Median graphs, parallelism and posets

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Abstract

A notion of parallelism is defined in finite median graphs and a number of properties about geodesics and the existence of cubes are obtained. Introducing sites as a double structure of partial order and graph on a set, it is shown that all median graphs can be constructed from sites and, in fact, that the categories of sites and pointed median graphs are equivalent, generalizing Birkhoff’s duality.

0. Introduction

In this paper we show that the category of pointed median graphs is equivalent to a category of sites (posets cum graphs); this generalizes the well-known Birkhoff’s duality between posets and distributive lattices to posets with obstructions and pointed median graphs (i.e. median semilattices, cf. [1]). The proof needs to focus on a notion of parallelism in graphs. In the case of median graphs, this notion can be derived from the Mulder’s isometric embedding of a median graph into a hypercube [7]. But we found interesting to discuss it from a more intrinsic viewpoint (i.e. without embedding in a cube). So, the first section of this paper is devoted to the notion of parallelism and the second establishes relationships between median graphs and sites.

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1. Median graphs and parallelism

1.1. Definitions

Let $G = (V, E)$ be a simple undirected and connected graph with vertex set $V$ and edge set $E$. For $V' \subseteq V$ and $E' \subseteq E$, $G_{V'}$ will denote the subgraph of $G$ induced by $V'$ and $G_{E'}$ the graph $(V, E')$. In this paper, we will consider only finite graphs. The distance $d_G(u, v)$ between two vertices $u, v$ is the length (i.e. the number of edges) of a shortest path, called a geodesic, between $u$ and $v$. The interval $[u, v]$ is the set of all vertices $w \in V$ belonging to a geodesic between $u$ and $v$, that is, $[u, v] = \{w \in V : d_G(u, v) = d_G(u, w) + d_G(w, v)\}$. $V' \subseteq V$ is convex if $[u, v] \subseteq V'$ for any $u, v \in V'$. The subgraph induced by a convex subset is a convex subgraph. An isometric embedding of $G$ in a graph $G' = (V', E')$ is a mapping $g : V \to V'$ such that $d_{G'}(g(u), g(v)) = d_G(u, v)$; in the case $V' \subseteq V$, if such a $g$ is the canonical embedding of $V$ into $V'$, we say that $G$ is an isometric subgraph of $G'$. A convex subgraph is clearly an isometric subgraph.

$G$ is a median graph if and only if, for all $u, v, w \in V$, the set $[u, v] \cap [v, w] \cap [w, u]$ contains exactly one element, called the median of $u, v, w$ and denoted as $m_G(u, v, w)$. This median is the unique solution of $\min_{x \in V} (d_G(x, u) + d_G(x, v) + d_G(x, w))$ and it is clear that $d_G(u, v) + d_G(v, w) + d_G(w, u) = 2(d_G(m, u) + d_G(m, v) + d_G(m, w))$, where $m = m_G(u, v, w)$. It is well-known (see, for example, [6]), and can be deduced from the preceding remark, that a median graph is connected and bipartite. The set of all $(0, 1)$-vectors of length $n$, two vectors being adjacent if they differ in exactly one coordinate, is a median graph called the $n$-cube. References on median graphs include, among others, [2, 3, 6]. The following proposition is easily proved.

**Proposition 1.1.** Let $G' = (V', E')$ be an isometric subgraph of the median graph $G$. Then the following statements are equivalent: (i) $G'$ is a median graph; (ii) for all $u, v, w \in V$, $m_G(u, v, w) \in V'$. In particular, a convex subgraph of $G$ is a median graph.

1.2. Decomposition in 4-cycles

We denote by $\oplus$ the sum in the usual algebra of cycles of a graph (cf. [4]). An $n$-cycle is a cycle of length $n$.

**Proposition 1.2.** Let $G$ be a median graph. For every cycle $C$ of $G$, there exist 4-cycles $C_1, C_2, \ldots, C_k$ such that $C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$.

