



Thermodynamics of flat FLRW universe in Rastall theory



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ABSTRACT

In this paper, after referring to the Rastall theory, we address some of its cosmological consequences. Moreover, bearing the Clausius relation in mind, using Friedman equations in Rastall theory and the Cai–Kim temperature, we obtain a relation for the apparent horizon entropy of a flat FLRW universe. In addition, we impose the entropy positivity condition on the obtained relation for the horizon entropy, to find some constraints on the Rastall parameters. Moreover, we investigate the second and generalized second laws of thermodynamics. The results of considering a dominated perfect fluid of constant state parameter are also addressed helping us familiarize with the Rastall theory.

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1. Introduction

Generalization of the $T^{\mu\nu}_{;\mu} = 0$ condition from flat to the curved spacetimes is one of the Einstein's basic assumptions to get the general relativity [1,2]. Indeed, by insisting on this generalization in his attempt to formulate the Mach principle, Einstein could get his famous tensor and thus the corresponding field equations leading to the second order equations of motion [1,2], which have also vast applications in the cosmological and astrophysical studies [2,3]. Many years after Einstein, Jacobson showed that one can reobtain Einstein equations by applying the Clausius relation on the local Rindler causal horizon [4]. In fact, the Jacobson work proposes that for spacetimes with a causal horizon the Einstein equations on the horizon may be considered as a thermodynamical equation of state if one generalizes the four law of black holes to the causal horizon, i.e. a causal horizon may be taken into account as a proper causal boundary. Moreover, the Jacobson's idea has been generalized to $f(R)$ theory by Eling et al. showing that terms other than the Einstein–Hilbert action produce entropy due to their non-equilibrium thermodynamical aspects [5], which leads to the modification of the event horizon entropy [5,6]. One can also use the Eling et al.'s proposal to get the equation of motion in scalar–tensor gravity theory [7]. It is useful to note here that one may use the thermodynamics laws and gravitational field equations to get the horizon entropy in the vast theories of gravity [8], which may satisfy the second law of thermodynamics [9]. The horizon entropy, indeed, is not the total entropy in a gravitational system. In fact, the total entropy of a gravitational system, includ-

ing the sum of horizon entropy and the entropy of fields confined by the horizon, should increase during every gravitational process. The latter referred to as the generalized second law of thermodynamics [10,11].

Introducing the unified first law of thermodynamics, Hayward could show that the Einstein field equations on the trapping horizon of a dynamic black hole are nothing but the unified first law of thermodynamics [12–15]. This shows that the trapping horizon may be considered as a causal boundary for non-stationary spherically symmetric spacetimes. In fact, extending the Hayward method to the apparent horizon of a FLRW universe, as its causal boundary [16], we can get the apparent horizon entropy and Friedman equations in the various theories of gravity [6,7,17,18]. This approach may also be employed to investigate the thermodynamic properties of apparent horizon of the FLRW universe in some braneworld models such as the Gauss–Bonnet and warped DGP braneworlds [19,20]. Moreover, one can also use the Hayward proposal to study the effects of interactions between the dark energy and other parts of cosmos on the horizon entropy in the Einstein and quasi-topological theories [21,22]. Here, it is worthwhile to mention that this approach does not lead to the Friedman equations in the scalar–tensor theory, a result which is due to the non-equilibrium thermodynamic aspect of the scalar–tensor theory [7].

In another approach, Cai et al. applied the Clausius relation to the apparent horizon of the FLRW universe, and used the horizon entropy relation to get Friedman equations in various theories of gravity [17,23]. In addition, one can also use the Friedman equations as well as the Clausius relation to get an expression for the effects of interactions between the dark energy and other parts of cosmos on the horizon entropy in the Einstein and

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quasi-topological theories [21,22,24]. Although, following such approaches, the definition of temperature differs from that of the Hayward–Kodama temperature, the result for the horizon entropy is equal to that of the approaches by which authors use the Hayward–Kodama temperature and the unified first law of thermodynamics [21–23,25]. Besides, it seems that the Cai–Kim temperature is more suitable than that of the Hayward–Kodama for investigating the horizon thermodynamics [26]. Indeed, their definition of temperature is equal to that of a Hawking radiation for a locally defined apparent horizon of the FLRW universe [27], and may be used to investigate the mutual relation between Friedman equations and the thermodynamic properties of apparent horizon in various gravitational theories [17,21–26]. Here, it is useful to note that the thermodynamic analysis of the total entropy of universe, including the horizon entropy and the entropy of fields confined by it, signals us to a universe satisfying thermodynamical equilibrium conditions, in vast models of gravity [28].

