is justified.

Furthermore, we may assume the functions $f_j$ to be the same. Replacing $n$ with $n + 1$, if necessary, we may assume that $b \neq 0$. For each $\varepsilon > 0$ there is a positive $z_1$ so that for all large $x$ the interval $1 \leq n \leq x$ contains at least $(1 - \varepsilon)x$ integers for which

$$|f_1(an + b) - f_2(An + B) - \alpha(x)| \leq z_1.$$

Replacing $n$ by $bBn$ and employing the complete additivity of the $f_j$, the same may be said with respect to the inequality

$$|f_1(abn + 1) - f_2(ab(An + B) - \alpha(x))| \leq z_1 + |f_1(b)| + |f_2(B)|.$$

Arguing with $(ab + 1)n + 1$ in place of $n$ we see that there are at least $(1 - 2\varepsilon)x$ integers $n$ not exceeding $x$ for which we may further assert that

$$|f_1(abn + 1) - f_2(ab(ab + 1)n + ab + 1) - \alpha(x)| \leq z_2,$$

with $z_2 = z_1 + |f_1(b)| + |f_2(B)| + |f_1(ab + 1)|$. Elimination of the $f_1$ term between these last two inequalities gives

$$|f_2(abn + 1) - f_2(ab(ab + 1)n + ab + 1)| \leq 2z_2.$$

Since

$$\det \begin{pmatrix} Ab & 1 \\ Ab(ab + 1) & Ab + 1 \end{pmatrix} = Ab(ab - ab) \neq 0,$$

our second assertion is justified.

These specializations will not play a vital rôle in the following analysis, but will allow some simplification.

5

Theorem 1 is largely contained in the following result.

**Lemma 7.** Let $a > 0$, $A > 0$, $b$, $B$ be integers for which $\Delta = ab - Ab \neq 0$, $x \geq 2$ real. Then there are positive constants $c_2 < 1$, $c_3$ with the following property:

Let $f$ be a completely additive function for which $g(n) = \exp(itf(n))$ satisfies

$$\sup_{n \leq y \leq x} \sum_{n \leq y} |g(an + b) - e^{it\tau(y)}g(An + B)| \leq \delta$$
for some real \( v(y) \) and some \( \delta \) in the range \((\log x)^{-c_2} \leq \delta \leq \frac{1}{3}\), uniformly for \( |t| \leq t_0, t_0 > 0 \). Assume further that
\[
\sum_{p \leq x} p^{-1} |1 - g(p)|^2 \leq \frac{1}{2} \log \frac{1}{\delta}
\]
holds uniformly for \(|t| \leq t_0\). Then there is real \( \mu, |\mu t_0| \leq 1 \), so that
\[
\sum_{p \leq x} p^{-1} |1 - g(p) p^{it}|^2 \leq c_3
\]
holds uniformly for \(|t| \leq t_0\). The constant \( c_3 \) depends at most on the four initial integers; \( c_2 \) is absolute.

Note that \( c_3 \) does not depend on \( t_0 \).

The proof of Lemma 7 occupies the next four sections. In fact it is instructive to begin with the more general hypothesis.

\[
\sup \min y^{-1} \sum_{n \leq y} |g_1(an+b) - \gamma g_2(An+B)|^2 \leq \delta,
\]
where \( \lambda \) is a positive real and the \( g_j \) are completely multiplicative functions with values on the unit circle. We employ a dual of the Turán–Kubilius inequality.

**Lemma 8.** The inequality
\[
\sum_{p \leq x} p \left| \sum_{n \leq x} d_n - \frac{1}{p} \sum_{n \equiv 0 \pmod{p}} d_n \right|^2 \leq 16x \sum_{n \leq x} |d_n|^2
\]
holds uniformly for all \( x \geq 1 \) and complex numbers \( d_n, 1 \leq n \leq x \).

**Proof.** This is Lemma (4.7) of Elliott [1].

We replace \( d_n \) by zero unless \( n \) is of the form \( am+b \). Straightforward applications of the Cauchy–Schwarz inequality show that if we increase the constant 16, then we may replace
\[
p^{-1} \sum_{n \leq x} d_n \quad \text{by} \quad (p-1)^{-1} \sum_{n \leq x, (Am+B,p) = 1} d_n.
\]
In the resulting inequality we set \( d_n = g_1(am+b) - \gamma g_2(An+B) \). Then
\[
\sum_{p \leq x} p \left| \sum_{am+b \leq x} g_1(am+b) - \frac{1}{p-1} \sum_{am+b \equiv 0 \pmod{p}} g_1(am+b) - \gamma \psi(p) \right|^2 \leq x^2 d(x) \quad (9)
\]
with

\[ d(x) = \frac{c_4}{x} \sum_{am+b \leq x} |g_1(am+b) - \gamma g_2(2Am+B)|^2 \]

for a certain constant \(c_4\), and

\[ \psi(p) = \sum_{am+b \leq x \atop am+b \equiv 0 \pmod{p}} g_2(2Am+B) \frac{1}{p-1} \sum_{am+b \leq x \atop (Am+B,p) = 1} g_2(2Am+B). \]

If a prime \(p\) divides both \(am+b\) and \(2Am+B\), then it divides \(\Delta = aB - Ab = a(2Am + B) - A(am+b)\). By fixing \(\beta, 0 < \beta < \frac{1}{2}\), and reducing the summation in (9) to primes \(p \leq x^{1/2}\) with possibly \(2|AA|\) removed, including those which divide \(a\), we may apply Lemma 4 to obtain

\[ \sum_{p \leq x^{1/2}} \left| \sum_{am+b \leq x \atop am+b \equiv 0 \pmod{p}} g_1(am+b) \right|^2 \leq x^2 d(x) + c_5 x^2 (\log x)^{-c_6} \]

for certain constants \(c_6\) (absolute) and \(c_5\) (depending upon \(a, b, A, B,\) and \(\lambda\)). Along the lines of Elliott [2, Chap. 8], I have removed one of the multiplicative functions from consideration. For ease of notation I drop the suffix 1.

The second inner sum at (10) can be expressed in the form

\[ \sum_{n \leq x \atop n \equiv b \pmod{a}} g(n) - \sum_{n \leq x \atop n \equiv b \pmod{a}} g(n). \]

At the expense of removing at most \(q(|A|)\) further prime moduli from the summation at (10), we may again apply Lemma 4 to replace this last sum by

\[ \frac{1}{p-1} \sum_{n \leq x \atop (n,p) = 1} \frac{g(n)}{p-1} \sum_{n \leq x \atop n \equiv b \pmod{a}} g(n) \frac{1}{p-1} \sum_{pm \leq x \atop pm \equiv b \pmod{a}} g(pm). \]

Further removing the prime \(p=2\) from the moduli at (10) we see that by adjusting the constant \(c_5\), if necessary,

\[ \sum_{p \leq x^{1/2}} \frac{1}{p} \left| g(p) \sum_{m \leq x/p \atop pm \equiv b \pmod{a}} g(m) - \sum_{n \leq x \atop n \equiv b \pmod{a}} g(n) \right|^2 \leq 2 d(x) + c_5 (\log x)^{-c_6}. \]
We define \( D = (a, b), a_1 = aD^{-1}, b_1 = bD^{-1} \),
\[
M(y, r) = y^{-1} \sum_{n \leq y \atop n \equiv r (\text{mod } a_1)} g(n), \quad M(y) = y^{-1} \sum_{n \leq y} g(n).
\]
For \( (p, a) = 1 \) we may factor \( g(D) \) from a typical inner sum
\[
\frac{p}{x} \sum_{m \leq x/p \atop pm \equiv b (\text{mod } a)} g(m) = \frac{p}{x} \sum_{w \leq x/(pD)^{-1} \atop w = bp_1 (\text{mod } a_1)} g(D) M \left( \frac{x}{pD}, \tilde{p}b_1 \right),
\]
where \( p\tilde{p} \equiv 1 (\text{mod } a_1) \). In this way, with possibly further adjustments to the constants \( c_5 \) and \( c_6 \), we reach
\[
\sum_{p \leq x^\beta} \left| \frac{1}{p} \left( g(p) M \left( \frac{x}{pD}, \tilde{p}b_1 \right) - M \left( \frac{x}{D}, b_1 \right) \right) \right|^2 \leq 2D^2 d(x) + c_5 (\log x)^{-\epsilon_0}. \tag{11}
\]
Here the prime moduli \( p \) run over all those not exceeding \( x^\beta \), with at most \( 4|aA| \) omitted, including those which divide \( 2aA \). Our aim is to vary \( x \) and solve this approximate functional equation for the \( M(y, r) \) as far as possible. To some extent these \( \varphi(a_1) \) unknown functions may be reduced to a single one.

