Nilpotent Lie Algebras of Classical Simple Type

G. Favre and L. Santharoubane

Mathématique Bât 425, Université de Paris Sud, 91405 Orsay Cedex, France

Communicated by Georgia Benkart

Received February 1, 1994

INTRODUCTION

Although the classification of complex semi-simple Lie algebras was achieved as earlier as 1890 by Killing and Cartan, the classification of solvable Lie algebras is still a hopeless problem. In 1945, Malcev reduced the solvable case to the nilpotent one.

In [San 83] the second author associated canonically a Kac–Moody Lie algebra \( g(A) \) to each nilpotent Lie algebra; thus breaking the big hopeless problem into small hopeless ones (of type \( A \) each).

The “first” Kac–Moody Lie algebras are the simple Lie algebras of type: \( A_n, B_n, C_n, D_n \); in this work we find the number of associated nilpotent Lie algebras which are \( h \)-modules (\( h \) being the Cartan subalgebra of \( g(A) \)).

Several authors are investigating other cases: \( E_6, E_7, E_8 \) (Agrafiotou and Tsagas [A-T]), \( F_4 \) (Favre and Tsagas [F-T]), \( A_2^{(1)}, B_2^{(1)}, G_2^{(1)} \) (Kanagavel [K]), \( D_4^{(3)} \) (Agrafiotou [A]). There is also a computer approach by Callegari [C]. See the bibliography for related works.

1. NILPOTENT LIE ALGEBRAS OF MAXIMAL RANK AND OF CLASSICAL SIMPLE TYPE

1.1. Let \( \mathfrak{m} \) be a finite-dimensional nilpotent Lie algebra, \( \text{Der} \mathfrak{m} \) its derivation algebra and \( \text{Aut} \mathfrak{m} \) its automorphism group. A torus on \( \mathfrak{m} \) is a commutative subalgebra of \( \text{Der} \mathfrak{m} \) whose elements are semi-simple. Mostow's Theorem 4.1 [Mos] says that all maximal (for the inclusion) tori on \( \mathfrak{m} \) are conjugate under \( \text{Aut} \mathfrak{m} \); their common dimension is called the rank of \( \mathfrak{m} \). By [F] the rank \( r \) of \( \mathfrak{m} \) is less than the dimension \( l \) of
one says that $\mathfrak{m}$ is of maximal rank if

$$r = l.$$ 

1.2. By 1.7 [San 83] one can associate to $\mathfrak{m}$ a generalized Cartan matrix $A = (a_{ij})_{1 \leq i, j \leq l}$ (1.1 [San 82]) and one says that $\mathfrak{m}$ is of type $A$ in [San 82] $l$ was rather the type. Let $\text{Nilp}(\text{max}, A)$ be a set of representatives for the isomorphism classes of nilpotent Lie algebras of maximal rank and of type $A$. From now on, we assume that $A$ is of classical simple type, i.e. $A \in \{A_l, B_l, C_l, D_l\}$ [B].

1.3. Let $\mathfrak{q}(A)$ be the simple Lie algebra of type $A$, $\mathfrak{n}_+(A)$ be its positive part, $R_+(A)$ be the set of positive roots, $\mathfrak{g}^\alpha$ be the root space associated to $\alpha \in R_+(A)$, $\mathfrak{Z}(A)$ be the automorphism group of the Dynkin diagram of $A$ and $(\alpha_1, \ldots, \alpha_l)$ be a base [B].

1.4. One says that $I$ is an ideal of $R_+(A)$ if $I$ is a subset of $R_+(A)$ such that $\alpha + \alpha_i \in I$ for all $\alpha \in I$ and all $i = 1, \ldots, l$ whenever $\alpha + \alpha_i \in R_+(A)$ (we remark that $\phi$ is an ideal of $R_+(A)$). Let $R_{++}(A)$ be the complement of the basic chains

$$\alpha_i, \alpha_j, \ldots, \alpha_i - \alpha_j, \quad 1 \leq i \neq j \leq l,$$

and let $\mathcal{I}(A)$ be the set of ideals of $R_+(A)$ contained in $R_{++}(A)$.

1.5. Theorem (8.4 [San 82]). The mapping

$$\mathfrak{Z}(A).I \mapsto \mathfrak{n}_+(A) \bigg/ \bigoplus_{\alpha \in I} \mathfrak{g}^\alpha$$

is a bijection from the set of $\mathfrak{Z}(A)$-orbits of $\mathcal{I}(A)$ onto $\text{Nilp}(\text{max}, A)$.

2. ALGEBRAS OF TYPE $A_l$ ($l \geq 1$)

2.1. Let $V$ be a complex vector space of dimension $l + 1$ and let $\mathfrak{g}$ be the algebra $\mathfrak{sl}(V)$ of endomorphisms of $V$ with zero trace. One knows that $\mathfrak{g}$ is simple. Denote by $E_{ij}$ the matrix having 1 in the $(i, j)$ position and 0 elsewhere. The matrices

$$E_{ij} \quad (1 \leq i, j \leq l + 1, i \neq j).$$

$$E_{ii} - E_{i+1, i+1} \quad (1 \leq i \leq l)$$
form a basis of \( g \). Let \( \mathfrak{h} \) be the space of all traceless diagonal matrices. Define \( \varepsilon_i \in \mathfrak{h}^* \ (1 \leq i \leq l + 1) \) by

\[
\varepsilon_i \begin{pmatrix}
a_1 & \cdots & 0 \\
0 & \ddots & \vdots \\
& \ddots & 0 \\
& & \ddots & a_{l+1}
\end{pmatrix} = a_i.
\]

Then \( \mathfrak{h} \) is a Cartan subalgebra of \( g \) and the roots of \( (g, \mathfrak{h}) \) are

\[
\varepsilon_i - \varepsilon_j \quad (1 \leq i, j \leq l + 1, i \neq j).
\]

The root system \( R \) of \( (g, \mathfrak{h}) \) is of type \( A_l \) with

\[
A_l = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & & \\
& & \ddots & \\
& & & -1 & 2
\end{pmatrix}.
\]

One has

\[
\mathfrak{h}^{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}.
\]

Set

\[
\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \quad \alpha_l = \varepsilon_l - \varepsilon_{l+1};
\]

then \( (\alpha_1 \cdots \alpha_l) \) is a base of \( R \).

2.2. The set of positive roots relative to the base \( (\alpha_1 \cdots \alpha_l) \) is

\[
R_+(A_l) = \{ \varepsilon_i - \varepsilon_j; 1 \leq i < j \leq l + 1 \}
\]

and the corresponding positive part \( \mathfrak{n}_+(A_l) \) is the subalgebra of strictly upper triangular matrices. Explicitly,

\[
\mathfrak{n}_+(A_l) = \bigoplus_{1 \leq i < j \leq l + 1} \mathbb{C} E_{ij},
\]

\[
[E_{ij}, E_{jk}] = E_{ik}, \quad 1 \leq i < j < k \leq l + 1;
\]

all the other brackets are either 0 or deduced from the ones above by antisymmetry (see [B] for all details).