**Proof.** $G$ being bipartite, we will show the result by induction on the length $2p$ of a cycle $C$ of $G$. For $p = 2$, this is clear. Let $p > 2$. If $u$ and $a$ are vertices of $C$ such that $d_C(u, a) = p$ and $b, c$ are those two vertices of $C$ with $d_C(u, b) = d_C(u, c) = 1$, then, for $x = b$ and $x = c$, we have $d_C(x, a) + d_C(x, b) + d_C(x, c) = p + 1$, whence, by the unicity of the median, $d_G(m, a) + d_G(m, b) + d_G(m, c) < p + 1$, where $m = m_G(a, b, c)$. If $m$ is on $C$, then, since $d_C(m, a) + d_C(m, b) + d_C(m, c) \geq p + 1$, at least one of the three inequalities
\(d_G(m, a) < d_c(m, a), \quad d_G(m, b) < d_c(m, b), \quad d_G(m, c) < d_c(m, c)\) is true. In each case we get a decomposition \(C = C_1 \oplus C_2\), with \(|C_1| < |C|\) and \(|C_2| < |C|\) (Fig. 1a). On the other hand, if \(m\) is not on \(C\), then considering cycles \(C_1, C_2, C_3\) of \(G\), going respectively through \(u, b, m, c\), through \(b, m, a\) and through \(c, m, a\), we get \(C = C_1 \oplus C_2 \oplus C_3\), where \(|C_1| = 4, |C_2| \leq 2(p-1), |C_3| \leq 2(p-1)\) (Fig. 1b). Hence the result by induction. \(\square\)

**Proposition 1.3.** In a median graph, two distinct 4-cycles have at most one common edge.

**Proof.** Two distinct 4-cycles cannot have three edges in common. Suppose now that they have exactly two common edges. If these edges are incident, we get two cycles \(abcd\) and \(abce\): the graph is not median (Fig. 2a), since \(a\) and \(c\) are solutions of \(\min_{x \in V} (d_G(x, d) + d_G(x, b) + d_G(x, e))\); if the two edges are not incident, then \(G\) is not bipartite (Fig. 2b). \(\square\)

The converse of the preceding proposition is false as shown by the graph in Fig. 2c, where the cycles are generated by 4-cycles and two distinct 4-cycles have at most one common edge. Note also that the decomposition in 4-cycles is not unique, in general.

### 1.3. Tunnels

Every 4-cycle defines 4 pairs of incident edges and 2 pairs of opposite edges (i.e. without incident vertex). Write \(e \text{Opp}(C)e'\) if \(e\) and \(e'\) are two opposite edges of the 4-cycle \(C\). More generally, if \(C = C_1 \oplus C_2 \oplus \ldots \oplus C_k\), where all the \(C_i\) are 4-cycles, write \(e \text{Opp}(C)e'\) if and only if there exists \(e''\) such that \(e \text{Opp}(C_1 \oplus C_2 \oplus \ldots \oplus C_{k-1})e''\) and \(e'' \text{Opp}(C_k)e'\). A tunnels between the edges \(e\) and \(e'\) is any sequence \(C_1, C_2, \ldots, C_k\) of 4-cycles such that \(e \text{Opp}(C_1 \oplus C_2 \oplus \ldots \oplus C_k)e'\). The integer \(k > 0\) is the length of the tunnel. The following lemma is immediate.

![Fig. 1](image1.png)

![Fig. 2](image2.png)
Lemma 1.4. Let \( e = uv \) and \( e' = u'v' \) be two edges of a graph \( G = (V, E) \). Then the following statements are equivalent:

(i) There exists a tunnel of length \( k \) between \( e \) and \( e' \);

(ii) we can choose an endpoint of each \( e \) and \( e' \), say \( u \) and \( u' \), such that there are paths \( \alpha : u = u_0u_1 \cdots u_k \) and \( \beta : v = v_0v_1 \cdots v_k \), of length \( k \), with \( u_iv_{i+1}u_{i+1} \) a 4-cycle of \( G \), for \( i = 0, 1, 2, \ldots, k-1 \).

We then say that \( (\alpha, \beta) \) defines a tunnel between \((u, v)\) and \((u', v')\) along \( \alpha \) and \( \beta \). Note that, in general, there can exist a tunnel between incident edges and even incident edges of a 4-cycle (Fig. 3).

Proposition 1.5. Let \( G \) be a median graph and \( e = uv \), \( e' = u'v' \) two edges of \( G \); if \((\alpha, \beta)\) defines a tunnel between \((u, v)\) and \((u', v')\), then every vertex of \( G \) strictly nearer to \( u \) than to \( v \) is strictly nearer to \( u' \) than to \( v' \).

Proof. We proceed by induction on the length \( k \) of the tunnel defined by \((\alpha, \beta)\). Suppose that \( k = 1 \) and \( x \) is such that \( d(x, u) < d(x, v) \) and \( d(x, v') < d(x, u') \) (equality is impossible since the graph is bipartite). With \( d(x, u) = t \) and \( d(x, v') = s \) (Fig. 4a), we have \( t+1 = d(x, v) \leq d(x, v') + d(v', v) = s + 1 \leq d(x, u') \leq d(x, u) + d(u, u') = t+1 \); so, \( s = t \). But \( d(x, u) + d(u', u) + d(v, u) = t+2 \) and \( d(x, v') + d(u', v') + d(v, v') = t+2 \) and this value is minimal since \( d(x, u') + d(u', v') + d(x, v) = (t+1) + 2 + (t+1) = 2(t+2) \). So, \( u = v' \), which is impossible.