Let \( \rho(x) \) denote the upper bound at (11). Without loss of generality I shall assume that \( 12c_6 \leq 1 \). For each \( r \) prime to \( a_1, r \neq 1 \), let \( p_r \) be the first prime \( p \equiv r (\text{mod } a_1) \) counted in the sum " at (11). Set \( p_1 = 1 \) and denote by \( p_0 \) the maximum of the complete set of \( p_r \). Note that as a consequence of Lemmas 3 and 4, the condition " may be assumed to remove the same moduli even when \( x \) in (11) is replaced by any \( x_1, \exp((\log x)^{\eta/10}) \leq x_1 \leq x \).

**Lemma 9.** The estimate
\[
M \left( \frac{x}{p_rD}, \tilde{r}b_1 \right) = g(p_r) M \left( \frac{x}{D}, b_1 \right) + O(\rho(x))
\]
holds uniformly in \( r \) prime to \( a_1 \).

**Proof.** It follows from (11) that
\[
\left| g(p_r) M \left( \frac{x}{p_rD}, \tilde{r}b_1 \right) - M \left( \frac{x}{D}, b_1 \right) \right| \leq p_r \rho(x).
\]

In the solution of the approximate functional equation (11) we employ the following result.

**Lemma 10.** For real \( Q \geq 2, K \geq 10 \) let \( a_n \) be complex numbers, \( |a_n| = 1 \),
one for each prime \( p \equiv r(\text{mod } b) \), \( (r, b) = 1 \), \( b \geq 1 \), in the range \( Q < p \leq Q^K \).

Let \( \psi(w) \) be a complex-valued function, defined for real \( w \), \( 1 \leq w \leq Q^{2K} \), which satisfies

\[
|\psi(w') - \psi(w)| \leq \varepsilon, \quad 1 \leq w \leq w' \leq w(1 + Q^{-1/3}) \leq Q^{2K}
\]

and

\[
\sum_{1 \leq w \leq Q^K} \frac{1}{p} |a_p \psi(wp) - \psi(w)| \leq \varepsilon \log K, \quad 1 \leq w \leq Q^K.
\]

Then there is a real number \( \alpha, |\alpha| \leq Q \), for which

\[
\psi(w) = \psi(1) w^{i\alpha} + O(\varepsilon)
\]

with an implied constant that depends at most on \( b \), uniformly for \( 1 \leq w \leq Q \).

**Proof.** This is Lemma 2 of Hildebrand [5], save that I have incorporated the condition \( p \equiv r(\text{mod } b) \). For this I employ the estimate for the number of such primes in short intervals provided by Prachar [8, Satz (3.2), p. 323], bearing in mind the remarks which he makes following the statement of Satz (3.3) on the same page. Lemma 10 represents a multiplicative variant of the approximate functional equation appearing in Elliott [2, Chap. 9].

I shall apply Lemma 10 to (11) by applying Lemma 9 to essentially reduce the number of unknown functions to one. There is an alternative procedure. One may establish a version of Lemma 10 involving an equation.

\[
\sum_{1 \leq w \leq Q^K} \frac{1}{p} |a_p \rho(wp) - \psi(w)| \leq \varepsilon \log K
\]

with two unknown functions \( \rho \) and \( \psi \), and the primes \( p \) restricted to a reduced residue class (mod \( a, \)). This we may then apply \( \phi(a, \) \) times. In these matters the remark made in Elliott [2, Chap. 9, p. 184] is relevant.

We set \( Q = x^{1/40}, K = 10 \). After an application of the Cauchy–Schwarz inequality and the elementary bound

\[
\sum_{Q < p \leq Q^K} \frac{1}{p} = \log K + O((\log Q)^{-1}),
\]

we deduce from (11) that, provided \( \beta \geq \frac{1}{3} \),

\[
\sum_{Q < p \leq Q^K} \frac{1}{p} g(p) M \left( \frac{x}{tpD}, \bar{p}b_1 \right) - M \left( \frac{x}{tD^2} b_1 \right) \ll \left( d \left( \frac{x}{t} \right) + (\log x)^{\gamma} \right)^{1/2}
\]

uniformly for \( 1 \leq t \leq Q^K \).
Setting $\psi(w) = M(x/(Dw), b_1)$ and restricting ourselves to the primes $p \equiv 1 \pmod{a_1}$, we see that condition (13) of Lemma 10 is satisfied with $a_p = g(p)$ and

$$c = c_7 \left( \sup_{x^{1/2} \leq y \leq x} d(y)^{1/2} + (\log x)^{-c_6/2} \right).$$

Moreover, we have trivially that

$$M(y(1 + \theta), b_1) = \frac{1}{1 + \theta} M(y, b_1) + O \left( \frac{\sum}{y, y < n \leq x(1 + \theta)} 1 \right)$$

$$= M(y, b_1) + O(y^{-1} + \theta)$$

uniformly for $y \geq 1$ and $0 \leq \theta \leq 1$. By adjusting upwards the value of the constant $c_7$, if necessary, the condition (12) will also be satisfied. Alternatively, we may apply (3) of Lemma 5. We have therefore the representation

$$M \left( \frac{x}{wD}, b_1 \right) = M \left( \frac{x}{D}, b_1 \right) w^{\alpha x} + O(\varepsilon) \quad (14)$$

with $|\alpha| \leq x^{1/40}$, uniformly for $1 \leq w \leq x^{1/40}$.

**Lemma 11.** For each $x \geq 2$ there is a real $\alpha$, $|\alpha| \leq 1$, so that

$$M \left( \frac{x}{p, wD}, \bar{r}b_1 \right) = M \left( \frac{x}{p, D}, \bar{r}b_1 \right) w^{\alpha x} + O \left( \sup_{x^{1/2} \leq y \leq x} d(y)^{1/2} + (\log x)^{-c_6} \right)$$

$$= M \left( \frac{x}{D}, b_1 \right) w^{\alpha x} + O(\varepsilon) \quad (15)$$

and

$$M \left( \frac{x}{wD} \right) = M \left( \frac{x}{D} \right) w^{\alpha x} + O \left( \sup_{x^{1/2} \leq y \leq x} d(y)^{1/2} + (\log x)^{-c_6} \right) \quad (16)$$

hold uniformly for $(r, a_1) = 1$, $1 \leq w \leq x^{1/40}$. Moreover

$$M \left( \frac{x}{D}, b_1 \right) = \gamma M \left( \frac{x}{D}, b_1 \right) + O(\varepsilon_1) \quad (17)$$

with

$$\gamma = \prod_{p \mid a_1} \left( 1 - \frac{g(p)}{p} \right)^{-1} \sum_{\substack{r = 1 \\ (r, a_1) = 1}}^{a_1} g(p_r) \left( \frac{p_0}{p_r} \right)^{\alpha x}$$
and \( \varepsilon_1 \) the upper bound appearing at (15). In particular \( |\gamma| \leq a_1 \). The implied constants depend at most on \( a_1 \).