2.3. Using the values of the entries \( a_{ij} \) of \( A_l \) we notice that the chains

\[
\alpha_i, \alpha_i + \alpha_j, \ldots, \alpha_i - a_{ji} \alpha_j, \quad 1 \leq i, j \leq l,
\]
are
\[ \alpha_i, \alpha_i + \alpha_{i+1}, \quad 1 \leq i \leq l; \]

that is,
\[ e_i - e_{i+1}, e_i - e_{i+2}, \quad 1 \leq i \leq l. \]

Therefore, the complement in \( R_+ (A_i) \) of the above chains is
\[ R_{++} (A_i) = \{ e_i - e_j; 3 \leq i + 2 < j \leq l + 1 \}. \]

2.4. Lemma. Let \( I \) be an ideal of \( R_+ (A_i) \):

1. if \( e_i - e_j \in I \) then \( e_i - e_s \in I \) for \( r \leq i \) and \( s \geq j \)
2. if \( e_i - e_j \notin I \) then \( e_r - e_i \notin I \) for \( r \geq i \) and \( s \leq j \).

Proof. (1) If \( e_i - e_j \in I \), \( r \leq i \), and \( s \geq j \) then
\[ e_r - e_i = \alpha_r + \alpha_{r+1} + \cdots + \alpha_{i-1} + (e_i - e_j) + \alpha_j + \cdots + \alpha_{i-1} \in I \]
by the definition of an ideal.

(2) comes from (1).

2.5. For \( 1 \leq m \leq l - 2 \), we define the subset of \( R_{++} (A_i) \)
\[ R_m (A_i) = \{ e_i - e_j; 1 \leq i \leq m, 4 \leq j \leq l + 1, i + 2 < j \}; \]
then \( R_{l-2} (A_i) = R_{++} (A_i) \).

Let \( \mathcal{J}_m^r (A_i) \) be the set of ideals of \( R_+ (A_i) \) contained in \( R_m (A_i) \); then
\[ \mathcal{J}_{l-2}^r (A_i) = \mathcal{J} (A_i). \]
Let \( s_m \) (resp. \( s_r \)) be the number of elements of \( \mathcal{J}_m^r (A_i) \) (resp. \( \mathcal{J} (A_i) \)); then \( s_{l-2, i} = s_i \). If we define
\[ \mathcal{J}_m^r (A_i) = \{ I \in \mathcal{J}_m^r (A_i); e_m - e_{i+1} \notin I \} \]
\[ \mathcal{J}_m^r (A_i) = \{ I \in \mathcal{J}_m^r (A_i); e_m - e_{i+1} \in I \} \]
then
\[ \mathcal{J}_m^r (A_i) = \mathcal{J}_m^r (A_i) \cup \mathcal{J}_m^r (A_i) \]
is a partition. And if \( I \in \mathcal{J}_m^r (A_i) \) then \( e_m - e_j \notin I \) for \( j = m, m+1, \ldots, l + 1 \) (by 2.4). Thus \( I \subset R_{m-1}^r (A_i) \) (with \( R_r (A_i) = \phi \) for \( l \geq 2 \)) which gives
\[ \mathcal{J}_m^r (A_i) = \mathcal{J}_{m-1}^r (A_i) \] (with \( \mathcal{J}_r (A_i) = \{ \phi \} \) for \( l \geq 2 \)) and \# \( \mathcal{J}_m^r (A_i) = s_{m-1, i} \) (with \( s_{m-1, i} = 1 \) for \( l \geq 2 \)). Next, if \( I \in \mathcal{J}_m^r (A_i) \) then \( e_i - e_{i+1} \in I \) for \( i = 1, \ldots, m \) (by 2.4) and \( J = I \setminus \{ e_i - e_{i+1}; 1 \leq i \leq m \} \) is included in \( R_m (A_i) \) (we identify \( R_+ (A_{i-1}) \) with a subset of \( R_+ (A_i) \) and put
$R_{l-2}(A_{l-1}) = R_{l-3}(A_{l-1})$. The mapping $I \mapsto J$ being obviously a bijection from $S_2^m(A_i)$ onto $S_m(A_{l-1})$ (with $S_2^m(A_{l-1}) = S_m(A_{l-1})$), we get
$\# S_2^m(A_i) = s_{m,l-1}$ (with $s_{l-2,l-1} = s_{l-3,l-1}$). Therefore,

$$(S_{ml}): \quad \begin{align*}
    s_{ml} &= s_{m-1,l} + s_{m,l-1}, & 1 \leq m \leq l - 2, \\
    s_{0l} &= 1, & l \geq 2, \\
    s_{l-2,l-1} &= s_{l-3,l-1}, & l \geq 3.
\end{align*}$$

2.6. **Lemma.** $s_{ml} = \left( \frac{m + l - 2}{m} \right) - \left( \frac{m + l - 2}{m - 2} \right)$, $0 \leq m \leq l - 2$.

**Proof.** From $S_{0l} = 1$ and $(S_{1l})$, we get $s_{1l}$; from $S_{2l}$ and $(S_{2l})$ we get $s_{2l}$. And continue until we guess the correct formula. Then we assume the lemma for $m_1 < m$, $l_1 < l$, and we compute $s_{ml}$ by using $(S_{ml})$:

$$s_{ml} = \left( \frac{m + l - 3}{m - 1} \right) - \left( \frac{m + l - 3}{m - 3} \right) + \left( \frac{m + l - 3}{m} \right) - \left( \frac{m + l - 3}{m - 2} \right)$$

$$= \left( \frac{m + l - 2}{m} \right) - \left( \frac{m + l - 2}{m - 2} \right).$$

2.7. It follows that $s_{l} = \# S(A_i) = s_{l-1,l}$ is given by

$$s_{l} = \left( \frac{2l - 4}{l - 2} \right) - \left( \frac{2l - 4}{l - 4} \right), \quad l \geq 2,$$

$$s_{1} = 1.$$

2.8. The automorphism group of the Dynkin diagram

```
   1 2 3 4 5 6
```

is $\mathbb{Z}(A_i) = (1, \sigma)$ with $\sigma i = l + 1 - i$ for $1 \leq i \leq l$. This group acts on $R_{+,}(A_i)$ by $\sigma (e_i - e_j) = e_{l+1-i} - e_{l+1-j}$. We can extend this action to $S(A_i)$ by $\sigma I = \{\alpha I; \alpha \in I\}$; denote by $S(A_i)$ the set of fixed elements of $S(A_i)$ and by $t_l$ the number of elements of $S(A_i)$. For $1 \leq m \leq l - 2$ we define the subset of $R_{+,}(A_i)$

$$R_{m}(A_i) = \{e_i - e_j; 1 \leq i \leq m, l + 2 - m \leq j \leq l + 1, i + 2 < j\};$$

then $R_{l-2}^m(A_i) = R_{+,}(A_i)$.

It is easy to check that $R_{m}(A_i)$ is stable under $\sigma$. Let $S(A_i)$ be the set of ideals of $R_{+,}(A_i)$ contained in $R_{m}(A_i)$ and fixed by $\sigma$; then $S_{l-2}(A_i) = S(A_i)$. Let $t_{ml}$ be the number of elements of $S_{m}(A_i)$; then $t_{l-2,l} = t_l$. If we define