For the general case, we use the notation of Lemma 1.4 (Fig. 4b). If \( d(x, u) < d(x, v) \), then \( d(x, u_1) < d(x, v_1) \) and, by induction, \( d(x, u') < d(x, v') \). \( \square \)

Corollary 1.6. In a median graph \( G \), if there is a tunnel between the edges \( e \) and \( e' \), then \( e = e' \) or they are not incident.

Proof. Let \( e = uv \) and \( e' = u'v' \) be distinct edges with \( u' = v \). A tunnel between \( e \) and \( e' \) is defined either by a pair of paths between \((u, v)\) and \((u', v')\) or by a pair of paths between \((u, v)\) and \((v', u')\) (Fig. 5). In the first case, we have \( d(u', v) < d(u', u) \) and \( d(u', v') > d(u', u') \); in the second case, \( d(u, u) < d(u, v) \) and \( d(u, v') > d(u, u') \). Both cases contradict the theorem.
**Proposition 1.7.** Let $G$ be a median graph. For every cycle $C$ of $G$ and every edge $e$ of $C$, there is another edge $e'$ of $C$ with a tunnel between $e$ and $e'$.

**Proof.** We proceed by induction on the smallest number $m$ of 4-cycle necessary to generate $C$ (Proposition 1.2). If $m = 1$, this is clear; now, let $\mathcal{C}$ be a set of $m > 1$ cycles of length 4 generating $C$. Take $C_1 \in \mathcal{C}$ with $e \in C_1$ and $e_1$ the edge of $C_1$ such that $e_1 \text{Opp}(C_1)e$. If $e_1 \in C$, the result is clear; if not, $C' = C \oplus C_1$ is a cycle (the sum of two cycles intersecting in a path is a cycle) that can be generated by $\mathcal{C} \setminus \{ C_1 \}$ and we apply the induction hypothesis to $C'$ and $e_1$. □

1.4. Parallelism

Two edges $e = uv$ and $e' = u'v'$ of a graph are called **parallel** if $d(u, u') = d(v, v')$ and $d(u, v') = d(v, u')$. This is generally not an equivalence relation, not even in a bipartite graph (see Fig. 6).

**Lemma 1.8.** If the edges $e = uv$ and $e' = u'v'$ of the median graph $G$ are parallel, we can choose endpoints of $e$ and $e'$, say $u$ and $u'$, in such a way that $d(u, u') = d(v, v')$ and $d(u, v') = d(v, u') = d(u, u') + 1$. With this choice, if $\beta: v = v_0 v_1 \ldots v_n = v'$ is any geodesic between $v$ and $v'$, with $n > 0$, there exists a geodesic $\alpha: u = u_0 u_1 \ldots u_n = u'$ such that $(\alpha, \beta)$ defines a tunnel between $(u, v)$ and $(u', v')$.

**Proof.** The first statement is clear since the graph is bipartite. For the second statement, we proceed by induction on $n > 0$. Then the vertices $u, u', v, v'$ are distinct. For $n = 1$ the result is clear. For the general case, let $\alpha: u = u_0 u_1 \ldots u_n = u'$ be a geodesic between $u$ and $u'$. First we note that $\alpha$ and $\beta$ cannot have a common vertex: If $t$ is such a vertex, with $m = d(t, u)$, $m' = d(t, v)$, then $m' + n - m > n + 1$ and $m + n > m' + n + 1$, which is impossible (Fig. 7a).
Now if \( u_1v_1 \in E \), then \( u_1v_1 \) and \( u'v' \) are clearly parallel and we apply the induction hypothesis to obtain a tunnel of length \( n - 1 \) along the geodesic \( v_1 \ldots v_n = v' \); with the 4-cycle \( uu_1v_1v \), we get a tunnel between \((u, v)\) and \((u', v')\) along \( \beta \). If \( u_1v_1 \notin E \), we have \( d(u_1, v_1) = 3 \); otherwise, there would be an odd cycle (Fig. 7b). Since \( d(u, u) + d(u, u') + d(u_1, v_1) = n + 2 \) and \( d(v_1, u) + d(v_1, u') + d(v_1, v_1) = n + 2 \), then, from the unicity of the median \( m = m(u, u', v_1) \), we have \( n \leq d(m, u) \quad d(m, u') \quad d(m, v_1) \leq n + 1 \). But the value of the sum cannot be \( n \) since this would give \( m = v_1 \) and \( v_1 \) would be on a geodesic between \( u \) and \( u' \), which is false. So, \( d(m, u) + d(m, u') + d(m, v_1) = n + 1 \) and \( mv_1 \in E \); but \( 1 < d(m, u) + d(m, v_1) \leq 2 \) since \( m \neq u \); consequently, \( mu \in E \). Moreover, \( n = d(u, u') = d(u, m) + d(m, u') \) and \( d(m, u') = n - 1 = d(v_1, u') \); also \( n \leq d(m, v') \leq d(m, v_1) + d(v_1, v') = 1 + n - 1 \); whence, \( d(m, v') = n = d(v_1, u') \). So, by the induction hypothesis, there is a tunnel between \((m, v_1)\) and \((u', v')\); with the 4-cycle opposing \( uv \) and \( mu_1 \), we get the result.  