**Proof.** In view of (14) the asymptotic estimate (15) follows from Lemma 9.

For each \( r \) prime to \( a_1 \), by (15), and Lemma 9,

\[
M \left( \frac{x}{p_0 D}, \tilde{r}b_1 \right) = \left( p_0 \right)^{i_p} M \left( \frac{x}{p_r D}, \tilde{r}b_1 \right) + O(\varepsilon_1)
\]

\[
= g(p_r) \left( \frac{p_0}{p_r} \right)^{i_p} M \left( \frac{x}{D}, b_1 \right) + O(\varepsilon_1).
\]

By means of this estimate there is a representation

\[
\left( \sum_r g(p_r) \left( \frac{p_0}{p_r} \right)^{i_p} \right) M \left( \frac{x}{D}, b_1 \right) = \sum_{r=1 \atop (r, a_1) = 1}^{a_1} M \left( \frac{x}{p_0 D}, \tilde{r}b_1 \right) + O(\varepsilon_1).
\]

The sum on the right-hand side has the alternative form

\[
\frac{p_0 D}{x} \sum_{n \leq x/(p_0 D)} g(n)
\]

which we estimate by applying Lemmas 6 and 5. To this end we note that if the constant \( c_2 \) in the statement of Lemma 7 is given a sufficiently small value, then by the Cauchy–Schwarz inequality

\[
L \leq \left( \sum_{r \leq x} \sum_{p \leq x} \frac{|1 - g(p)|^2}{p} \right)^{1/2} \leq \frac{1}{40} \log \log x + O(1).
\]

In particular, \( \exp(L/3) \ll (\log x)^{1/100} \). Hence the above sum can be estimated by

\[
\prod_{p | a_1} \left( 1 - \frac{g(p)}{p} \right) M \left( \frac{x}{D} \right) + O(\log x)^{-1/9},
\]

and we have established (17).

The estimate (16) follows from (17) together with the case \( r = 1 \) of (15), again employing Lemma 5. It remains to establish the bound on \( x \). This I shall do following Lemma 12.

It is interesting to compare (16) of Lemma 11 with (3) of Lemma 5. According to that result

\[
M \left( \frac{x}{w} \right) = M(x) + O \left( e^{L/8} \left( \frac{\log 2w}{\log x} \right)^{1/8} \right)
\]
uniformly for \(1 \leq w \leq x\). This is useful only if \(w\) does not reach a power of \(x\). Lemma 11, which is reached via the dual of the Turán–Kubilius inequality, asserts a result with a much larger useful uniformity in \(w\). It is paid for by introducing the requirement that the mean differences \(d(x)\) be small, in order to gain something from applying the Turán–Kubilius dual; moreover, the parameter \(\alpha\) is only partially located.

It should be remarked that if \(a, b, A, B\) have highest common factor 1, then either \(a_1\) or \(A(A, B)^{-1}\) will not be a multiple of the exceptional modulus \(D_0\) of Lemma 3. Then without loss of generality

\[
M(x, r) = \frac{1}{\phi(a_1)} \prod_{p | a_1} \left(1 - \frac{g(p)}{p}\right) M(x) + O(\varepsilon_1)
\]

holds uniformly for \((r, a_1) = 1\).

**Lemma 12.** If \(\delta\) is sufficiently small and \(x\) is sufficiently large (in terms of the integers \(a, b, A, B\) and of \(\lambda\)), then

\[
\left| M\left(\frac{x}{D}\right) \right| \geq \delta^{1/4}.
\]

**Proof.** Provided the constant \(c_2\) in the statement of Lemma 7 is taken sufficiently (absolutely) small, it follows from the second part of Lemma 11 that

\[
|M(u)| - |M(v)| \leq \delta
\]

uniformly for \(v^{39/40} \leq u \leq v, x^{36} \leq v \leq x/D\). Arguing by induction we deduce that

\[
|M(y)| - \left| M\left(\frac{x}{D}\right) \right| \leq \delta \log \frac{1}{\delta} \leq \delta^{3/4}
\]

for \(x^{39} \leq y \leq x/D\).

Consider now the Dirichlet series

\[
G(\sigma) = \sigma \int_1^\infty M(y) y^{-\sigma} dy
\]

with \(\sigma = 1 + (\sqrt{\delta} \log x)^{-1}\). The ranges \(1 \leq y \leq x^{36}, y > x/D\) trivially contribute

\[
\leq \delta \log x + \frac{x^{1-\sigma}}{\sigma - 1} \leq (\sigma - 1)^{-1} (\sqrt{\delta} + e^{-1/\sqrt{\delta}})
\]
to the integral. The above estimate for \( |M(y)| \) shows that the range \( x^{3\delta} \leq y \leq x/D \) contributes

\[
\ll (\sigma - 1)^{-1} \left( \left| M \left( \frac{x}{D} \right) \right| + \delta^{3/4} \right).
\]

In terms of the Riemann zeta function \( \zeta(\sigma) \) we can express this in the form

\[
|G(\sigma)| \zeta(\sigma)^{-1} \ll \left| M \left( \frac{x}{D} \right) \right| + \delta^{1/2}.
\]

The ratio of these Dirichlet series has the alternative representation

\[
\exp \left( - \sum_{\nu} \frac{1}{\nu^{\sigma}} (1 - \text{Re} g(p)) + O(1) \right)
\]

derived from their Euler products. The well-known Chebyshev estimate \( \ll w/\log w \) for the number of primes up to \( w \) shows that the terms in this exponential with \( p > x^{\sqrt{\delta}} \) contribute a bounded amount. The terms with \( p \leq x^{\sqrt{\delta}} \) contribute

\[
\frac{1}{2} \sum_{p \leq x^{\sqrt{\delta}}} \frac{1}{p^{\sigma}} \left| 1 - g(p) \right|^2 \ll \frac{1}{4} \log \frac{1}{\delta}
\]

by an hypothesis of Lemma 7.

This completes the proof of Lemma 12. It is a consequence of this result, (17), and Lemma 9 that \( |M(x/p, D, r^{\alpha})| \gg \delta^{1/4} \) for each \( r \) prime to \( a_1 \).

We can now complete the proof of Lemma 11. Suppose (to the contrary) that \( |\alpha| > 1 \). From (16) with \( w = 1 + (2|\alpha|)^{-1} \) and (3) of Lemma 5 (continuing in the notation of Lemma 11),

\[
M \left( \frac{x}{D} \right) \left[ \left( 1 + \frac{1}{2|\alpha|} \right)^{\alpha} - 1 \right] \ll \varepsilon_1 \ll \delta.
\]

Here the coefficient of \( M(x/D) \) has the absolute value

\[
2 \left| \sin \frac{|\alpha|}{2} \log \left( 1 + \frac{1}{2|\alpha|} \right) \right| \geq \frac{2|\alpha|}{\pi} \log \left( 1 + \frac{1}{2|\alpha|} \right) \geq \frac{1}{2\pi},
\]

so that \( M(x/D) \ll \delta \). For sufficiently small \( \delta \) this contradicts the assertion of Lemma 12.
In this section the mean-values $M$ in the inequality (11) will be largely factored out.

**Lemma 13.** For each $r$ prime to $a_1$, there is a $w_r$, $|w_r| = 1$, so that

$$\sum_{\rho \equiv r \pmod{a_1}} \frac{1}{\rho} |1 - w_r g(p) p^{|\rho|}|^2 \ll 1.$$ 

Moreover, there is a continuous real function $\psi(t)$ so that $w_r = \exp(i\psi(t)) + O(\delta^{3/4})$ uniformly for $|t| \leq t_0$.