$$S_{m}(A_i) = \{I \in S(A_i); e_m - e_{l+1} \not\in I\}$$

$$S_{m}(A_i) = \{I \in S(A_i); e_m - e_{l+1} \in I\}$$

then
\[ \mathcal{F}_m(A_t) = \mathcal{F}_m^o(A_t) \cup \mathcal{F}_{m+1}^1(A_t) \]
is a partition. Now if \( I \in \mathcal{F}_m^n(A_t) \) then \( e_m - e_{t+1} \notin I \) and \( e_1 - e_{t+2-m} = \sigma(e_m - e_{t+1}) \notin \sigma I \) lead to \( e_m - e_{j+1} \notin I \) for \( j = m + 3, \ldots, l + 1 \) and
\[ e_i - e_{t+2-m} \notin I \] for \( i = 1, \ldots, l - 1 - m \) (by 2.4). Thus \( I \in R_m^{l-1}(A_t) \) (with \( R_m^l(A_t) = \phi \) for \( l \geq 1 \)) which gives \( \mathcal{F}_m(A_t) = \mathcal{F}_m^o(A_t) \) (with \( \mathcal{F}_m(A_t) = \{ \phi \} \) for \( l \geq 1 \)) and \( \#\mathcal{F}_m^o(A_t) = t_{m-1, l} \) (with \( t_{m-1, l} = 1 \) for \( l \geq 1 \)). Next, if \( I \in \mathcal{F}_m(A_t) \) then
\[ e_m - e_{t+1} \in I \] and \( e_1 - e_{t+2-m} = \sigma(e_m - e_{t+1}) \in \sigma I \) lead to \( e_j - e_{t+1} \in I \) for \( j = 1, \ldots, m \) and \( e_1 - e_i \in I \) for \( j = l + 2 - m, \ldots, l + 1 \) (by 2.4). Thus \( J = I \setminus \{(e_i - e_{t+1}); 1 \leq i \leq m\} \cup \{e_1 - e_i; l + 2 - m \leq j \leq l + 1\} \) is included in \( R_m^{l-1}(A_{t-2}) \) (we identify \( R_m^{l-1}(A_{t-2}) \) with a subset of \( R_m^l(A_t) \) by the mapping \( e_i - e_i \mapsto e_{i+1} - e_{i+1} \) \( 1 \leq i \leq l - 2, 2 \leq j \leq l - 1 \) and put \( R_m^{l-1}(A_{t-2}) = R_m^{l-1}(A_{t-2}) \) for \( l \geq 4 \)) and is fixed by \( \sigma \). The mapping \( I \mapsto J \) being obviously a bijection from \( \mathcal{F}_m^o(A_t) \) onto \( \mathcal{F}_{m-1}(A_{t-2}) \) (with \( \mathcal{F}_{m-1}(A_{t-2}) = \mathcal{F}_{m-1}(A_{t-2}) \) for \( l \geq 4 \)), we get \( \#\mathcal{F}_m^o(A_t) = t_{m-1, l-2} \) (with \( t_{m-1, l-2} = t_{m-1, l-2} \) for \( l \geq 4 \)). Therefore,
\[
\begin{aligned}
(T_{ml}): & \quad t_{ml} = t_{m-1, l} + t_{m-1, l-2}, \quad 1 \leq m \leq l - 2, \\
& \quad t_{ol} = 1, \quad l \geq 1, \\
& \quad t_{l-3, l-2} = t_{l-4, l-2}, \quad l \geq 4.
\end{aligned}
\]

2.9. L EMMA. \( t_{m, 2l + r - 2} = \binom{m}{m-l'} + \binom{m}{m-l'+1} + \cdots + \binom{m}{m-l'+r-1}, \) \( 0 \leq m \leq 2l' + r - 2, 0 \leq r \leq l - 1. \)

Proof. From \( t_{01} = 1 \) and \((T_{11})\) we get \( t_{11} \); from \( t_{11} \) and \((T_{21})\) we get \( t_{21} \).

And we continue until we guess the correct formula. Then we assume the lemma for \( m_1 < m, l'_1 < l' \), and we compute the \( t_{ml} \) by using \((T_{ml})\):
\[
t_{m, 2l + r} = t_{m-1, 2l + r} + t_{m-1, 2l + (r-1) + r}
\]
\[
\begin{align*}
&= \left(\binom{m-1}{m-l'} + \binom{m-1}{m-l'+1} + \cdots + \binom{m-1}{m-l'+r-1}\right) \\
&\quad + \left(\binom{m-1}{m-l'} + \binom{m-1}{m-l'+1} + \cdots + \binom{m-1}{m-l'+r-2}\right) \\
&= \left(\binom{m}{m-l'} + \binom{m}{m-l'+1} + \cdots + \binom{m}{m-l'+r-1}\right).
\end{align*}
\]

2.10. If follows that \( t_{l} = t_{l-1, l} = \binom{l+1}{l+r-2} + \binom{l+1}{l+r-1} = \binom{l+1}{l'+r-1} \) for \( l \geq 2l' + r - 2 \) and \( 0 \leq r \leq l - 1 \) (while \( t_1 = 1 \)).
2.11. Theorem. The number \( a_i \) of isomorphism classes of nilpotent Lie algebras of maximal rank and of type \( A_i \) is
\[
a_i = \frac{1}{2} \left[ (2l - 4) - \left( \frac{2l - 4}{l - 2} \right) + \left( 1 - \frac{1}{l' + r - 1} \right) \right],
\]
\[l = 2l' + r \geq 2, \quad 0 \leq r \leq 1,
\]
\[a_1 = 1.
\]

Proof. By 1.5, \( a_i \) is equal to the number of \( \mathbb{Z}_2(A_i) \)-orbits of \( \mathcal{F}(A_i) \); since \( \# \mathbb{Z}_2(A_i) = 2 \), the class equation is \( a_i = \frac{1}{2}(s_j - t_i) + t_j = \frac{1}{2}(s_j + t_i) \), whence the conclusion by 2.7 and 2.10.

2.12. If we define
\[e_{ij} = E_{ij} + \bigoplus_{a \in I} g^a \quad \forall I \in \mathcal{F}(A_i), \quad \forall e_i - e_j \notin I,
\]
then the quotient algebra
\[A_{i,n_1} = u_+(A_i) \big/ \bigoplus_{a \in I} g^a
\]
ads the following presentation by generators and relations:
\[A_{i,n_1} = \bigoplus_{e_i - e_j \in R_+(A_i) \setminus I} C e_{ij}
\[\quad [e_{ij}, e_{jk}] = e_{ik}
\]
\[\forall e_i - e_j, e_j - e_k, e_i - e_k \in R_+(A_i) \setminus I, \quad 1 \leq i < j < k \leq l + 1.
\]

2.13. Examples (\( l = 1, 2, 3, 4 \)). Obviously,
\[R_+(A_1) = R_+(A_2) = \phi, \quad R_+(A_3) = \{ e_1 - e_4 \},
\]
\[R_+(A_4) = \{ e_1 - e_4, e_2 - e_5, e_1 - e_5 \}
\]
and we get at once:
\[\mathcal{F}(A_1) = \mathcal{F}(A_2) = \{ \phi \}
\]
\[\mathcal{F}(A_3) = \mathcal{F}(A_4) = \{ \phi, \{ e_1 - e_4 \} \}
\]
\[\mathcal{F}(A_3) = \mathcal{F}(A_4) = \{ \phi, \{ e_1 - e_3, e_1 - e_5, e_2 - e_5, e_1 - e_5 \},
\]
\[\{ e_1 - e_4, e_2 - e_5, e_1 - e_5 \}
\]
\[\mathcal{F}(A_4) = \{ \phi, \{ e_1 - e_3, e_1 - e_4, e_2 - e_5, e_1 - e_5 \}\}
\]
Therefore the two-by-two nonisomorphic nilpotent Lie algebras of maximal rank and of type $A_l$ ($l = 1, 2, 3, 4$) are:

1. $l = 1$, $A_1 = Ce_{12}$
2. $l = 2$, $A_2 = Ce_{12} \oplus Ce_{23} \oplus Ce_{13}, [e_{12}, e_{23}] = e_{13}$
3. $l = 3$, $A_{3,1} = Ce_{12} \oplus Ce_{23} \oplus Ce_{34} \oplus Ce_{13} \oplus Ce_{24}, [e_{12}, e_{23}] = e_{13}[e_{23}, e_{34}] = e_{24}$
   
   $A_{3,2} = A_{3,1} \oplus Ce_{14}$ with the brackets of $A_{3,1}$ along with $[e_{12}, e_{24}] = [e_{13}, e_{34}] = e_{24}$.