In 1972, by relating $T^{\mu\nu}{}_{;\mu}$ to the derivative of Ricci scalar, Rastall proposed a new formulation for gravity [29], which converges to the Einstein formulation in the flat background (empty universe). Indeed, he argued that the $T^{\mu\nu}{}_{;\mu} = 0$ assumption, made by Einstein to obtain his field equations, is questionable in the curved spacetimes [29]. In fact, the $T^{\mu\nu}{}_{;\mu} \neq 0$ condition is phenomenologically confirmed by particle creation in cosmology [30–32]. Indeed, in gravitational systems, quantum effects lead to the violation of the classical condition $T^{\mu\nu}{}_{;\mu} = 0$ [33]. Therefore, since $T^{\mu\nu}{}_{;\mu}$ is related to the Ricci scalar, the Rastall theory may be considered as a classical formulation for the particle creation in cosmology [34], and it helps us in investigating the possibility of coupling the geometry to the matter fields in a non-minimal way. For the first time, Smalley tried to get a lagrangian for this theory [35]. In addition, it seems that astrophysical analysis, including the Neutron stars evolution, and cosmological data do not reject this theory [36–38]. Recently, this theory attracts more investigators to itself, and its combinations with the Brans–Dicke and Scalar–Tensor theories of gravity can be found in [39,40]. More investigations on the various aspects of the Rastall theory in the context of current phase of the universe expansion can also be found in [34,41–46]. It is also shown that this theory reproduces some loop quantum cosmological features of the universe expansion [47].

Our aim in this paper is to study the thermodynamics of the FLRW universe in the Rastall theory. For this propose, after referring to the Rastall theory, we address some of its cosmological features. In addition, by using Friedmann equations in the Rastall theory, attributing the Cai–Kim temperature to horizon and applying the Clausius relation on the apparent horizon of the FLRW universe, we get a relation for the apparent horizon entropy in the Rastall theory. Since the entropy of a physical system is to be a positive quantity [48], we study the effects of applying this condition to the horizon entropy. The second and generalized second laws of thermodynamics are also investigated. Our investigation shows that the thermodynamic analysis of the FLRW universe imposes some restrictions on the Rastall theory parameters which are in line with previous studies. The results of considering a universe filled by a perfect fluid with constant state parameter are also addressed.

The paper is organized as follows. In the next section, after referring to the Rastall theory, we derive the corresponding Friedman, Raychaudhuri and continuity equations and point to some of the cosmological consequences of the Rastall theory. In addition, some general remarks of the FLRW universe are also addressed. Bearing the Cai–Kim temperature together with the Clausius relation in mind, we use the Friedman and continuity equations to get the horizon entropy in the Rastall theory, in section 3. The

results of imposing the entropy positivity condition on the obtained relation for the horizon entropy are also studied. We also investigate the second and generalized second laws of thermodynamics in the third section. Throughout the paper, the results of considering a perfect fluid with constant state parameter filling the universe, are investigated in more details. Section 4 is devoted to a summary and concluding remarks. Throughout this paper we set $G = \hbar = c = 1$ for the sake of simplicity.

2. Rastall theory and basic assumptions

Rastall questioned the assumption $T^{\mu\nu}{}_{;\mu} = 0$ in curved spacetime, and gets a new theory for gravity by proposing $T^{\mu\nu}{}_{;\mu} = \lambda R^{;\nu}$, where λ is an unknown constant which should be specified from observations and other parts of physics [29]. Therefore, for a spacetime metric $g_{\mu\nu}$, the corresponding gravitational field equations can be written as

$$G_{\mu\nu} + k\lambda g_{\mu\nu} R = kT_{\mu\nu}, \quad (1)$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and energy–momentum tensors, respectively [29]. Moreover, R is Ricci scalar, and k is also gravitational constant in Rastall theory and should probably be specified from other parts of physics and observations [29]. It is obvious that for $\lambda = 0$ and $k = 8\pi$ the Einstein field equations are reobtained wherever $T^{\mu\nu}{}_{;\mu} = 0$ [29].