**Proof.** With $x_1 = x^{1/20}$, define

$$\tilde{M}(y, r) = \frac{1}{\log x_1} \int_{r_1}^{x_1} M\left(\frac{y}{w}, r\right) w^{-r\alpha} dw, \quad y > 0,$$

formally adopting the representation of Lemma 11. The integral represents an averaging on the multiplicative group of positive reals. Set $z = x/(p_0 D)$. In view of Lemmas 9 and 11

$$\tilde{M}(z, r) = \frac{1}{\log x_1} \int_{r_1}^{x_1} (M(z, r) + O(\delta)) \frac{dt}{t} = M(z, r) + O(\delta) \quad (18)$$

so that $\tilde{M}(z, r)$ and $M(z, r)$ are close for each $r$ prime to $a_1$. In particular, $\delta^{1/4} \ll |\tilde{M}(z, r)| \ll |\tilde{M}(z, b_1)|$.

If now $1 \leq y \leq x_1$, then

$$\tilde{M}\left(\frac{z}{y}, r\right) = \frac{1}{\log x_1} \int_{r_1}^{x_1} M\left(\frac{z}{t}, r\right) t^{-r\alpha} dt - \frac{y^{r\alpha}}{\log x_1} \int_{r_1}^{x_1} M\left(\frac{z}{t}, r\right) t^{-r\alpha} dt$$

$$= \tilde{M}(z, r) y^{r\alpha} + O\left(\frac{\log y}{\log x_1} \max_{1 \leq t \leq x_1} |M\left(\frac{z}{t}, r\right)|\right)$$

$$= \tilde{M}(z, r) y^{r\alpha} + O\left(\frac{\log y}{\log x_1} |\tilde{M}(z, r)|\right) \quad (19)$$

uniformly in $r$ prime to $a_1$.

The functions $\tilde{M}(y, r)$ satisfy an approximate functional equation similar to that satisfied by the $M(y, r)$:

$$\sum_{\rho \equiv x^{1/3} p} \frac{1}{p} g(p) \tilde{M}\left(\frac{x}{pD}, \rho b_1\right) - \tilde{M}\left(\frac{x}{D}, b_1\right)^2 \ll \delta. \quad (20)$$
Indeed
\[ \left| g(p) \tilde{M} \left( \frac{x}{tpD}, r \right) - \tilde{M} \left( \frac{x}{D}, b_1 \right) \right|^2 \]
\[ = \left| \frac{1}{\log x_1} \int_1^{x_1} \left( g(p) M \left( \frac{x}{tpD}, r \right) - M \left( \frac{x}{tD}, b_1 \right) \right) t^{-ix} \frac{dt}{t} \right|^2 \]
\[ \leq \frac{1}{\log x_1} \int_1^{x_1} \left| g(p) M \left( \frac{x}{tpD}, r \right) - M \left( \frac{x}{tD}, b_1 \right) \right|^2 \frac{dt}{t}. \]

Setting \( r = \tilde{p}b_1 \) for the prime \( p \), multiplying by \( p^{-1} \), and summing over the range \( p \leq x^{1/3} \) with the restriction of inequality (11), we bound the sum at (20) by
\[ \leq \frac{1}{\log x_1} \int_1^{x_1} \sum_{p \leq x^{1/3}} \frac{1}{p} \left| g(p) M \left( \frac{x}{tpD}, pb_1 \right) - M \left( \frac{x}{tD}, b_1 \right) \right|^2 \frac{dt}{t} \]
\[ \leq \sup_{1 \leq t \leq x^{1/3}} d \left( \frac{x^2}{t} \right) + (\log x)^{-\epsilon}, \]
provided \( \beta \geq \frac{3}{2} \), say.

We apply the Cauchy–Schwarz inequality for each prime \( p \):
\[ \left| \tilde{M} \left( z, b_1 \right) - g(p) p^{i\alpha} \tilde{M} \left( \frac{z}{p}, \tilde{p}b_1 \right) \right|^2 \]
\[ \leq 2 \left| \tilde{M} \left( z, b_1 \right) - g(p) \tilde{M} \left( \frac{z}{p}, \tilde{p}b_1 \right) \right|^2 + 2 \left| \tilde{M} \left( \frac{z}{p}, \tilde{p}b_1 \right) - p^{i\alpha} \tilde{M} \left( z, \tilde{p}b_1 \right) \right|^2. \]

Restricting ourselves to the primes \( p \equiv r \pmod{a_1} \) we see from (20), (19), and (18) that
\[ \sum_{\substack{p \leq x_1 \\ p \equiv r \pmod{a_1}}} \frac{1}{p} \left| \tilde{M} \left( z, b_1 \right) - g(p) p^{i\alpha} \tilde{M} \left( \frac{z}{p}, \tilde{p}b_1 \right) \right|^2 \]
\[ \leq \delta + \sum_{\substack{p \leq x_1 \\ p \equiv r \pmod{a_1}}} \left( \frac{\log p}{\log x_1} \right) \left| \tilde{M} \left( z, \tilde{r}b_1 \right) \right|^2 \leq \left| \tilde{M} \left( z, b_1 \right) \right|^2. \] (21)

Since \( |u| - |v| \leq |u - v| \), a similar upper bound is obtained for the sum
\[ \sum_{\substack{p \leq x_1 \\ p \equiv r \pmod{a_1}}} \frac{1}{p} \left| \left| \tilde{M} \left( z, b_1 \right) \right| - \left| \tilde{M} \left( z, \tilde{r}b_1 \right) \right| \right|^2. \]
Define \( w_r \) to be \( \exp(i(\arg \hat{M}(z, \bar{r}b_1) - \arg \hat{M}(z, b_1))) \). We see that

\[
\sum_{\substack{p \leqslant x \leqslant x_1 \\ p \equiv r \pmod{a_1}}} \frac{1}{p} |w_r \hat{M}(z, b_1) - \hat{M}(z, \bar{p}b_1)|^2 \ll |\hat{M}(z, b_1)|^2,
\]

which together with (21) gives

\[
|\hat{M}(z, b_1)|^2 \sum_{\substack{p \leqslant x \leqslant x_1 \\ p \equiv r \pmod{a_1}}} \frac{1}{p} |1 - w_r g(p) p^{\alpha_2}|^2 \ll |\hat{M}(z, b_1)|^2.
\]

Since \( |M(z, b_1)| \gg |M(x/D, b_1)| \gg \delta^{1/4} \), this justifies the first assertion of Lemma 13, for the condition \( x \) may clearly be omitted, and the extra range \( x_1 \leqslant p \leqslant x \) contributes a bounded amount. The second assertion, concerning \( w_r \), is justified by defining a continuous \( \psi(t) \) so that \( \exp(i\psi(t)) = \exp(i(\arg M(z, \bar{r}b_1) - \arg M(z, b_1))) \) and applying (18).

This completes the proof of Lemma 13.

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Until now the parameter \( r \) has played no rôole whatsoever in the argument, which could have been carried out over any set of \( t \)-values, including a single value. In this and the following section, I specialize \( g \) to \( \exp(itf(n)) \), and examine the dependence of the parameters \( M \), and \( \alpha \) on \( t \). I employ approximate functional equations.

**Lemma 14.** For \( s = \sigma + it \), \( \sigma = \text{Re}(s) \),

\[
\log L(s, \chi) \ll \log \log(|t| + 4)
\]

in \( \sigma \geqslant 1 \) if \( \chi \) is a non-principal Dirichlet character, and in the region \( \sigma \geqslant 1, |t| \geqslant \tau_0 > 0 \) if \( \chi \) is a principal character. The implied constant then depends upon \( \tau_0 \).

We define \( \log L(s, \chi) \) continuously along the path \( 2 \rightarrow 2 + it \rightarrow \sigma + it \), with the principal value of the logarithm at \( s = 2 \).

The assertion of Lemma 14 is not uniform in the modulus of \( \chi \).