4. $l = 4$, $A_{4,1} = Ce_{12} \oplus Ce_{23} \oplus Ce_{34} \oplus Ce_{45} \oplus Ce_{13} \oplus Ce_{24} \oplus Ce_{35}, [e_{12}, e_{23}] = e_{13}[e_{23}, e_{34}] = e_{24} [e_{34}, e_{45}] = e_{35}$
   
   $A_{4,2} = A_{4,1} \oplus Ce_{14}$ with the brackets of $A_{4,1}$ along with $[e_{12}, e_{24}] = [e_{13}, e_{34}] = e_{24}$.

3. ALGEBRAS OF TYPE $B_l$ ($l \geq 2$)

3.1. Let $V = Ce_1 \oplus \cdots \oplus Ce_l \oplus Ce_o \oplus Ce_l \oplus \cdots \oplus Ce_1$ be a complex vector space of dimension $2l + 1$ and $\psi$ the nondegenerate symmetric bilinear form on $V$ whose matrix is

$$S = \begin{pmatrix} 0 & 0 & s \\ 0 & -2 & 0 \\ s & 0 & 0 \end{pmatrix},$$

where $s$ is a $l \times l$ matrix. Let $g$ be the set of all endomorphisms $\chi$ of $V$ such that

$$\psi(\chi v, v') + \psi(v, \chi v') = 0 \quad \forall v, v' \in V.$$ 

One knows that $g$ is simple.

Let $h$ be the set of diagonal elements of $g$. It is a commutative subalgebra of $g$ having for a basis the elements

$$H_i = E_{ii} - E_{-i, -i} \quad (1 \leq i \leq l).$$
Recall that $E_{ij}$ is the matrix having 1 in the $(i,j)$ position and 0 elsewhere. Let $(e_i)$ be the basis of $h^*$ dual to $(H_i)$ and set
\[ R = \{ \pm e_i; 1 \leq i \leq l \} \cup \{ \pm e_i \pm e_j; 1 \leq i < j \leq l \}. \]

Then $h$ is a Cartan subalgebra of $g$ and the roots of $(g,h)$ are the elements of $R$. The root system $R$ is of type $B$ with
\[ B_l = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & & & \\
& & \ddots & & \\
& & & 0 & \\
& & & & -1 \end{pmatrix}. \]

Set $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\ldots$, $\alpha_{l-1} = e_{l-1} - e_l$, $\alpha_l = e_l$ then $(\alpha_1, \ldots, \alpha_l)$ is a base of $R$. 

**3.2.** The set of positive roots relative to the base $(\alpha_1, \ldots, \alpha_l)$ is
\[ R^+(B_l) = \{ e_i; 1 \leq i \leq l \} \cup \{ e_i \pm e_j; 1 \leq i < j \leq l \}. \]

Let
\[ X_{ij} = E_{ij} - E_{-j,-i}, \quad 1 \leq i < j \leq l, \]
\[ Y_{ij} = E_{-i,-j} - E_{-j,-i}, \quad 1 \leq i < j \leq l, \]
\[ Z_i = 2E_{i,-i} + E_{-i,-i}, \quad 1 \leq i \leq l. \]

Then by simple calculations we get
\[ g^{e_i - e_j} = CX_{ij}, \quad 1 \leq i < j \leq l, \]
\[ g^{e_i + e_j} = CY_{ij}, \quad 1 \leq i < j \leq l, \]
\[ g^{e_i} = CZ_i, \quad 1 \leq i \leq l. \]

One easily checks that $(X_{ij}, Y_{ij}, Z_i)$ is a basis of $\mathfrak{u}^+(B_l)$. Some little computations give the brackets of $\mathfrak{u}^+(B_l)$ (the others are either 0 or deduced by antisymmetry):
\[ [X_{ij}, X_{jk}] = X_{ik}, \quad 1 \leq i < j < k \leq l, \]
\[ [X_{ij}, Y_{jk}] = Y_{ik}, \quad 1 \leq i < j < k \leq l; \]
\[ [X_{ik}, Y_{jk}] = -Y_{ij}, \quad 1 \leq i < j < k \leq l; \]
\[ [X_{jk}, Y_{ij}] = Y_{ij}, \quad 1 \leq i < j < k \leq l; \]
\[ [X_{ij}, Z_i] = XZ_i, \quad 1 \leq i < j \leq l, \]
\[ [Z_i, Z_j] = 2Y_{ij}, \quad 1 \leq i < j \leq l. \]
3.3. Using the values of the entries $a_{ij}$ of $B_t$ we notice that the chains
\[
\alpha_i, \alpha_i + \alpha_{i+1}, \ldots, \alpha_i - a_{ji} \alpha_j, \quad 1 \leq i, j \leq l,
\]
are
\[
\alpha_i, \alpha_i + \alpha_{i+1}(1 \leq i \leq l - 2); \quad \alpha_{i-1}, \alpha_{i-1} + \alpha_i, \alpha_{i-1} + 2 \alpha_i, \alpha_i;
\]
that is,
\[
e_i - e_{i+1}, e_i - e_{i+2}(1 \leq i \leq l - 2); \quad e_{i-1} - e_i, e_{i-1} + e_i, e_i.
\]
Therefore, the complement in $R_+(B_t)$ of the above chains is
\[
R_{+-}(B_t) = \{e_i; 1 \leq i \leq l - 2\} \cup \{e_i - e_j; 3 \leq i + 2 < j \leq l\}
\]
\[
\cup \{e_i + e_j; 1 \leq i < j \leq l, (i, j) \neq (l - 1, l)\}.
\]

3.4. Lemma. Let $I$ be an ideal of $R_+(B_t)$:

(1) if $e_i \in I$ then $e_r \in I$ for $r \leq i$ and $e_r + e_s \in I$ for $r \leq i, s \geq 2$,

(2) if $e_i - e_j \in I$ then $e_r - e_s \in I$ for $r \leq i, s \geq j$; $e_r \in I$ for $r \leq i$; and

$e_r + e_s \in I$ for $r \leq i, s \geq 2$,

(3) if $e_i + e_j \in I$ then $e_r + e_s \in I$ for $r \leq i, s > r$.

Proof. Just apply the definition of an ideal as in 2.4.

3.5. For $1 \leq m \leq l - 2$, we define the subset of $R_{+-}(B_t)$
\[
R_m(B_t) = \{e_i; 1 \leq i \leq m\} \cup \{e_i - e_j; 1 \leq i \leq m, 4 \leq j \leq l, i + 2 < j\}
\]
\[
\cup \{e_i + e_j; 1 \leq i \leq m, 2 \leq j \leq l, i < j\};
\]
then $R_{l-2}(B_t) = R_{+-}(B_t)$.