Consider a cosmological background described by FLRW metric:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]. \quad (2)$$

Here, $a(t)$ and κ denote the scale factor and the curvature parameter, respectively, while $\kappa = -1, 0, 1$ denotes the open, flat and closed universes, respectively [3]. Apparent horizon as the marginally trapped surface of FLRW universe is defined as

$$\partial_\alpha \tilde{r} \partial^\alpha \tilde{r} = 0 \rightarrow r_A, \quad (3)$$

where $\tilde{r} = a(t)r$, leading to

$$\tilde{r}_A = a(t)r_A = \frac{1}{\sqrt{H^2 + \frac{\kappa}{a(t)^2}}}, \quad (4)$$

for the physical radii of apparent horizon. In fact, it seems that the apparent horizon can play the role of causal boundary for the FLRW spacetime [14–16,19,20]. Since cosmological data points to a flat universe [3], we only consider the flat case ($\kappa = 0$) throughout this paper. For a perfect fluid source ($T^\nu_\mu = \text{diag}(-\rho, p, p, p)$), by using Eqs. (1) and (2), we get the corresponding Friedmann equations in the Rastall theory

$$(12k\lambda - 3)H^2 + 6k\lambda\dot{H} = -k\rho, \quad (5)$$

and

$$(12k\lambda - 3)H^2 + (6k\lambda - 2)\dot{H} = kp, \quad (6)$$

where ρ and p are the energy density and pressure of the energy–momentum source, respectively. The Rastall field equations can also be written as

$$G_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{\kappa\lambda T}{4\kappa\lambda - 1} g_{\mu\nu} \right), \quad (7)$$

in which T is the trace of energy–momentum tensor. Bianchi identity implies $G^{\mu\nu}{}_{;\mu} = 0$, which finally leads to [41]

$$\left(\frac{3k\lambda - 1}{4k\lambda - 1} \right) \dot{\rho} + \left(\frac{3k\lambda}{4k\lambda - 1} \right) \dot{p} + 3H(\rho + p) = 0, \quad (8)$$

as the continuity equation in the Rastall theory. It is worth mentioning here that one can rediscover the Friedmann and continuity equations in the Einstein theory by inserting $\lambda = 0$ and $k = 8\pi$ in the above formulas. Moreover, by combining Eqs. (5) and (6) with each other, one may get the Raychaudhuri equation

$$\dot{H} = -\frac{k}{2}(\rho + p), \quad (9)$$

which is independent of λ , and it is indeed the same as that of the standard cosmology, which is based on the Einstein theory and the FLRW metric. For a fluid with state parameter $\omega = \frac{p}{\rho}$, by combining this equation with Eq. (5) one can get

$$H^2 = \frac{k(3k\lambda(1+\omega) - 1)}{3(4k\lambda - 1)}\rho. \quad (10)$$

Moreover, for a fluid, the $\rho(a)$ relation may be obtained by inserting $\omega = \frac{p}{\rho}$ into Eq. (8) and taking integral from the result. This leads to $\rho = \rho_0 a^{\frac{-3(1+\omega)(4k\lambda-1)}{3k\lambda(1+\omega)-1}}$ for a fluid of constant state parameter. It is easy to check that the famous first Friedman equation in the Einstein relativity framework is reobtained by inserting $\lambda = 0$ and $k = 8\pi$ into Eq. (10). Moreover, by combining Eqs. (5) and (6) one reaches