**Theorem 1.9.** Let \( e \) and \( e' \) be edges of \( G \); then \( e \) is parallel to \( e' \) if and only if \( e = e' \) or there is a tunnel between \( e \) and \( e' \). 

**Proof.** It remains only to prove that, if there is a tunnel between \( e = uv \) and \( e' = u'v' \), then \( e \) and \( e' \) are parallel. But \( u \) being nearer to \( u \) than to \( v \), we have (Proposition 1.5) \( d(u, u') < d(u, v') \leq d(u, v) + d(v, v') = 1 + d(v, v') \); so, \( d(u, u') \leq d(v, v') \). A similar argument gives \( d(v, v') \leq d(u, v') \), whence equality.

Suppose now \( d(u, v') < d(u, u') \). Since \( d(v, u') \leq d(v, v') + d(v, u') = d(v, v') + 1 \), we have \( d(u, v') \leq d(v, v') \); equality is impossible since the graph is bipartite. So, we have \( d(u, v') < d(v, v') \) and Proposition 1.5 gives \( d(u', v') < d(v', v') \), which is absurd. Hence, \( d(u, v') = d(v, v') \).  

**Corollary 1.10.** In a median graph, parallelism is an equivalence relation on the edge set. Moreover, if \( e \) is an edge of a cycle, this cycle contains another edge parallel to \( e \). 

This is immediate from Theorem 1.9 and Proposition 1.7.

An equivalence class for this relation is called a parallelism class of \( G \) and \( P_G \) denotes the set of all parallelism classes of \( G \). An edge belonging to the parallelism class \( b \) of \( G \) is called a \( b \)-edge and a tunnel between two \( b \)-edges is a \( b \)-tunnel. A \((b, c)\)-cycle or \((b, c)\)-cube is a 4-cycle whose edges belong to the distinct classes \( b \) and \( c \); more generally, a \((b_1, b_2, \ldots, b_n)\)-cube of \( G \) is an isometric subgraph of \( G \) isomorphic to an \( n \)-cube and whose edges belong to the distinct classes \( b_1, b_2, \ldots, b_n \). A \((b_1, b_2, \ldots, b_n)\)-vertex is a vertex adjacent to a \( b_i \)-edge, for \( i = 1, 2, \ldots, n \). 

**Lemma 1.11.** If \( e = uv \) is an edge of \( G \) and \( x \) a vertex such that \( d(x, u) < d(x, v) \), then a geodesic between \( x \) and \( u \) cannot contain an edge parallel to \( w \).
Proof. Let $z$ be a geodesic from $x$ to $u$ going through the edge $e'=u'v'$ with $d(u,u')=d(v,v')$ and $d(u,u')=d(u,u')+1$. Then $d(x,u')<d(x,v')$ (Proposition 1.5); so, $d(x,v')=d(x,u')+1$ and $d(x,u)=d(x,v')+d(v',u)=d(x,u')+1+d(u',u)+1=d(x,u)+2$, which is impossible. □

Theorem 1.12. A geodesic between two vertices $x$ and $y$ cannot contain two parallel edges and, on any two such geodesics, the same parallelism classes appear.

Proof. The first statement is an easy consequence of Lemma 1.11. The second statement then follows from Corollary 1.10. □

The next theorem can, in essence, be found in [6] once it is recognized that parallelism classes correspond to the sets denoted by $F_\varphi$ in [6].

Theorem 1.13. If $\varphi$ is a parallelism class of the median graph $G=(V,E)$, then the graph $G^{E_\varphi}$ has two connected components $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$, called the $\varphi$-components; these components are convex subgraphs of $G$. Moreover, for any edge $uv\in V_\varphi$, $V_1$ and $V_2$ are given by \{x: d(x,u)<d(x,v)\} and \{x: d(x,v)<d(x,u)\}.