Let $\mathcal{J}_m(B_t)$ be the set of ideals of $R_+(B_t)$ contained in $R_m(B_t)$; then
\[
\mathcal{J}_{l-2}(B_t) = \mathcal{J}(B_t).\]

Let $b_m$ be the number of elements of $\mathcal{J}_m(B_t)$; then
\[
\# \mathcal{J}(B_t) = b_{l-2},\]
If we define
\[
\mathcal{J}_m^0(B_t) = \{I \in \mathcal{J}_m(B_t); e_m + e_{m+1} \notin I\}
\]
\[
\mathcal{J}_m^1(B_t) = \{I \in \mathcal{J}_m(B_t); e_m + e_{m+1} \in I\}
\]
then
\[
\mathcal{J}_m(B_t) = \mathcal{J}_m^0(B_t) \cup \mathcal{J}_m^1(B_t)\]
is a partition. Now, if \( I \in \mathcal{J}_m^0(B_i) \) then \( e_m + e_j \not\in I \) for \( j \geq m + 1 \), \( e_m \not\in I \), and \( e_m - e_j \not\in I \) for \( j \geq m + 3 \) (by 3.4). Thus \( I \in \mathcal{R}_{m-3}(B_i) \) (with \( R_0(B_i) = \phi \)) which gives \( \mathcal{J}_m^0(B_i) = \mathcal{J}_{m-3}(B_i) \) (with \( \mathcal{J}_0(B_i) = \{ \phi \} \)) and \( \# \mathcal{J}_{m+1}(B_i) = b_{m-1,1} \) (with \( b_{m1} = 1 \)). Next, if \( I \in \mathcal{J}_m^1(B_i) \) then \( e_r + e_s \in I \) for \( 1 \leq r < s \leq m + 1 \) (by 3.4) and \( J = I \setminus \{ e_r + e_s; 1 \leq r < s \leq m + 1 \} \) is included in

\[
\mathcal{R}_m^1(B_i) = \{ e_i; 1 \leq i \leq m \} \cup \{ e_i - e_j; 1 \leq i \leq m, 4 \leq j \leq l, i + 2 < j \}
\]

\[
\cup \{ e_i + e_j; 1 \leq i \leq m, m + 2 \leq j \leq l, i < j \}.
\]

The set \( \mathcal{R}_m^1(B_i) \) is an ideal of \( \mathcal{R}_m^1(B_i) \) and can be identified to the ideal \( \mathcal{R}_m(A_{2l-1-1}^0) \) of \( \mathcal{R}_m(A_{2l-1-1}^0) \) by the following isomorphism:

\[
e_i - e_j \mapsto e_i - e_j, \quad 1 \leq i \leq m, 4 \leq j \leq l, i + 2 < j, \]

\[
e_i \mapsto e_i - e_{i+1}, \quad 1 \leq i \leq m, \]

\[
e_i + e_j \mapsto e_i - e_{2l+2-j}, \quad 1 \leq i \leq m, m + 2 \leq j \leq l, i < j.
\]

Then \( J \) can be identified to an ideal of \( \mathcal{R}_m(A_{2l-1-1}^0) \) contained in \( \mathcal{R}_m(A_{2l-1-1}^0) \), the mapping \( I \mapsto J \) being obviously a bijection from \( \mathcal{J}_m(B_i) \) onto \( \mathcal{J}_m(A_{2l-1-1}^0) \) we get \( \# \mathcal{J}_m(B_i) = s_{m,2l-1} \). Therefore,

\[
b_{ml} = b_{m-1,l} + s_{m,2l-1-m}, \quad 1 \leq m \leq l - 2,
\]

\[
b_{0l} = 1, \quad l \geq 2.
\]

3.6. Lemma. \( b_{ml} = (\frac{2l-2}{m}), 0 \leq m \leq l - 2. \)

Proof. From 2.6 we know that \( s_{m,2l-1-m} = (\frac{2l-3}{m}) - (\frac{2l-3}{m-2}) \) for \( 0 \leq m \leq 2l - 3 - m \), i.e. \( 0 \leq m \leq l - 2. \) Therefore,

\[
(R_{ml}): b_{ml} = b_{m-1,l} + \left( \frac{2l-3}{m} \right) - \left( \frac{2l-3}{m-2} \right), \quad 1 \leq m \leq l - 2.
\]

If we add up the relations \( (R_{1l}), (R_{2l}), \ldots, (R_{ml}) \) we get

\[
b_{ml} = b_{0l} + \left( \frac{2l-3}{m-1} \right) + \left( \frac{2l-3}{m} \right) - \left( \frac{2l-3}{m-1} \right) - \left( \frac{2l-3}{0} \right)
\]

\[
= \left( \frac{2l-2}{m} \right), \quad 1 \leq m \leq l - 2,
\]

and this holds for \( m = 0 \), too.
3.7. It follows that \( \#\mathfrak{F}(B_l) = b_{l-2,l} \) is given by

\[
\#\mathfrak{F}(B_l) = \left( \frac{2l - 2}{l - 2} \right), \quad l \geq 2.
\]

3.8. Theorem. The number \( b_l \) of isomorphism classes of nilpotent Lie algebras of maximal rank and of type \( B_l \) is

\[
b_l = \left( \frac{2l - 2}{l - 2} \right).
\]

Proof. The automorphism group of the Dynkin diagram

\[
\begin{align*}
&\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \\
&1 \quad 2 \quad 3 \quad \ldots \quad l-1 \quad l
\end{align*}
\]

is reduced to the identity; therefore by 1.5 one has \( b_l = \#\mathfrak{F}(B_l) \), whence the conclusion from 3.7.

3.9. Proposition. The 2-by-2 nonisomorphic nilpotent Lie algebra of maximal rank and of type \( B_l \) \((l = 2, 3)\) are

(a) \( l = 2 \),

\[
B_2 = \mathbb{C}X_{12} \oplus \mathbb{C}Z_1 \oplus \mathbb{C}Z_2 \oplus \mathbb{C}Y_{12}
\]

\[
[X_{12}, Z_2] = Z_1, \quad [Z_1, Z_2] = 2Y_{12}.
\]

(b) \( l = 3 \),

\[
B_{3,1} = \mathbb{C}X_{12} \oplus \mathbb{C}X_{23} \oplus \mathbb{C}X_{13} \oplus \mathbb{C}Z_2 \oplus \mathbb{C}Z_3 \oplus \mathbb{C}Y_{23}
\]

\[
[X_{12}, X_{23}] = X_{13}, \quad [X_{23}, Z_3] = Z_2, \quad [Z_2, Z_3] = 2Y_{23};
\]

\[
B_{3,2} = B_{3,1} \oplus \mathbb{C}Z_1 \text{ with the brackets of } B_{3,1} \text{ along with}
\]

\[
[X_{12}, Z_2] = [X_{13}, Z_3] = Z_1;
\]

\[
B_{3,3} = B_{3,1} \oplus \mathbb{C}Y_{13} \text{ with the brackets of } B_{3,2} \text{ along with}
\]

\[
[X_{12}, Y_{23}] = Y_{13}, \quad [Z_1, Z_3] = 2Y_{13};
\]

\[
B_{3,4} = B_{3,1} \oplus \mathbb{C}Y_{12} \text{ with the brackets of } B_{3,3} \text{ along with}
\]

\[
-\left[ X_{13}, Y_{23} \right] = [X_{23}, Y_{13}] = Y_{12}, \quad [Z_1, Z_2] = 2Y_{12}.
\]

Proof is left to the reader.
4. ALGEBRAS OF TYPE $C_l (l \geq 3)$

**Theorem.** The number $c_l$ of isomorphism classes of nilpotent Lie algebras of maximal rank and the type $C_l$ is

$$c_l = \left( \frac{2l-2}{l-2} \right).$$

**Proof.** One knows that $B_l$ and $C_l$ are dual to each other and it is easy to check that the isomorphism $\alpha \mapsto \alpha^\vee$ from $R_\pm(B_l)$ onto $R_\pm(C_l)$ realizes a bijection from $\mathcal{I}(B_l)$ onto $\mathcal{I}(C_l)$; therefore $b_l = c_l$.