$$kT = 2(12k\lambda - 3)(2H^2 + \dot{H}), \quad (11)$$

in which $T = 3p - \rho$ is the trace of energy-momentum source. For a traceless energy-momentum tensor, this equation leads to $H = \frac{1}{2t}$ which is nothing but the Hubble parameter of radiation dominated era [3]. Therefore, its prediction about the Hubble parameter of the radiation dominated era is the same as that of Einstein theory (the Friedman equations). In addition, for a pressureless source ($p = 0$), Eq. (5) yields $H = \frac{(6k\lambda-2)}{(12k\lambda-3)t}$ which, only for $\lambda = 0$, covers the Friedman results about the matter dominated era ($H = \frac{2}{3t}$). Briefly, apart of a coefficient, this theory predictions about the Hubble parameter of the matter dominated era is the same as that of Einstein theory. Now, regarding a universe of a constant density ($\rho = \rho_0$) in which $\omega = -1$ that leads to a constant density, we can rewrite Eq. (10) as

$$H = \sqrt{\frac{-k}{3(4k\lambda - 1)}}\rho_0. \quad (12)$$

Therefore, for the $k > 0$ case, a Rastall theory with $\lambda < \frac{1}{4k}$ leads to a constant positive Hubble parameter if $\rho_0 > 0$. Additionally, a source with $\rho_0 < 0$ may also lead to a constant positive Hubble parameter whenever $k > 0$ and $\lambda > \frac{1}{4k}$. In sum, the Rastall theory with $k > 0$ may cover the primary inflationary era ($\omega = -1$ and $\rho_0 > 0$), if $\lambda < \frac{1}{4k}$. In fact, since H^2 is a positive quantity ($H^2 > 0$), for the $k > 0$ case, the RHS of Eq. (10) will be positive for a fluid with $-1 \leq \omega \leq \frac{1}{3}$ and $\rho > 0$, if λ either satisfies $\lambda < \frac{1}{4k}$ or $\lambda \geq \frac{1}{3k(1+\omega)}$. These results indicate that the Rastall theory may also be used to describe the current accelerating phase [34,41–46], as well as the primary inflationary era [47].

3. Thermodynamics of apparent horizon

Since the Rastall gravitational field equations (1) differ from those of the Einstein theory, the Rastall lagrangian also differs from that of Einstein [35]. Therefore, one may expect that the horizon entropy in this theory differs from the Bekenstein entropy. Now, by using the Cai–Kim approach [23], we try to get an expression for the horizon entropy. Indeed, we apply the Clausius relation on the apparent horizon and use the Cai–Kim approach together with the Friedmann equations in the Rastall theory, to get a relation for

the horizon entropy in this theory. Following that, we study the results of imposing the entropy positivity condition on the obtained entropy relation. Finally, we point to the second law of thermodynamics [9] and required conditions for meeting this law in the corresponding Rastall cosmology.

3.1. The entropy of apparent horizon

In the Cai–Kim approach, horizon temperature meets the $T = \frac{H}{2\pi}$ relation in the flat FLRW universe, while the volume change of universe in the infinitesimal time dt is neglected ($dV \approx 0$) [18,23,27]. The projection of the total four-dimensional energy-momentum tensor T_a^b in the normal direction of the two-dimensional sphere with radii \tilde{r} is defined as [18,23,27]

$$\psi_a = T_a^b \partial_b \tilde{r} + W \partial_a \tilde{r}, \quad (13)$$

where $W = \frac{\rho-p}{2}$ is the work density, and $a, b = t, r$ [18,23,27]. The energy flux (δQ^m) crossing the apparent horizon during the infinitesimal time dt and small radius change dr is defined as

$$\delta Q^m \equiv A \psi_a dx^a. \quad (14)$$

A being the surface area of two-dimensional sphere with radii \tilde{r} [18,23,27]. Simple calculations lead to [18,21–24,27]

$$\delta Q^m = -\frac{3V(\rho+p)H}{2}dt + \frac{A(\rho+p)}{2}(d\tilde{r} - \tilde{r}Hdt). \quad (15)$$

In obtaining this equation we have used $d\tilde{r} = rda + adr$ and $A\tilde{r} = 3V$. By applying the $d\tilde{r} \approx 0$ approximation to this result, we get

$$\delta Q^m = -3VH(\rho+p)dt. \quad (16)$$

Now, combining Eq. (16) with the Clausius relation ($TdS_A = \delta Q^m$) and the Cai–Kim temperature ($T = \frac{H}{2\pi}$), we reach

$$dS_A \equiv -\frac{\delta Q^m}{T} = 6\pi V(\rho+p)dt, \quad (17)$$

in which the extra minus sign comes from the universe expansion [18,21–24,27]. Now, using Eq. (8), one can rewrite this equation as

$$dS_A = -\frac{2\pi}{(4k\lambda - 1)H}((3k\lambda - 1)d\rho + (3k\lambda)dp). \quad (18)$$