**Proof.** In what follows we may clearly assume \( |t| \) to be sufficiently large. A classical result in analytic number theory asserts that for a suitable positive \( c \), \( L(s, \chi) \neq 0 \) in the disc \( |s - 1 - it| \leqslant c/\log |t| \). It is anyway non-zero in the half plane \( \text{Re}(s) > 1 \). Moreover, within this same disc \( L(s, \chi) \ll \log |t| \) (Prachar [8, Chap. IV, Sect. 5, in particular Satz (5.4)]).
ADDITIVE ARITHMETIC FUNCTIONS

Let \( s_0 = 1 + c(4 \log |\tau|)^{-1} + it \). Then

\[
\log L(s_0, \chi) = \log |L(s_0, \chi)| + \arg L(s_0, \chi)
\]

\[
\ll -\log |\text{Re}(s_0) - 1| - \sum_p \arg(1 - \chi(p) p^{-s_0}) + O(1)
\]

\[
\ll \log \log |\tau| + \sum_p p^{-\text{Re}(s_0)} \ll \log \log |\tau|.
\]

We now apply the Borel–Carathéodory theorem (e.g., Prachar [8, Satz (4.2), p. 383]) to deduce that \( \log L(s, \chi) \ll \log \log |\tau| \) holds within the smaller disc \(|s - s_0| \leq c(4 \log |\tau|)^{-1}\). This gives the asserted bound for the region \( 1 \leq \sigma \leq 1 + c(4 \log |\tau|)^{-1} \). For \( \sigma > 1 + c(4 \log |\tau|)^{-1} \) the above elementary treatment of \( L(s_0, \chi) \) leads to a similar bound.

**Lemma 15.** Let \( k \geq 1, r \) be comprime integers, \( x \geq 2 \). Suppose that

\[
\sum_{p \leq x, \, p \equiv r (\text{mod } k)} \frac{1}{p} |1 - \eta p^{i\beta}|^2 \leq d
\]

holds for a complex \( \eta, |\eta| \leq 1 \), and real \( \beta, |\beta| \leq (\log x)^{d_1} \). Then either

\[
\log x \leq \exp(6d \phi(k)^2)
\]

or

\[
|\beta| \log x \leq e^{2d \phi(k)^2} \quad \text{and} \quad |1 - \eta|^2 \log \log x \leq e^{4d \phi(k)^2}.
\]

The implied constants depends only upon \( k \) and \( d_1 \).

The same conclusion may be drawn with a rather weaker upper bound restriction on \( |\beta| \).

**Proof.** Let \( \sigma = 1 + (\log x)^{-1} \). The hypothesis implies that

\[
\sum_{p \equiv r (\text{mod } k)} \frac{1}{p^\sigma} |1 - \eta p^{i\beta}|^2 \leq d + O(1).
\]

Introducing the Dirichlet characters (mod \( k \)) we can express this in the form

\[
\sum_{\chi (\text{mod } k)} \overline{\chi}(r) \sum_{p} (1 + |\eta|^2 - \eta p^{i\beta} - \overline{\eta} p^{-i\beta}) \chi(p) p^{-\sigma} \leq d \phi(k) + O(1).
\]

Since we are in the half-plane \( \text{Re}(s) > 1 \), each of the series \( L(s, \chi) \) has an Euler product, so that we may further write

\[
\sum_{\chi (\text{mod } k)} \overline{\chi}(r) \left( \{1 + |\eta|^2\} \log L(\sigma, \chi) - \eta \log L(\sigma - i\beta, \chi) - \overline{\eta} \log L(\sigma + i\beta, \chi) \right) \leq d \phi(k) + O(1).
\]
If $|\beta| > 1$, then by Lemma 14 the terms with a non-principal character $\chi$ contribute $\ll \log \log \log x$. Since for the principal character $\chi_0$, $\log L(\sigma, \chi_0) = -\log(\sigma - 1) + O(1) = \log \log x + O(1)$, we reach $\log x \ll \exp(2d \varphi(k))$. We may therefore assume that $|\beta| \leq 1$, so that our main inequality further simplifies to

$$(1 + |\eta|^2) \log \frac{1}{\sigma - 1} - \eta \log \zeta(\sigma - i\beta) - \bar{\eta} \log \zeta(\sigma + i\beta) \leq d \varphi(k) + O(1).$$

On the line-segment $\Re(s) = \sigma$, $2(\sigma - 1) \leq \tau \leq \Im(s) \leq 1$, we have

$$\frac{\zeta'}{\zeta}(\sigma + iw) = -\frac{1}{\sigma + iw - 1} + O(1) = -\frac{1}{iw} + O\left(\frac{\sigma - 1}{w^2 + 1}\right).$$

Thus by integration

$$\log \zeta(\sigma + i\tau) - \log \zeta(\sigma + i) = -\int_1^{i\tau} \frac{dw}{w} + O\left((\sigma - 1)^{\int_1^{i\tau} \frac{dw}{w^2 + 1}}\right)$$

and

$$\log \zeta(\sigma + i\tau) = \log \frac{1}{|\tau|} + O(1).$$

A similar argument can be made for negative values of $\tau$, so that this estimate holds whenever $2(\sigma - 1) \leq |\tau| \leq 1$.

Altogether, if $|\beta| \geq 2(\sigma - 1)$, then

$$(1 + |\eta|^2) \log \frac{1}{\sigma - 1} - 2\Re \eta \log \frac{1}{|\beta|} \leq d\varphi(k) + O(1).$$

If, further, $\Re \eta \leq 0$, then again $\log x \ll \exp(2d \varphi(k))$. Without loss of generality $\Re \eta > 0$, and we write

$$|1 - \eta|^2 \log \frac{1}{\sigma - 1} + 2\Re \eta \log \frac{\beta}{\sigma - 1} \leq d \varphi(k) + O(1).$$

It follows that

$$|1 - \eta|^2 \log \log x \leq d \varphi(k) + O(1).$$

Applying the Cauchy–Schwarz inequality we may now replace $\eta$ in the hypothesis by 1, provided that $d$ is inflated to $2d(1 + \varphi(k)) + O(1)$. Retracing our steps, this time with $\eta = 1$, we reach

$$\frac{\beta}{\sigma - 1} = \exp\left(\log \frac{\beta}{\sigma - 1}\right) \leq \exp(2d \varphi(k^2)).$$
Moreover, if $|\beta| \geq 2(\sigma - 1)$ fails, then we obtain a similar inequality. This gives one of the remaining assertions of Lemma 15.

With the estimate for $|\beta|$ to hand a further application of the Cauchy–Schwarz inequality enables us to replace $\beta$ in the hypothesis by zero:

$$|1 - \eta|^2 \sum_{p \leq x \atop p \equiv r(\text{mod } k)} \frac{1}{p}$$

$$\leq 2 \sum_{p \leq x \atop p \equiv r(\text{mod } k)} \frac{1}{p} |1 - \eta p^{\beta}|^2 + 2 \sum_{p \leq x \atop p \equiv r(\text{mod } k)} \frac{|\beta \log p|^2}{p} \leq \exp(4d \varphi(k)^2).$$

An application of a sufficiently strong version of Dirichlet's theorem on primes in arithmetic progression (e.g., Prachar [8, IV Satz (7.5)]) now completes the proof of Lemma 15.

**Lemma 16.** Let $h(t)$ be a real-valued function defined on the real interval $[-t_0, t_0], t_0 > 0$, and satisfying $|h(t_1 + t_2) - h(t_1) - h(t_2)| \leq \varepsilon$ whenever $t_1, t_2$, and $t_1 + t_2$ belong to that interval. Then with $H = h(t_0)/t_0$ it satisfies $|h(t) - Ht| \leq 3\varepsilon$ uniformly on the same interval.