5. ALGEBRAS OF TYPE $D_l (l \geq 4)$

5.1. Let

$$V = \mathbb{C}e_{-l} \oplus \cdots \oplus \mathbb{C}e_{-1} \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_l$$

be a complex vector space of dimension $2l$ and $\Psi$ be a nondegenerate symmetric bilinear form on $V$ whose matrix is the $2l \times 2l$ one:

$$S = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

Let $\mathfrak{g}$ be the set of all endomorphisms $x$ of $V$:

$$\Psi(xv, v') + \Psi(v, xv') = 0 \quad \forall v, v' \in V,$$

one knows that $\mathfrak{g}$ is a simple Lie algebra.

Let $\mathfrak{h}$ be the set of diagonal elements of $\mathfrak{g}$. It is a commutative subalgebra of $\mathfrak{g}$ having for basis the elements:

$$H_i = E_{ii} - E_{-i,-i} \quad (1 \leq i \leq l).$$

(Recall that $E_{ij}$ is the matrix having 1 in the $(i, j)$-position and 0 elsewhere).

Let $(e_i)$ be the basis of $\mathfrak{h}^*$ dual to $(H_i)$ and set

$$R = \{ \pm e_i \pm e_j; 1 \leq i < j \leq l \};$$
then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and the roots of $(\mathfrak{g}, \mathfrak{h})$ are the elements of $R$. The root system $R$ is of type $D_l$ with

$$D_l = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}. $$

Set $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3, \ldots, \alpha_{l-1} = e_{l-1} - e_l$, $\alpha_l = e_{l-1} + e_l$. Then $(\alpha_1, \ldots, \alpha_l)$ is a base $B$ of $R$.

**5.2.** The set of positive roots relative to $B$ is

$$R_+(D_l) = \{e_i \pm e_j; 1 \leq i < j \leq l\}.$$  

Let

$$X_{ij} = E_{ij} - E_{-j, -i}, \quad 1 \leq i < j \leq l,$$

$$Y_{ij} = E_{i, -j} - E_{j, -i}, \quad 1 \leq i < j \leq l.$$  

Then by simple computations we get

$$\mathfrak{g}^{e_i - e_j} = \mathbb{C} X_{ij}, \quad 1 \leq i < j \leq l,$$

$$\mathfrak{g}^{e_i + e_j} = \mathbb{C} Y_{ij}, \quad 1 \leq i < j \leq l.$$  

One easily checks that $(X_{ij}, Y_{ij})$ is a basis of $\mathfrak{n}_+(D_l)$. Some little calculations give the brackets

$$[X_{ij}, X_{jk}] = X_{ik}, \quad 1 \leq i < j < k \leq l,$$

$$[X_{ij}, Y_{jk}] = Y_{ik}, \quad 1 \leq i < j < k \leq l,$$

$$[X_{ik}, Y_{jk}] = -[X_{jk}, Y_{ik}] = -Y_{ij}, \quad 1 \leq i < j < k \leq l$$

(the other ones are either 0 or deduced by antisymmetry).

**5.3.** Using the values $a_{ij}$ of $D_l$ we notice that the chains

$$\alpha_i, \alpha_i + \alpha_j, \ldots, \alpha_i - a_{ji} \alpha_j, \quad 1 \leq i, j \leq l,$$

are

$$\alpha_i (1 \leq i \leq l), \alpha_i + \alpha_{i+1} (1 \leq i \leq l - 2), \alpha_{l-2} + \alpha_l;$$
that is,
\[ e_i - e_{i+1} (1 \leq i \leq l - 1), \quad e_i - e_{i-2} (1 \leq i \leq l - 2), \quad e_{i-1} + e_i, \quad e_{i-2} + e_i. \]

Therefore the complement in \( R_+ (D_l) \) of the above chains is
\[
R_{++} (D_l) = \{ e_i - e_j; 3 \leq i + 2 < j \leq l \}
\cup \{ e_i + e_j; 1 \leq i \leq l, (i, j) \neq (l - 1, l), (l - 2, l) \}.
\]

One can partition \( R_{++} (D_l) \) as
\[
R_{++} (D_l) = R^a (D_l) \cup R^o (D_l) \cup R^l (D_l),
\]
where
\[
R^a (D_l) = \{ e_i - e_j; 3 \leq i + 2 < j \leq l - 1 \}
\]
\[
R^o (D_l) = \{ e_i \pm e_j; 1 \leq i \leq l - 3 \}
\]
\[
R^l (D_l) = \{ e_i + e_j; 1 \leq i < j \leq l - 1 \}.
\]

Obviously, one can identify \( R^a (D_l) \) with \( R_{++} (A_{l-2}) \).

**5.4.** For \( l \geq 5 \) (we will treat separately the case \( l = 4 \)), the automorphism group of the Dynkin diagram:

\[
\begin{array}{cccccccc}
& & o & & o & & o & & o \\
1 & 2 & 3 & \cdots & 0_{l-1} & & & \\
& & & & o & & & \\
& & & & & 0_i & & \\
\end{array}
\]

is \( S_l (D_l) = \{ 1, \sigma \} \) with
\[
\sigma i = i, \quad 1 \leq i \leq l - 2,
\]
\[
\sigma (l - 1) = l
\]
\[
\sigma l = l - 1.
\]

This group acts on \( R_+ (D_l) \) by
\[
\sigma (e_i - e_j) = e_i - e_j, \quad 1 \leq i < j \leq l - 1,
\]
\[
\sigma (e_i + e_j) = e_i + e_j, \quad 1 \leq i \leq l - 1,
\]
\[
\sigma (e_i + e_j) = e_i - e_j, \quad 1 \leq i \leq l - 1,
\]
\[
\sigma (e_i + e_j) = e_i + e_j, \quad 1 \leq i < j \leq l - 1.
\]
We can extend this action to $\mathcal{Z}(D_l)$ by

$$\sigma I = \{\sigma \alpha; \alpha \in I\}.$$ 

5.5. Lemma. Let $I \in \mathcal{Z}(D_l)$:

1. if $e_i - e_j \in I$ (for $3 \leq i + 2 < j \leq l - 1$) then $e_i - e_j \in I$ (for $3 \leq r + 2 < i + 2 < j \leq s \leq l$) and $e_i + e_j \in I$ (for $1 \leq r \leq i, r < s \leq l$).
2. if $e_i - e_j \in I$ (for $1 \leq i \leq l - 3$) then $e_i + e_j \in I$ (for $1 \leq r \leq i, r < s \leq l - 1$).
3. if $e_i + e_j \in I$ (for $1 \leq i < j \leq l$) then $e_i + e_j \in I$ (for $1 \leq r \leq i, r < s \leq j \leq l$).

Proof. Just apply the definition of an ideal as in 2.4.

5.6. Let $I \in \mathcal{Z}(D_l)$ and define $m_i, n_i \in \{0, 1, \ldots, l - 3\}$ by

$$m_i = \text{Sup}\{i; 1 \leq i \leq l - 3, e_i - e_i \in I\},$$
$$n_i = \text{Sup}\{i; 1 \leq i \leq l - 3, e_i + e_i \in I\},$$

with the convention $\text{Sup} \phi = 0$.