Proof. By Corollary 1.10, if $uv\in V_\varphi$, $u$ and $v$ cannot be in the same component. So, $G^{E_\varphi}$ has at least two components, and, by Lemma 1.11, it has at most two. Let $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ be these two components and suppose $u\in V_1$ and $v\in V_2$. If $x$ is such that $d(x,u)<d(x,v)$, then (Lemma 1.11) every geodesic of $G$ from $x$ to $u$ avoids all edges in $\varphi$ and, so, $x\in V_1$. If $d(x,u)>d(x,v)$, we have similarly $x\in V_2$. Since $d(x,u)=d(x,v)$ is impossible, $V_1=\{x: d(x,u)<d(x,v)\}$ and $V_2=\{x: d(x,v)<d(x,u)\}$. Let $x,y\in V_i$, for $i=1,2$; if a geodesic between $x$ and $y$ contains a $\varphi$-edge, then, since there is a path in $V_i$ between $x$ and $y$, there will be a cycle with only one $\varphi$-edge, which is impossible (Corollary 1.10). Therefore, $[x,y]\subseteq V_1$ and the proof is complete. □

Corollary 1.14. (1) If $\varphi$, $\psi$ are distinct parallelism classes of $G$ and there is no $(\varphi,\psi)$-cycle, then all the $\psi$-edges belong to the same $\varphi$-component;

(2) for $i=1,2$, let $x_i$ be a $\varphi$-vertex and $C_i$ a geodesic from the vertex $y$ to $x_i$. Let $Q_i=\{a\in P_G: a\neq \varphi, C_i has an a-edge and G has no (\varphi,a)-cycle\}; then $Q_1=Q_2$;

(3) every vertex on a geodesic between two $\varphi$-vertices is a $\varphi$-vertex.

Proof. For (1), two $\psi$-edges are linked by a $\psi$-tunnel and this tunnel must be entirely in $G^{E_\varphi}$ since there is no $(\varphi,\psi)$-cycle. For (2), if there is no $(\varphi,\psi)$-cycle, then a geodesic $D$ between $x_1$ and $x_2$ has no $\psi$-edge. If $C_1$ has an $a$-edge, there must be another one on the cycle $C_1+D+C_2$, and, so, on $C_2$. For (3), let $C$ be a geodesic between the $\varphi$-vertices $x$ and $y$. If these vertices are in the same $\varphi$-component, then (Lemma 1.8) there is a $\varphi$-tunnel along $C$ and the result is clear. If they are not in the same $\varphi$-component, $C$ is made of a $\varphi$-edge and of two geodesics between $\varphi$-vertices and the first case applies. □
Lemma 1.15. Let $u, v, w$ be vertices of $G$. If $uv$ is a $b$-edge and $vw$ a $c$-edge and if there exists a $(b, c)$-cycle in $G$, then there is one containing the edges $uv$ and $vw$.

Proof. Let $u'v'w's'$ be a $(b, c)$-cycle. One of the four vertices of this cycle, say $v'$, can be chosen in such a way that there is a $b$-tunnel between the $b$-edges given by the couples $(u, v), (u', v'), (s', w')$ and a $c$-tunnel between the $c$-edges given by $(v, w), (v', w'), (u', s')$ (Fig. 8). Note that, because of parallelism, $d(u, s') = d(v, w') = d(v, v') + 1 = d(v, u') = d(w, s')$; consequently, if $m = m(u, w, s')$, then $m # u$ and $m # w$. Since $2 = d(u, w) = d(u, m) + d(m, w)$, $um$ and $wm$ are edges of $G$. Now $d(s', v) = d(s', w) + 1$; so, $v$ cannot be on a geodesic between $s'$ and $w$ and $m # v$. The vertices $u, v, w, m$ give the required $(b, c)$-cycle.

Theorem 1.16. Let $b_1, b_2, ..., b_n$ be distinct elements of $P_G$. If, for all $i # j$, there exists a $(b_i, b_j)$-cube, then there exists a $(b_1, b_2, ..., b_n)$-vertex and all such vertices belongs to a $(b_1, b_2, ..., b_n)$-cube.

Proof (by induction on $n$). The case $n = 2$ follows from the preceding lemma. Let $n \geq 3$ and suppose the theorem is true for $n - 1$ parallelism classes. Then there is a $(b_2, b_3, ..., b_n)$-cube $Q_1$, a $(b_1, b_3, b_4, ..., b_n)$-cube $Q_2$ and a $(b_1, b_2, b_4, ..., b_n)$-cube $Q_3$. Let $x_i$ be an arbitrary vertex of $Q_i$, for $i = 1, 2, 3$ and $m = m(x_1, x_2, x_3)$ be the median of $x_1, x_2, x_3$. For every $i = 1, 2, ..., n$, $m$ is on a geodesic between two $b_i$-vertices and, therefore, by Theorem 1.14(3), $m$ is a $(b_1, b_2, ..., b_n)$-vertex. At any such vertex $u$, there exists, by the induction hypothesis, a $(b_1, b_2, ..., b_{i-1}, b_{i+1}, ..., b_n)$-cube, for every $i$; these cubes have, between themselves, $2^n - 1$ vertices and we see easily that there is a $(b_1, b_2, ..., b_n)$-cube at $u$.