**Proof.** This is proved by Ruzsa [9], who deduces it from an analogous but not localized result of Hyers concerning Banach spaces.

**Lemma 17.** There is real $\mu, |\mu t_0| \leq 1$, and for each $r$ prime to $a_1$ a real $\theta_r$, so that

$$\sum_{p \leq x \atop p \equiv r(\text{mod } a_1)} \frac{1}{p} |1 - e^{i\theta_r} g(p) p^{\mu t}|^2 \leq 1$$

holds uniformly for $|t| \leq t_0$.

**Proof.** I employ the inequality

$$|z_3 \bar{z}_1 \bar{z}_2 - 1|^2 \leq 3 \sum_{j=1}^{3} |1 - z_j|^2,$$

valid for all complex numbers $z_j$ with $|z_j| = 1$, which may be readily obtained by applications of the Cauchy–Schwarz inequality. Set $z_3 = w_r(t_1 + t_2) \exp(i\{(t_1 + t_2) f(p) + \alpha(t_1 + t_2) \log p\}), z_j = w_r(t_j) \exp(i\{t_j f(p) + \alpha(t_j) \log p\}), j = 1, 2, 2$. Then it follows from three applications of Lemma 13 that the hypotheses of Lemma 15 are satisfied with $k = a_1, \eta = w_r(t_1 + t_2)$ $w_r(t_1) w_r(t_2), \beta = \alpha(t_1 + t_2) - \alpha(t_1) - \alpha(t_2)$.
For sufficiently large values of $x$, independent of the value of $t_0$, Lemma 16 shows that $|\alpha(t) - \mu t| \ll (\log x)^{-1}$ with $\mu = \alpha(t_0) t_0^{-1}$ holds uniformly for $|t| \leq t_0$. In particular, $|\mu t_0| \leq 1$ follows from Lemma 11.

For each real $t$ in the interval $[-t_0, t_0]$, let $w_r = \exp(2\pi i y(t))$ where $y(t)$ is a real for which the distance to $\psi(t)/(2\pi)$, where $\psi(t)$ is the function appearing in Lemma 13, is minimal (mod 1).

Let $\|u\|$ denote the distance of the real $u$ from a nearest integer. This satisfies $|e^{2\pi i u} - 1| = 2|\sin \pi \|u\|| \geq 4\|u\|$. By the choice of the $y(t)$, $|y(t) - \psi(t)/(2\pi)| = \|y(t) - \psi(t)/(2\pi)\| \leq \delta^{3/4}$ may be assumed. Moreover, our upper bound on $|1 - e^\eta|^2$ shows that with $\epsilon$ a suitable constant multiple of $(\log\log x)^{-1/2}$, $|y(t_1 + t_2) - y(t_1) - y(t_2)| \leq \epsilon$. For each pair $(t_1, t_2)$ there is an integer $N$ such that

$$|N - \{y(t_1 + t_2) - y(t_1) - y(t_2)\}| \leq \epsilon.$$

Changing the $t_i$ to $t'_i$ we get a similar inequality with $N'$ in place of $N$. In particular

$$|N - \frac{1}{2\pi} \{\psi(t_1 + t_2) - \psi(t_1) - \psi(t_2)\}| \leq \epsilon + O(\delta^{3/4}).$$

Since $\psi$ is a continuous function of $t$, if the change from $t_j$ to $t'_j$ is sufficiently small, with $\delta$ small and $x$ large, we have $|N - N'| < 1$. Thus $N = N'$ and the integer $N$ has the same value for all permitted values of the $t_j$. The function $v(t) = y(t) - N$ now satisfies the hypothesis of Lemma 16, and with $\theta_r = 2\pi v(t_0)/t_0$, we have

$$|w_r - e^{i\theta_r}|^2 \leq 4\pi^2 |v(t) - tv(t_0) t_0^{-1}|^2 \leq 50\epsilon^2.$$

As in the proof of Lemma 15 we may use these results, along with the Cauchy–Schwarz inequality, to replace $\alpha$ and $w_r$ in the assertion of Lemma 13 by $\mu t$ and $\theta_r t$, respectively, so completing the proof of Lemma 17.

In this section I justify replacing the $\theta_r$ appearing in Lemma 17 with zero, by relating the difference of the additive function $f$ to the difference of another additive function concerning whose value distribution we can give reasonably sharp information. It is in this section that the full weight of the uniformity in $t$ of Lemma 17 is applied.
Lemma 18. The inequality

\[ \sum_{n \leq x} |h(n) - \sum_{p^m \leq x} h(p^m) p^{-m}|^2 \leq x \sum_{p^m \leq x} |h(p^m)|^2 p^{-m} \]

holds uniformly in all complex additive functions \( h \), and real \( x \geq 2 \).

Proof. For a proof of this version of the Turán–Kubilius inequality see Elliott [1, Chap. 4].

An additive function \( k(n) \) is said to be strongly additive if it satisfies \( k(p^m) = k(p) \) for \( m \geq 1 \). In this next lemma \( k(n) \) is a real strongly additive function which is zero on the primes dividing \( AB - Ab \), and which satisfies \( k(p) = 0 \) when \( p > x \). Let \( x \geq 2 \),

\[ B_1(x) = \left( 2 \sum_{p \leq x} \frac{|k(p)|^2}{p} \right)^{1/2} \geq 0, \]

and define

\[ \tau_x = \max_{p \leq x} |k(p)|B_1(x)^{-1}. \]

For each real \( z \), let \( J_x(z) \) denote the frequency, among the integers \( n \) not exceeding \( x \), of those for which \( an + b, An + B \) are both positive, and \( k(an + b) - k(An + B) \leq zB_1(x) \).

Lemma 19. There is a number \( x_1 \), depending at most upon the four integers \( a, A, b, B \), so that

\[ J_x(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} dw + O(\tau_x^{1/2}) \]

holds uniformly for all real \( z \) and \( x \geq x_1 \).

Proof. A result of this type was obtained by Kubilius [7], and with a sharper error term. For the sharpest result presently available see Elliott [1, Theorem (20.1)]. There attention is paid to the dependence of the error terms upon the various parameters and upon the function \( k \), although ‘absolute’ in that context is understood to mean ‘independent of the additive functions \( f_j \) and of the real \( z \).’

In all of these proofs the additive functions involved are simulated by sums of independent random variables, and this is only possible since with (say) \( \tau_x \) assumed small, the contribution towards \( k(n) \) arising from the larger prime divisors of \( n \) may be neglected. In the present situation we cannot study the difference \( f_1(an + b) - f_2(An + B) \) in this manner since the
$f_j(p)$ may very well be large on the primes near to $x$. After Lemma 17 we shall be able to “change ends” and study our function $f$ through the eye of the $\theta_i$.

**Lemma 20.** Let the real numbers $d_p$, one for each prime $p$ not exceeding $x$, satisfy

$$\sum_{p \leq x} \frac{1}{p} \left| 1 - e^{itd_p} \right|^2 \leq c$$

uniformly for $|t| \leq t_0$. Then

$$\sum_{p \leq x} \frac{1}{p} \leq c, \quad \sum_{p \leq x} \frac{|d_p|^2}{p} \leq 3ct_0^{-2}.$$  

**Proof (cf. Lemma 2).** For real $\beta \neq 0$, $T > 0$

$$\frac{1}{2T} \int_{-T}^{T} \left| 1 - e^{it\beta} \right|^2 dt = 2 \left( 1 - \frac{\sin T\beta}{T\beta} \right).$$

Hence

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{2t_0} \sum_{-t_0}^{t_0} \frac{|1 - e^{it\beta}|^2}{p} dt \leq c.$$  

Moreover, if $|\beta t| \leq \pi$, then $|1 - e^{i\beta t}| \geq 4 \beta t(2\pi)^{-1} = |\beta t| 2/\pi$, so that

$$\sum_{p \leq x} \frac{|d_p|^2}{p} \leq \frac{\pi^2}{4t_0^2} \sum_{p \leq x} \frac{|1 - e^{itd_p}|^2}{p} \leq \frac{\pi^2 c}{4t_0^2}.$$  

This completes the proof of the lemma.