For $m, n \in \{0, \ldots, l - 3\}$ define the subset of $\mathcal{Z}(D_l)$:

$$\mathcal{Z}^{mn}(D_l) = \{I \in \mathcal{Z}(D_l); (m_i, n_i) = (m, n)\}.$$ 

Then, obviously, one gets a partition of $\mathcal{Z}(D_l)$:

$$\mathcal{Z}(D_l) = \bigcup_{(m, n) \in \{0, \ldots, l - 3\}^2} \mathcal{Z}^{mn}(D_l).$$

It is clear that

$$\sigma \mathcal{Z}^{mn}(D_l) = \mathcal{Z}^{\sigma m}(D_l), \quad l \geq 5.$$ 

Therefore,

$$\mathcal{Z}(D_l) = \bigcup_{0 \leq m, n \leq l - 3} \mathcal{Z}^{mn}(D_l), \quad l \geq 5,$$

is a fundamental domain for $\mathcal{Z}(D_l)$, so

$$d_l = \sum_{0 \leq m, n \leq l - 3} d_l^{mn}, \quad l \geq 5,$$

where

$d_l$ is the number of $\mathcal{Z}(D_l)$-orbits in $\mathcal{Z}(D_l)$,

$d_l^{mn} = \# \mathcal{Z}^{mn}(D_l)$. 

5.7. Lemma. Let \( 0 \leq m \leq n \leq l - 3 \) and \( I \in \mathfrak{N}^{m,n}(D_i) \). Then one can partition \( I \) as
\[
I = I^a \cup I^0 \cup J^1 \cup K^1,
\]
where
\[
\begin{align*}
I^a & \text{ can be identified with an ideal of } R_m(A_{l-2}) \\
I^0 &= \{e_i - e_j; 1 \leq i \leq m\} \cup \{e_i + e_j; 1 \leq i \leq m\} \\
J^1 &= \{e_i + e_j; 1 \leq r \leq n, r < s \leq l - 1\} \\
\end{align*}
\]
\( K^1 \) is an ideal of \( R_n(D_i) = \{e_i + e_j; 0 < i < j \leq l - 1\} \)
(in the sense: \( (B + K^1) \cap R_n(D_i) \subset K^1 \)).

Proof. Let
\[
I^a = I \cap R^a(D_i), \quad I^0 = I \cap R^0(D_i), \quad I^1 = I \cap R^1(D_i).
\]
Then
\[
I = I^a \cup I^0 \cup I^1
\]
is a partition. If there exists \((i, j)\) such that \( 2 \leq m + 2 < i + 2 < j \leq l - 1 \) and \( e_i - e_j \in I^a \) then \( e_i - e_j \in I^a \) by 5.5(1), contradicting the maximality
of \( m \). Therefore, \( e_i - e_j \notin I \) for \( 2 \leq m + 2 < i + 2 < j \leq l - 1 \); it follows
that \( I^a \subset R_m^a(D_i) \).

Obviously, \( R_n^a(D_i) \) can be identified with \( R_m(A_{l-2}) \) and \( I^a \) with an ideal
of \( R_m(A_{l-2}) \).

The description of \( I^0 \) is clear.

Finally, \( e_n + e_i \in I \) gives \( e_i + e_j \in I \) for \( 1 \leq r \leq n, r < s \leq l - 1 \) by 5.5(3),
thus \( e_i + e_j \in I^1 \) for \( 1 \leq r \leq n, r < s \leq l - 1 \); in other words, \( J^1 \subset I^1 \).

Let \( K^1 \) be the complement of \( J^1 \) in \( I^1 \); then \( I^1 = J^1 \cup K^1 \) and
\( K^1 \subset R_n^1(D_i) \); obviously \( K^1 \) is an ideal of \( R_n^1(D_i) \).

5.8. Lemma. The set of ideals \( \mathfrak{N}^1_n(D_i) \) of \( R_n^1(D_i) \) has exactly \( 2^{l-2-n} \) elements.

Proof. Let \( X_{nl} \) be the number of elements of \( \mathfrak{N}^1_n(D_i) \). As usual we
write a partition,
\[
\mathfrak{N}^1_n(D_i) = \mathfrak{N}^{10}_n(D_i) \cup \mathfrak{N}^{11}_n(D_i),
\]
where
\[
\begin{align*}
\mathfrak{N}^{10}_n(D_i) &= \{I \in \mathfrak{N}^1_n(D_i); e_{n+1} + e_{i-1} \notin I\} \\
\mathfrak{N}^{11}_n(D_i) &= \{I \in \mathfrak{N}^1_n(D_i); e_{n+1} + e_{i-1} \in I\}
\end{align*}
\]
If \( I^{10} \in \mathfrak{N}_{n}^{10}(D_i) \) then \( e_i + e_{i-1} \notin I^{10} \) for \( n + 1 \leq i \leq l - 2 \) by 5.5(3) thus \( I^{10} \) is in fact included in the following subset of \( R^2_n(D_i) \):

\[
R^2_n(D_i) = \{ e_i + e_j ; n < i < j < l \}
\]

We can identify \( R^2_n(D_i) \) with

\[
R^1_{n+1}(D_i) = \{ e_r + e_s ; n + 1 < r < s < l - 1 \}
\]

by the bijection

\[
e_i + e_j \rightarrow e_{i+1} + e_{j+1}
\]

(indeed, the condition \( n < i < j \leq l - 2 \) is transformed into \( n + 1 < i + 1 < j + 1 \leq l - 1 \) and \( I^{10} \) becomes an ideal of \( R^1_{n+1}(D_i) \). In other words, \( \mathfrak{N}_{n}^{10}(D_i) \) can be identified with \( \mathfrak{N}_{n+1}^{1}(D_i) \) so \( \# \mathfrak{N}_{n}^{10}(D_i) = \chi_{n+1,i} \).

Next, if \( I^{11} \in \mathfrak{N}_{n}^{11}(D_i) \) then \( e_{n+1} + e_j \in I^{11} \) for \( n + 1 < j \leq l - 1 \) by 5.5(3). If we remove these elements from \( I^{11} \) then we obtain an ideal \( J^{11} \) of \( R^1_{n+2}(D_i) \). Obviously,

\[
I^{11} \rightarrow J^{11} \mathfrak{N}_{n}^{11}(D_i) \rightarrow \mathfrak{N}_{n+1}^{1}(D_i)
\]

is a bijection; therefore \( \# \mathfrak{N}_{n}^{11}(D_i) = \chi_{n+1,i} \).

In conclusion, the partition gives

\[
\chi_{nl} = \chi_{n+1,l} + \chi_{n+1,l} = 2\chi_{n+1,l}, \quad 0 \leq n \leq l - 4,
\]

and starting from \( \chi_{l-3,l} = 2^2 \) (since \( \mathfrak{N}_{l-3}^{1}(D_i) = \{ e_{l-2} + e_{l-3} \} \) an induction leads to

\[
\chi_{nl} = 2^{l-2-n}.
\]

5.9. **Lemma.** \( d^m_{ln} = \lceil \left( \frac{m + l - 4}{m} \right) - \left( \frac{m + l - 4}{m - 2} \right) \rceil 2^{l-2-n} \).