2. Median graphs and sites

In this section, we introduce the notion of a site and show how to construct median graphs from sites, generalizing a classical method of constructing median graphs from the initial sets of a poset, in such a way as to obtain all median graphs.

Fig. 8.
Recall that with every vertex \( x \) of a median graph \( G \) is associated a canonical order \( \leq_x \) defined by \( u \leq_x v \) if and only if \( u \in [x, v] \). The covering graph of a poset \((P, \leq)\) is the undirected graph with vertex set \( P \) such that two vertices are adjacent if and only if one covers the other in \((P, \leq)\). A median semilattice is a meet semilattice \((P, \leq)\) such that (i) for every \( a \), \( \{x \in P \mid x \leq a\} \) is a distributive lattice, and (ii) any three elements have a least upper bound in \( P \) whenever each pair of them does.

**Theorem 2.1** (Avann [1]). The covering graph of any median semilattice is a median graph. Conversely, every median graph gives a median semilattice with respect to any canonical order \( \leq_x \).

This enables us to consider median semilattice and pointed median graph interchangeably.

### 2.1. From sites to median graphs and return

Consider a triple \( T = (Q, \leq, E) \), where \((Q, \leq)\) is an ordered set and \((Q, E)\) a simple graph. \( A \subseteq Q \) is an initial set of \( Q \) if \( y \leq x \in A \) implies \( y \in A \). The subset \( A \) is a stable subset of the graph \((Q, E)\) if \( \{x, y\} \subseteq A \) implies \( xy \notin E \). Let \( \mathcal{M}(T) \) be the set of all initial sets that are stable subsets of \( Q \). For \( A, B \in \mathcal{M}(T) \), define \( A \) and \( B \) to be adjacent if they differ in exactly one point, i.e. the symmetric difference \( A \triangle B \) has exactly one element.

**Theorem 2.2.** For any \( T = (Q, \leq, E) \), the set \( \mathcal{M}(T) \) of all stable initial sets of \( Q \) is a median graph.

**Proof.** Let \( A, B, C \in \mathcal{M}(T) \), with \( A \cup B, B \cup C, C \cup A \in \mathcal{M}(T) \); then \( A \cup B \cup C \in \mathcal{M}(T) \). For it is clearly an initial set and if \( u, v \in A \cup B \cup C \) with, for example, \( u \in A \) and \( v \in B \), then \( \{u, v\} \subseteq A \cup B \in \mathcal{M}(T) \) and \( uv \notin E \). So, \( A \cup B \cup C \) is stable. Therefore, \((\mathcal{M}(T), \subseteq)\) is a median semilattice and the associated covering graph is a median graph (Theorem 2.1). \( \Box \)

It is easy to see that, for \( x, y, u, v \in Q \),

1. if \( xy \in E \) and \( x \leq u, y \leq u \), then, with \( T_1 = (Q_1, \leq_1, E_1) \), where \( Q_1 = Q \setminus \{t \in Q : 1 \geq u\} \) and \( \leq_1 \) and \( E_1 \) are the restrictions of \( \leq \) and \( E \) to \( Q_1 \), we have \( \mathcal{M}(T) = \mathcal{M}(T_1) \);
2. if \( xy \in E \) and \( x \leq u, y \leq v \), then, with \( T_1 = (Q, \leq, E \cup \{uv\}) \), we have \( \mathcal{M}(T) = \mathcal{M}(T_1) \). So, we define a site to be a triple \( T = (Q, \leq, E) \), where
   - \((Q, \leq)\) is a partially ordered set,
   - \((Q, E)\) is a graph;
   - For all, not necessarily distinct, \( x, y, u, v \in Q \), if \( xy \in E \) and \( x \leq u, y \leq v \), then \( uv \in E \).

Condition S3 implies, in particular, that if \( xy \in E \) then \( x \) and \( y \) cannot have a common upper bound and, so, cannot be comparable. As we have just seen, this condition does not restrict the median graphs that can be obtained from triples by the
construction of Theorem 2.2. As a matter of fact, we will prove that every median graph can be constructed from sites. More precisely, every vertex \( \alpha \) of a median graph will define a site describing precisely how the graph is seen when looked from \( \alpha \). This is, by the way, why the word "site" was chosen.