Also, applying Eqs. (5) and (6) we get

$$d\rho = \frac{-1}{k}[2(12k\lambda - 3)HdH + 6k\lambda d\dot{H}], \quad (19)$$

and

$$dp = \frac{1}{k}[2(12k\lambda - 3)HdH + (6k\lambda - 2)d\dot{H}], \quad (20)$$

respectively. Inserting these two last equations into Eq. (18) leads to

$$dS_A = -\frac{8\pi^2}{kH^3}dH + \frac{32\pi^2\lambda(3k\lambda - 1)}{4k\lambda - 1}\frac{d\dot{H}}{H^4}. \quad (21)$$

Taking integral from this equation and using the $A = \frac{4\pi}{H^2}$ relation one obtains

$$S_A = \frac{2\pi A}{k} + \frac{32\pi^2\lambda(3k\lambda - 1)}{4k\lambda - 1} \int \frac{d\dot{H}}{H^4}. \quad (22)$$

This is nothing but the apparent horizon entropy in the Rastall theory. It is easy to check that the Bekenstein limit (Einstein result) is also deducible by inserting $\lambda = 0$ and $k = 8\pi$.

3.2. The entropy positivity condition

As we know, entropy is a positive quantity [48] and therefore, since Eq. (9) indicates $\dot{H} = 0$ for a universe filled by a perfect fluid with $\omega = -1$, it is apparent that the $S_A \geq 0$ condition is met if $k > 0$. Therefore, we take into account $k > 0$ now on. Applying the $S_A \geq 0$ condition to Eq. (22), we get

$$\frac{4k\lambda(1-3k\lambda)H^2}{4k\lambda-1} \int \frac{d\dot{H}}{H^4} \leq 1. \quad (23)$$

As further investigations on the $S_A \geq 0$ condition are considered, we confine ourselves to a universe filled by a perfect fluid with constant state parameter. Combining Eqs. (9) and (10) with each other, inserting the result into Eq. (22) and taking integral, one reaches

$$S_A = \frac{2\pi A}{k} + \frac{4\pi^{\frac{1}{2}}\lambda(3k\lambda-1)(1+\omega)}{3k\lambda(1+\omega)-1} A^{\frac{3}{2}}, \quad (24)$$

for the apparent horizon entropy of the flat FLRW universe filled by a perfect fluid source with constant state parameter ω . This relation is similar to the entropy of apparent horizon in Dvali–Gabadadze–Porrati (DGP) model [19,49–51]. It is also useful to note a similarity between this equation and the dark energy correction to the horizon entropy in the Einstein framework [21,22,24,50,51]. Based on Eq. (24), it is obvious that the second right hand side term of this equation vanishes for a perfect fluid with $\omega = -1$. In the Einstein relativity framework, only dark energy candidates with state parameter $\omega \neq -1$ affect the Bekenstein entropy [21,22,24]. Therefore, for a source with $\omega = -1$, the apparent horizon entropy in the Rastall theory is the same as that of the apparent horizon in the Einstein theory if $k = 8\pi$ and thus, we see that the $\omega = -1$ case again necessitates $k > 0$. In fact, it is shown that, in the Einstein relativity framework, a dark energy candidate with energy density proportional to the Hubble parameter and state parameter $\omega \neq -1$, leads to the similar relation for the horizon entropy [21,22]. Moreover, in order to get a positive entropy for all values of H , the coefficient of $A^{\frac{3}{2}}$ should be positive. For the perfect fluids with $-1 < \omega \leq 0$ and $0 \leq \omega$, the latter leads to

$$0 \leq \lambda \leq \frac{1}{3k}, \text{ and } \frac{1}{3k(1+\omega)} < \lambda, \quad (25)$$