**Proof.** With the notation of 5.7, it is obvious that \( I \rightarrow (I^u, K^1) \), \( \mathfrak{N}^m(D_i) \rightarrow \mathfrak{N}^m(A_{l-2}) \times \mathfrak{N}^1_n(D_i) \) is a bijection. Therefore,

\[
d^m_{ln} = \# \mathfrak{N}^m(A_{l-1}) \times \# \mathfrak{N}^1_n(D_i).
\]

From 2.6, we get

\[
\# \mathfrak{N}^m(A_{l-1}) = s_{m,l-2} = \left( \frac{m + l - 4}{m} \right) - \left( \frac{m + l - 4}{m - 2} \right)
\]

and from 5.8, we get

\[
\# \mathfrak{N}^1_n(D_i) = 2^{l-2-n}, \quad \text{whence the conclusion.}
\]
5.10. Theorem. The number \( d_l \) of isomorphism classes of nilpotent Lie algebras of maximal rank and of type \( D_l \) is
\[
d_l = 2 \left( \frac{2l - 4}{l - 3} \right) \quad \text{for } l \geq 5.
\]

Proof. From 5.6 and 5.9 we get
\[
d_l = \sum_{0 \leq m \leq n \leq l - 3} s_{m,l - 2} 2^{l - 2 - n} = \sum_{n = 0}^{l - 3} \left( \sum_{m = 0}^{n} s_{m,l - 2} \right) 2^{l - 2 - n}.
\]
But, by induction, \( \sum_{m = 0}^{n} s_{m,l - 2} = s_{n,l - 1} \)

(Indeed, \( \sum_{m = 0}^{0} s_{m,l - 2} = s_{0,l - 1} = 1 \) and \( \sum_{m = 0}^{n+1} s_{m,l - 2} = \sum_{m = 0}^{n} s_{m,l - 2} + s_{n+1,l - 2} = s_{n,l - 1} + s_{n+1,l - 2} = s_{n+1,l - 1} \) by 2.5.). Therefore,
\[
d_l = \sum_{n = 0}^{l - 3} s_{n,l - 1} 2^{l - 2 - n}
\]
\[
= 2 \sum_{n = 0}^{l - 3} \left( \frac{l - 3 + n}{n} \right) 2^{l - 3 - n} - 2 \sum_{n = 0}^{l - 3} \left( \frac{l - 3 + n}{n - 2} \right) 2^{l - 3 - n}.
\]

We now use one of the infinitely many combinatory identities,
\[
\sum_{n = 0}^{k} \left( \frac{p - k - 1 + n}{n} \right) 2^{k - n} = \sum_{n = 0}^{k} \left( \frac{p}{n} \right),
\]
which could be proved by induction. If we take \( k = l - 3 \) and \( p = 2l - 5 \) then we get
\[
\sum_{n = 0}^{l - 3} \left( \frac{l - 3 + n}{n - 2} \right) 2^{l - 3 - n} = \sum_{n = 0}^{l - 3} \left( \frac{2l - 5}{n} \right).
\]

Next,
\[
\sum_{n = 0}^{l - 3} \left( \frac{l - 3 + n}{n - 2} \right) 2^{l - 3 - n}
\]
\[
= \left( \frac{l - 3}{-2} \right) 2^{l - 3} + \left( \frac{l - 2}{-1} \right) 2^{l - 4} + \left( \frac{l - 1}{0} \right) 2^{l - 5} + \ldots + \left( \frac{2l - 6}{l - 5} \right)
\]
\[
= \sum_{n = 0}^{l - 5} \left( \frac{l - 1 + n}{n} \right) 2^{l - 5 - n}.
\]
and if we take $k = l - 5, p = 2l - 5$ in the above identity we get
\[
\sum_{n=0}^{l-5} \left( \frac{l - 1 + n}{n} \right) 2^{l-5-n} = \sum_{n=0}^{l-5} \left( \frac{2l - 5}{n} \right).
\]

Therefore,
\[
d_k = 2 \sum_{n=0}^{l-3} \left( \frac{2l - 5}{n} \right) - 2 \sum_{n=0}^{l-5} \left( \frac{2l - 5}{n} \right)
= 2\left( \frac{2l - 5}{l - 4} \right) + 2\left( \frac{2l - 5}{l - 3} \right) = 2\left( \frac{2l - 4}{l - 3} \right).
\]

### 5.11. Type $D_4$
With the notation of 5.1, 5.2, and 5.3 we get:

\[
\begin{align*}
[X_{12}, X_{23}] &= X_{13}[X_{12}, Y_{23}] = Y_{13}[X_{13}, Y_{23}] = -Y_{12}[X_{23}, Y_{13}] = Y_{12} \\
[X_{12}, X_{24}] &= X_{14}[X_{12}, Y_{24}] = Y_{14}[X_{14}, Y_{24}] = -Y_{12}[X_{24}, Y_{14}] = Y_{12} \\
[X_{13}, X_{34}] &= X_{14}[X_{13}, Y_{34}] = Y_{14}[X_{14}, Y_{34}] = -Y_{13}[X_{34}, Y_{14}] = Y_{13} \\
[X_{23}, X_{34}] &= X_{24}[X_{23}, Y_{34}] = Y_{24}[X_{24}, Y_{34}] = -Y_{23}[X_{34}, Y_{24}] = Y_{23}.
\end{align*}
\]

The automorphism group $G_{\text{D}_4}$ of the Dynking diagram,

\[\begin{array}{c}
\text{1} \\
\text{2} \\
\text{0} \\
\text{3} \\
\text{4}
\end{array}\]

is the subgroup (isomorphic to $G_{\text{D}_4}$) of $G_4$ which fixes 2. It acts on $R_+(D_4)$ by

\[
\sigma \left( \sum_{i=1}^{4} n_i \alpha_i \right) = \sum_{i=1}^{4} n_i \alpha_{\sigma(i)}.
\]

Each of the five diagonals

\[
R^+_1(D_4) = \{ \alpha \in R_+(D_4); |\alpha| = i \}, \quad 1 = 1, \ldots, 5
\]

(where $\sum_{i=1}^{4} n_i \alpha_i = \sum_{i=1}^{4} n_i$) is stabilized by $G_{\text{D}_4}$ (since $|\sigma \alpha| = |\alpha|$). Therefore $G_{\text{D}_4}$ acts on $R^+_+(D_4)$ by just permuting the diagonal:

\[
R^3_+(D_4) = \{ \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 \}.
\]
Finally, there are exactly six (two by two nonisomorphic) nilpotent Lie algebras of maximal rank and of type $D_4$:

$$D_{4,1} = (CX_{12} \oplus CX_{23} \oplus CX_{34} \oplus CY_{34}) \oplus (CY_{13} \oplus CX_{24} \oplus CY_{24})$$

$$[X_{12}, X_{23}] = X_{13}, \quad [X_{23}, X_{34}] = X_{24}, \quad [X_{23}, Y_{34}] = Y_{24}$$

$D_{4,2} = D_{4,1} \oplus CX_{14}$ with the brackets of $D_{4,1}$ along with

$$[X_{12}, X_{24}] = [X_{13}, X_{34}] = X_{14}$$

$D_{4,3} = D_{4,2} \oplus CY_{14}$ with the brackets of $D_{4,2}$ along with

$$[X_{12}, Y_{14}] = [X_{13}, Y_{34}] = Y_{14}$$

$D_{4,4} = D_{4,3} \oplus CY_{23}$ with the brackets of $D_{4,3}$ along with

$$[X_{24}, Y_{34}] = -Y_{23}$$

$D_{4,5} = D_{4,4} \oplus CY_{13}$ with the brackets of $D_{4,4}$ along with

$$[X_{12}, Y_{23}] = -[X_{14}, Y_{34}] = [X_{34}, Y_{14}] = Y_{13}$$

$D_{4,6} = D_{4,5} \oplus CY_{12}$ with the brackets of $D_{4,5}$ along with

$$[X_{13}, Y_{23}] = [X_{14}, Y_{24}] = -[X_{23}, Y_{13}] = -[X_{24}, Y_{14}] = -Y_{12}.$$