For a site \( T = (Q, \preceq, E) \), \( M(T) = \{ (T), \emptyset \} \) denotes the median graph \( M(T) \) pointed at the vertex \( \emptyset \in M(T) \) (or, equivalently, the median semilattice \( (M(T), \preceq) \)).

Conversely, starting from a median \( G \) and an arbitrary vertex \( \alpha \) of \( G \), we construct a site on the set \( P_G \) of parallelism classes of \( G \). For \( a, b, c \in P_G \), we say that \( a \) separates \( b \) from \( c \) if there is no \((a,b)\)-cycle in \( G \) and the \( b \)-edges are not in the same \( a \)-component as \( \alpha \); \( a \) separates \( b \) from \( c \) if there is no \((a,b)\)-cycle nor \((a,c)\)-cycle and the \( b \)-edges are not in the same \( a \)-component as the \( c \)-edges (Corollary 1.14). Let \( a \preceq_{G, a} b \) iff \( a = b \) or there is no \((a,b)\)-cycle in \( G \) and \( a \) separates \( b \) from \( \alpha \). Let \( ab \in E_{G, a} \) iff there is no \((a,b)\)-cycle in \( G \), \( a \) does not separate \( b \) from \( \alpha \) and \( b \) does not separate \( a \) from \( \alpha \).

**Theorem 2.3.** \( (P_G, \preceq_{G, \alpha}, E_{G, \alpha}) \) is a site. Moreover, \( \alpha \) is minimal in \( P_G \) if and only if \( \alpha \) is an \( a \)-vertex.

**Proof.** Let \( a \preceq_{G, \alpha} b \) and \( b \preceq_{G, \alpha} a \), with \( a \neq b \). Taking a shortest path \( C \) among all the paths between \( \alpha \) and the \( b \)-edges, there must be an \( a \)-edge on \( C \), since \( b \) separates \( \alpha \) from \( a \). But \( a \preceq_{G, \alpha} b \) and there is also a \( b \)-edge between \( \alpha \) and this \( a \)-edge, contradicting the minimality of \( C \). So, \( \preceq_{G, \alpha} \) is antisymmetric.

In particular, if \( a \preceq_{G, \alpha} b \), then \( \alpha \) and all the \( a \)-edges are in the same \( b \)-component.

Let \( a <_{G, \alpha} b \) and \( b \preceq_{G, \alpha} c \). Since \( \alpha \) is in the same \( b \)-component as all of \( a \) but not in the same \( b \)-component as \( c \), there is no \((a,c)\)-cycle. If \( a \) and a \( c \)-edge were in the same \( a \)-component, there will be a path between \( \alpha \) and the \( c \)-edges using no \( b \)-edge, since the \( b \)-edges are not in this \( a \)-component. This contradicts \( b \preceq_{G, \alpha} c \). So, \( a \) separates \( \alpha \) from \( c \) and \( a \preceq_{G, \alpha} c \).

For condition S3, let \( ab \in E_{G, \alpha} \) and \( a <_{G, \alpha} x, b <_{G, \alpha} y \); then \( a \) separates \( b \) and \( x \); so, there is a path from every \( x \)-edge to some \( a \)-edge using no \( b \)-edge and, consequently, a path from every \( x \)-edge to \( \alpha \) using no \( b \)-edge. This implies \( x \neq y \). If some \( y \)-edge and \( x \)-edge were in the same \( b \)-component, there would be a path from this \( y \)-edge to \( \alpha \) using no \( b \)-edge; this is impossible, since \( b <_{G, \alpha} y \). So, \( b \) separates \( x \) and \( y \), there is no \((x,y)\)-cycle and \( \alpha \) is in the same \( b \)-component as \( x \); consequently, \( y \) does not separate \( \alpha \) from \( x \). Symmetrically, \( x \) does not separate \( \alpha \) from \( y \) and, so, \( xy \in E_{G, \alpha} \). The cases \( a = x \) or \( b = y \) are easy.

For the second statement, if \( \alpha \in P_G \) is not minimal, then \( b <_{G, \alpha} a \), for some \( b \in P_G \) and \( b \) separates \( \alpha \) from \( a \); so, \( \alpha \) is not an \( a \)-vertex. Conversely, let \( \alpha \) be minimal. Choose an \( a \)-vertex \( u \) that is nearest to \( \alpha \). Suppose \( u \neq \alpha \). Let \( uv \) be the \( a \)-edge at \( u \) and \( tu \) be the last edge on a geodesic from \( \alpha \) to \( u \). If there is no \((a,d)\)-cycle, with \( d \) the parallelism class of \( tu \), then \( d <_{G, \alpha} a \), which is impossible. So, there is an \((a,d)\)-cycle and, by Lemma 1.15, there is such a cycle using the vertices \( t, u, v \). Hence, \( t \) is an \( a \)-vertex nearer to \( \alpha \) than \( u \), contradicting the choice of \( u \). \( \square \)
Fig. 9 gives an example of a pointed median graph and its associated site, the parallelism classes being denoted by 1, 2, ..., 7.