In the next lemma I seek to construct integers of the form $an + b$, $An + B$ which have as few factors as possible in common with $2aAD$. Continuing with our earlier notation we set $D = (a, b)$, $a_1 = D^{-1}a$, $b_1 = D^{-1}b$, and $D_1 = (A, B)$, $A_1 = D_1^{-1}A$, $B_1 = D_1^{-1}B$. For odd primes $p$ the conditions

$$a_1 n + b_1 \not\equiv 0 \pmod{p}, \quad A_1 n + B_1 \not\equiv 0 \pmod{p}$$

can be simultaneously satisfied, since there are at least three residue classes (mod $p$) to choose from. They can also be satisfied for $p = 2$ as well, unless $a_1 A_1$ is odd and $b_1, B_1$ are of opposite parity. Except in this case the Chinese Remainder Theorem guarantees a residue class $n$ (mod $2aA |A_1|$) for which both $D^{-1}(an + b)$, $D_1^{-1}(An + B)$ have no factor in common with
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2aA |A|. In the exceptional case we can arrange a residue class (mod 4aA |A|) so that $(2D)^{-1}(an+b)$, $D_{-1}(An+B)$ have no factor in common with $2aA |A|$. We call the pairs $\{D, D_l\}$ and $\{2D, D_l\}$ fixed-divisor pairs of the sequences $an + b$, $An + B$. For a more elaborate discussion of the possible such divisor pairs see Elliott [2, Chap. 3].

**Lemma 21.** Let $\{p_j\}$ be a sequence of primes in the interval $3 \leq p_j \leq y^{1/70}$, and not dividing $2aA$. Then for all $y$ sufficiently large in terms of the integers $a, A, b, B$, there are at least

$$(10^4 aA |A|)^{-1} \prod_j \left(1 - \frac{2}{p_j}\right)$$

integers $n$ for which $an + b$ and $An + B$ do not exceed $y$, are not divisible by any $p_j$, and have a (n appropriate) fixed-divisor pair. This result is uniform in the sets of primes $p_j$.

**Proof.** This is a lower bound sieve estimate, obtained with the method of Selberg, as in Elliott [1, Lemma (2.1)]. The constants appearing are nowhere near ‘best,’ but are explicit and readily available. In the notation used there we set $z = y^{1/4}$, $r = y^{1/70}$, and note that $\eta(p) = 2p^{-1}$. Thus

$$S \leq 2 \sum_{p_j \leq y^{1/70}} \frac{\log p_j}{p_j} + O(1) \leq \frac{\log y}{35} + O(1) < \frac{\log z}{8}$$

for all large $y$, and the conditions required to make $2H < \exp(-0.006)$ in that result are satisfied.

**Completion of the proof of Lemma 7.** Define an additive function $k$ by $k(p^j) = f(p) + \theta_r + \mu \log p_j$ if $p \equiv r(\text{mod } a_1)$ and $|f(p) + \theta_r + \mu \log p| < 2t_0^{-1}$, to be zero on other prime-powers. From Lemma 17, and Lemma 20 with $d_p = k(p)$, there is a constant $c_0$, depending at most upon the four integers $a, b, A, \text{and } B$, so that

$$\sum_{p \leq x \atop |k(p)| > 2t_0^{-1}} \frac{1}{p} \leq c_0.$$

Define

$$\gamma = (10^4 aA |A|)^{-1} \prod_{p \leq x \atop |k(p)| > 2t_0^{-1}} \left(1 - \frac{2}{p}\right),$$

so that

$$\gamma \geq (10^4 aA |A|)^{-1} \prod_{p > x} \left(1 - \frac{4}{p^2}\right) \exp(-c_0) > 0,$$
independent of $x$. For $m = 0, 1, \ldots$, set $x_m = x^{(1/70)^m}$. Then

$$\sum_{m=0}^{r} \sum_{x_m^{1/70} < p \leq x_m} \frac{1}{p} \leq c_0,$$

and if $r$ is fixed at a sufficiently large value, there will be an $m$, $0 \leq m \leq r$, such that

$$\sum_{x_m^{1/70} < p \leq x_m} \frac{1}{p} \leq \frac{c_0}{r+1} < \frac{\gamma}{4}.$$

Let $y = x_m$ for the first such $m$.

We apply Lemma 21, taking for the $p_j$ those primes, not dividing $2aA$ $|A|$, for which $|k(p)| > 2t_0^{-1}$. Since the number of integers $an + b$ or $An + B$, not exceeding $y$, which are divisible by such a prime greater than $y^{1/70}$, is at most

$$2 \sum_{y^{1/70} < p \leq y} \left( \frac{y}{p} + O(1) \right) < \frac{5y}{9} y$$

for large enough $x$ (and so $y$), we can find at least $4\gamma y/9$ integers $n$ for which $an + b, An + B$ do not exceed $y$, and have no prime-divisor $p \geq 3$ with $|k(p)| > 2t_0^{-1}$. Moreover, their prime factors in common with $2aA$ $|A|$ are confined to a fixed-divisor pair, $(\lambda_1, \lambda_2)$ say. I shall show that at the expense of a fixed proportion of the frequency $4\gamma y/9$, further convenient properties may be demanded of these integers.

With

$$H = \sum_{p^m \leq y} p^{-m}k(p^m)$$

we apply the Cauchy–Schwarz inequality

$$|k(an + b) - k(An + B)|^2 \leq 2 |k(an + b) - H|^2 + 2 |k(An + B) - H|^2$$

and the Turán–Kubilius inequality (Lemma 18) to obtain, again applying Lemma 17 with Lemma 20,

$$\sum_{\max(an + b, An + B) \leq y} |k(an + b) - k(An + B)|^2 \leq \gamma y t_0^{-2}.$$

Provided we fix $\varepsilon_1$ at a sufficiently small value, depending only upon the integers $a, b, A$, and $B$, there will be $\ll \varepsilon_1^2 y < \gamma y/9$ integers, counted in this sum, for which $|k(an + b) - k(An + B)| > (\varepsilon_1 t_0)^{-1}$. 
Moreover, if $\delta$ is sufficiently small and $x$ sufficiently large, $y \geq 2x^\delta \max(a, A)$. Then by an hypothesis of Lemma 7

$$\psi(t) = y^{-1} \sum_{\max(an + b, An + B) \leq y} |e^{itf(an + b) - f(An + B) - v(y)}| - 1 \leq \delta.$$ 

Arguing as in the proof of Lemma 2 we see that among the integers $n$ with $\max(an + b, An + B) \leq y$, those for which $|f(an + b) - f(An + B) - v(y)| > 2t_0^{-1}$ are not more than

$$(2y + O(1)) \cdot \frac{1}{2t_0} \int_{y}^{\infty} \psi(t) \, dt \ll \delta y < \frac{y^2}{9}$$

in number.