and

$$0 \leq \lambda < \frac{1}{3k(1+\omega)}, \text{ and } \frac{1}{3k} \leq \lambda, \quad (26)$$

respectively. Therefore, a negative λ does not respect the entropy positivity condition. In short, the positivity of entropy signals us to the $k > 0$ and $\lambda \geq 0$ conditions which are in line with previous studies [36–38] as well as the results obtained from Eq. (12). Bearing the results obtained from the $H^2 > 0$ condition in mind and comparing them with the above results, we find out if the state parameter of dominated perfect fluid meets, in the Rastall theory with $k > 0$, the $-1 \leq \omega \leq 0$ condition, where $3k(1+\omega) < 3k < 4k$, and wherever λ either satisfies $0 \leq \lambda \leq \frac{1}{4k}$ or $\lambda > \frac{1}{3k(1+\omega)}$ all of the above mentioned conditions will be satisfied. In addition, for the $0 \leq \omega \leq \frac{1}{3}$ case, where $3k \leq 3k(1+\omega) \leq 4k$, λ should either satisfy $0 \leq \lambda \leq \frac{1}{4k}$ or $\lambda > \frac{1}{3k}$ to meet the entropy positivity as well as the $H^2 > 0$ conditions.

3.3. The second and generalized second laws of thermodynamics

In order to investigate the second law of thermodynamics, we use Eq. (17) to get

$$\frac{dS_A}{dt} = 6\pi V(\rho + p), \quad (27)$$

i.e. the second law of thermodynamics ($\dot{S}_A \geq 0$) is obeyed if the perfect fluid, which supports the geometry, satisfies the $\rho + p \geq 0$ condition. Indeed, if we define the state parameter to be $\omega = \frac{p}{\rho}$, so ω is not necessary constant, then for $\omega \geq -1$ and $\rho \geq 0$ the second law of thermodynamics is obtainable. Here, it is also useful to note that this result is the same as those of the Einstein and quasi-topological gravity theories [21,22]. Finally, by inserting Eq. (9) into this equation, one obtains

$$\frac{dS_A}{dt} = -\frac{12\pi V \dot{H}}{k}. \quad (28)$$

Since cosmological data indicates $\dot{H} \leq 0$ [3], the second law of thermodynamics is met if $k > 0$ which is in agreement with the result of employing the entropy positivity condition. The generalized second law of thermodynamics states that the sum of the entropy of cosmos parts, which include the horizon and the fluids confined, should increase during the universe expansion [10,11]. In order to study this law, we need evaluate the entropy of perfect fluid supporting the background. Taking into account the apparent horizon as the boundary and using the Gibbs law [48] as

$$T_m dS_m = dE + p dV, \quad (29)$$

where S_m and T_m are the matter entropy and temperature, respectively, one obtains

$$T_m dS_m = (\rho + p) dV + V d\rho. \quad (30)$$

In obtaining this equation we used the $E = \rho V$ relation. From now on, we assume that horizon and perfect fluid confined by it are in thermal equilibrium i.e. $T_m = T = \frac{H}{2\pi}$. In fact, since, due to their temperature difference, an energy flux between horizon and the materials confined has not yet been observed, such assumption is not far from reality. Moreover, it is shown that there is a Hawking radiation with the Cai–Kim temperature for the fields near the apparent horizon of FLRW universe [27], and thus, such Hawking radiation may be considered as a mechanism for producing such assumed thermal equilibrium. Therefore, the $T_m = \frac{H}{2\pi}$ assumption is not an unlikely guess. Now, by combining this result with Eqs. (27) and (30) and using Eq. (9) together with the $V = \frac{4\pi}{3H^3}$ relation we obtain

$$T(dS_A + dS_m) = 3VH(\rho + p)\left(1 + \frac{k}{2H^2}\right)dt + Vd\rho, \quad (31)$$

yielding

$$\frac{dS_T}{dt} = 6\pi V(\rho + p)\left(1 + \frac{k}{2H^2}\right) + \frac{2\pi V}{H}\dot{\rho}, \quad (32)$$

where $S_T = S_A + S_m$ is the total entropy. Since the generalized second law of thermodynamics states that S_T should satisfy the $\dot{S}_T \geq 0$ condition [10], this law will be met if the

$$-3(\rho + p)H\left(1 + \frac{k}{2H^2}\right) \leq \dot{\rho}, \quad (33)$$

condition is obeyed by the perfect fluid source. Bearing the results obtained from investigating the second law in mind, it is obvious that for $k > 0$, $-1 \leq \omega$ and whenever $\dot{\rho}$ meets Eq. (33), the second law of thermodynamics and its generalized form are satisfied simultaneously. Combining this equation with Eq. (8) we get

$$\dot{\rho} \leq \frac{\rho + p}{2H\lambda}(3k\lambda - 2H^2\lambda - 1), \quad (34)$$

which expresses that the generalized second law of thermodynamics holds if the time derivative of the perfect fluid pressure satisfies

this equation. Now, we focus on a dominated fluid of constant state parameter. Since for a perfect fluid with constant state parameter, $\dot{\rho} = \omega \dot{\rho}$ (as is the case for radiation), one can reach

$$\frac{1 + 2H^2\lambda - 3k\lambda}{6H^2\lambda(1 + \frac{k}{2H^2})} \leq \omega, \quad (35)$$

by combining Eqs. (33) and (34). Now, for a fluid with $-1 \leq \omega$, which satisfies the second law, we get

$$\lambda \leq -\frac{1}{8H^2 + 6k}, \quad (36)$$

if $\frac{1+2H^2\lambda-3k\lambda}{6H^2\lambda(1+\frac{k}{2H^2})} \leq -1$. In this situation λ is negative in conflict with the results obtained in Eqs. (25) and (26). Moreover, if $-1 \leq \frac{1+2H^2\lambda-3k\lambda}{6H^2\lambda(1+\frac{k}{2H^2})}$ one reaches

$$-\frac{1}{8H^2 + 6k} \leq \lambda, \quad (37)$$

which covers the intervals obtained in Eqs. (25) and (26). Therefore, the validity of the second and generalized second laws of thermodynamics lead to a lower bound for the λ factor (37). This is in agreement with the requirements for covering the $H^2 > 0$ condition, the results obtained by applying the entropy positivity condition, and some previous works [36–38].

4. Summary and concluding remarks

After referring to the Rastall theory of gravity, we considered the flat FLRW universe, and find the corresponding Friedman equations in this theory. We also addressed some cosmological consequences of this theory. Thereinafter, we used the Cai–Kim approach to get the energy flux crossing horizon during the infinitesimal time dt . In addition, by applying the Clausius relation on the apparent horizon (as the causal boundary of system) and using the obtained Friedman equations, we got a relation for the apparent horizon entropy (Eq. (22)). As we have shown, the entropy positivity condition binds up the k and λ parameters to be positive quantities. Our study suggests that, if the state parameter of dominated perfect fluid meets the $-1 \leq \omega \leq 0$ condition for a Rastall theory with $k > 0$, and if λ either satisfies $0 \leq \lambda \leq \frac{1}{4k}$ or $\lambda > \frac{1}{3k(1+\omega)}$, then both the entropy positivity and $H^2 > 0$ conditions are also satisfied. In addition, for the $0 \leq \omega \leq \frac{1}{3}$ case, λ should either obey $0 \leq \lambda \leq \frac{1}{4k}$ or $\lambda > \frac{1}{3k}$ to meet the mentioned conditions. Moreover, we found out that the validity of the second law of thermodynamics requires that $k > 0$ and $\rho + p \geq 0$. The generalized second law of thermodynamics has also been studied. Our investigation shows that, in order to meet the generalized second law of thermodynamics, the time derivatives of energy density and pressure of source supporting the background should satisfy a lower and an upper bound, respectively. We also found out, for a universe filled by a perfect fluid of constant state parameter, the validity of the generalized second law leads to a lower bound for the state parameter of perfect fluid supporting the geometry. Finally, our study shows that, in a universe filled by a perfect fluid with constant state parameter, the validity of the generalized second law of thermodynamics leads to the $-\frac{1}{8H^2+6k} \leq \lambda$ condition for the λ parameter, which is in agreement with the results obtained by employing both the $H^2 > 0$ and entropy positivity conditions.

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