Set

$$\omega(n) = \sum_{(r, a_1) = 1} \theta_r \sum_{p \equiv r \pmod{a_1}} 1,$$

where the outer sum runs over a complete set of reduced residue classes (mod $a_1$). On the primes which further satisfy $|k(p)| \leq 2t_0^{-1}$ we have $\omega(p) = k(p) - f(p) - \mu \log p$. Altogether we have constructed a sequence of integers $n$, with $\max(an + b, An + B) \leq y$, at least $\gamma y/9$ in number, on which

$$|\omega(an + b) - \omega(An + B) + v(y) + \eta_1| \leq dt_0^{-1},$$

where $\eta_1$ is the sum of $\mu \log(a/A)$ and certain terms arising from the fixed-divisor pair $\{\lambda_1, \lambda_2\}$, and where the constant $d$ depends at most upon the four integers $a, b, A$, and $B$. If, as in the notation of Lemma 19, we set

$$t_0 z = -(v(y) + \eta_1) B_1(y) - \eta_2 z = d(t_0 B_1(y))^{-1},$$

then within the conventions of that lemma

$$J_+(\eta_2 + z) - J_-(\eta_2 - z) > \gamma/9.$$ 

In the present circumstances, $\tau_\nu \ll (\log \log x)^{-1/2}$, and for large enough $x$ the error term in Lemma 19 will not exceed $\gamma/18$. Hence

$$\frac{\gamma}{18} \leq \frac{1}{\sqrt{2\pi}} \int_{\eta_2 - z}^{\eta_2 + z} e^{-w^2/2} \, dw \leq 2z = \frac{2d}{t_0 B_1(y)},$$

and

$$t_0 \sum_{(r, a_1) = 1} \theta_r^2 \log \log x \ll 1.$$
Applications of the Cauchy–Schwarz inequality now enable us to replace the $\theta$, in the inequalities of Lemma 17 with zero, and the proof of Lemma 7 is complete.

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**Completion of the proof of Theorem 1.** I continue with the notation $g(n) = \exp(itf(n))$ for an additive function $f$. For $x \geq 2$, $\tau > 0$, $w \geq 0$ define

$$L(x, \tau, w) = \min_{|\mu| < w} \max_{|d| < \tau} \sum_{p \leq x} \frac{1}{p} |1 - g(p) p^{i\mu t}|^2.$$

**Lemma 22.** Let $a > 0$, $A > 0$, $b$, $B$ be integers for which $aB \neq Ab$, $x \geq 2$. There are constants $c_2$, $0 < c_2 < 1$, and $c_3$ so that if a completely additive function $f$ satisfies

$$\sup_{x^\delta \leq y \leq x} y^{-1} \left| \sum_{n \leq y} g(an+b) - e^{i\nu(y)} g(An+B) \right| \leq \delta \quad (22)$$

for some $\nu(y)$, and a $\delta$ in the range $(\log x)^{-c_2} \leq \delta \leq \frac{1}{3}$, uniformly for $|t| \leq t_0$, $t_0 > 0$, then

$$L(x, t_0, \log x) \leq \frac{1}{3} \log \frac{1}{\delta} \quad \text{implies} \quad L(x, t_0, 1 + \log x) \leq c_3.$$

The constant $c_2$ is absolute, $c_3$ depends at most upon the four initial integers.

**Proof.** Suppose that for a real $\mu$, $|\mu_1 t_0| \leq \log x$, the inequality

$$\sum_{p \leq x} \frac{1}{p} |1 - g(p) p^{i\mu_1 t}|^2 \leq \frac{1}{3} \log \frac{1}{\delta}$$

holds uniformly for $|t| \leq t_0$. For sufficiently large $x$

$$y^{-1} \sum_{n \leq y} \left| \exp(i\mu_1 t \{\log(an+b) - \log(An+B) - \log(a/A)\}) - 1 \right|$$

$$\leq y^{-1} \sum_{n \leq y} \frac{|\mu_1 t|}{n} \leq y^{-1} (\log x)^2 < \delta$$

uniformly for $x^\delta \leq y \leq x$. The function $g(n) n^{i\mu_1 t}$ therefore satisfies the hypotheses of Lemma 7 with $\nu(y) + \log(a/A)$ in place of $\nu(y)$, and $2\delta$ in
place of $\delta$. Here we have assumed, as we may, that $\delta \leq \frac{1}{6}$. According to that lemma, if $\delta$ is sufficiently small, there is a $\mu, |\mu t_0| \leq 1$, so that

$$\sum_{p \leq x} p^{-1} |1 - g(p) p^{\mu t_0}|^2 \leq c_3,$$

uniformly for $|t| \leq t_0$. Adjusting the value of $c_3$, if necessary, to take care of the remaining range of $\delta$-values, we complete the proof of Lemma 22.

**Proof of Theorem 1.** For all sufficiently large $x$, say $x \geq x_0$,

$$\sum_{\frac{x}{e} < p \leq x} \frac{1}{p} = \log \left( \frac{\log x}{\log x - 1} \right) + O((\log x)^{-1}) < 1.$$

Fix a positive value of $\delta$, not exceeding $\frac{1}{3}$, so that $\frac{1}{3} \log(1/\delta) \geq c_3 + 5$. For this value of $\delta$ there will be a positive $\tau$ and an $x_1$ so that the condition (22) of Lemma 22, with $v(y) = \eta(y)$, holds uniformly for $x > x_1$, for $|t| \leq \tau$. Let $x_2 = \max(x_0, x_1)$. Choose a positive $t_0 < \tau$ so that $L(x, t_0, 1 + \log x) \leq c_3$ holds for $x \leq x_2$. This can be done since every such $L$ does not exceed

$$\max_{|t| \leq t_0} \sum_{p \leq x_2} \frac{1}{p} |1 - \exp(itf(p))|^2,$$

where the sum is continuous in $t$ and is zero for $t = 0$. It will now follow that $L(x, t_0, 1 + \log x) \leq c_3$ holds for $x > x_2$ as well.

Suppose to the contrary that $L(x, t_0, 1 + \log x) > c_3$ for some $x > x_2$. Then by Lemma 22 whatever the value of $\mu$, so long as $|\mu t_0| \leq \log x$, we have

$$\max_{|t| \leq t_0} \sum_{p \leq x} \frac{1}{p} |1 - g(p) p^{\mu t}|^2 > \frac{1}{3} \log \frac{1}{\delta},$$

and therefore

$$\max_{|t| \leq t_0} \sum_{p \leq \frac{x}{e}} \frac{1}{p} |1 - g(p) p^{\mu t}|^2 > \frac{1}{3} \log \frac{1}{\delta} - 4 > c_3.$$

Hence

$$L(x/e, t_0, 1 + \log(x/e)) > c_3.$$

Continuing inductively in this manner we reach a $y < x_2$ for which $L(y, t_0, 1 + \log y) > c_3$, which is impossible.

For each $x \geq 2$ there is a real $\mu = \mu(x), |\mu t_0| \leq 1 + \log x$, so that

$$\sum_{p < x} \frac{1}{p} |1 - g(p) p^{\mu t}|^2 \leq c_3 \quad (23)$$
uniformly for $|t| \leq t_0$. Eliminating the $g(p), p \leq x$, between this and a similar inequality with $w$ in place of $x$, $x \leq w \leq x^2$ (cf. the proof of Lemma 17), we see that

$$\sum_{p \leq x} \frac{1}{p} |1 - p^{it} \mu(x) - \mu(w)|^2 \leq 4c_3.$$  

With $t = t_0$ an application of Lemma 15 shows that $t_0(\mu(x) - \mu(w)) \ll (\log x)^{-1}$, the implied constant depending at most upon $c_3$. For $m \geq n \geq 2$

$$t_0(\mu(2^{2^n}) - \mu(2^{2^n})) \leq \sum_{j = m}^{n} 2^{-i},$$

and by Cauchy's criterion $-\alpha = \lim_{n \to \infty} \mu(2^{2^n})$, as $n \to \infty$, exists. Moreover, every real $x \geq 2$ lies in some interval $2^{2^n} \leq x < 2^{2^{n+1}}$, so that $t_0(\mu(x) + x) \ll (\log x)^{-1}$. Using this to replace $\mu$ in (23) we reach

$$\sum_{p \leq x} \frac{1}{p} |1 - \exp(it(f(p) - \alpha \log p))|^2 \ll 1$$

uniformly in $|t| \leq t_0$. An appeal to Lemma 20 completes the proof of Theorem 1.

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