2.2. A categorical viewpoint.

Let $\mathcal{M}$ be the category of all median graphs as objects and of all isometric embeddings as morphisms. $\mathcal{M}_\ast$ is then the category of all pointed median graphs $(M, \alpha)$, where $\alpha$ is a vertex of $M$, the morphisms being those of $\mathcal{M}$ that associate the distinguished vertices. The category $\mathcal{S}$ of sites has for objects the sites and for morphisms $f: (Q, \leq E) \to (Q', \leq, E')$ all applications $f: Q \to Q'$ such that:

1. $f$ is injective;
2. $f(Q)$ is an initial set of $Q'$;
3. for all $x, y \in Q$, $f(x) \leq f(y)$ implies $x \leq y$;
4. for all $x, y \in Q, f(x)f(y) \in E'$ implies $xy \in E$.

We now define functors between the categories $\mathcal{M}_\ast$ and $\mathcal{S}$ and use them to show that these categories are equivalent [5].

**Proposition 2.4.** Let $\mathcal{M}_\ast \xrightarrow{\Phi} \mathcal{S}$ be given by $\Phi(T) = (\mathcal{M}(T), \emptyset)$ for a site $T = (Q, \leq, E)$ and by $\Phi(f) = \Phi(T) \to \Phi(T')$ for an $\mathcal{S}$-morphism $f: T \to T'$, where $\Phi(f)(A) = f(\Phi(T))$ for $A \in \mathcal{M}(T)$. Then $\Phi$ is a functor.

**Proof.** For $A, B \in \mathcal{M}(T)$, a geodesic $C$ from $A$ to $B$ is obtained by removing one by one elements of $A$ not in $B$, then adding elements of $B$ not in $A$. Since $f$ is injective and $f(A)$ an initial set of $\mathcal{M}(T')$, $f(C)$ is a geodesic between $f(A)$ and $f(B)$. So, $f$ is an isometric embedding. □

**Proposition 2.5.** Let $\sigma: \mathcal{M}_\ast \to \mathcal{S}$ be given by $\sigma(M, \alpha) = (P_M, \leq_M, E_M, \alpha)$ for an object $(M, \alpha)$ of $\mathcal{M}_\ast$ and by $\sigma(f) = f$ for a morphism $f: (M, \alpha) \to (M', \alpha')$, where $\tilde{f}: P_M \to P_{M'}$ is defined by $\tilde{f}(uw) = f(x)f(y)$, $uw$ denoting the parallelism class of the edge $uw$ in the corresponding graph. Then $\sigma$ is a functor.

**Proof.** Since an isometric embedding sends parallel edges into parallel edges, it is clear that $f$ is well-defined and injective. To show that $f(P_M)$ is an initial set of $P_{M'}$, let
For any geodesic $C$ from $\alpha$ to some end of an $a$-edge, $f(C)$ links $\alpha'$ to an $f(a)$-edge and, so, uses a $\tau$-edge. Hence, $\tau \in f(P_M)$. Let $f(a)f(b) \in E_{M',\tau}$. There is no $(a,b)$-cycle in $M$ since its image would be an $(f(a),f(b))$-cycle and there is none. For any $a$-edge $c$, take a geodesic $C$ of $M$ between $\alpha$ and $c$. Then $f(C)$ is a geodesic of $M'$ between $\alpha'$ and an $f(a)$-edge; since $f(a)f(b) \in E_{M',\tau}$, $f(C)$ uses no $f(b)$-edge and, so, $C$ uses no $\tau$-edge. Therefore, $b$ does not separate $\alpha$ from $\alpha'$. In the same way, $a$ does not separate $\alpha'$ from $b$ and $a < M, x b$. A similar argument shows that $f(a) \leq M, x f(b)$ implies $a \leq M, x b$. □

**Lemma 2.6.** Let $f: M_1 \rightarrow M$ be an isometric embedding of the median graphs $M, M_1$ such that $f(M_1)$ is a convex subgraph of $M$; with $\phi(f) = f', \alpha \in M_1$ and $b,c \in P_{M_1}$, we have

- if $f(m) = \phi(b)$-vertex, then $m$ is a $\phi(b)$-vertex;
- there is a $(b,c)$-cycle in $M$ iff there is an $(f(b),f(c))$-cycle in $M$.

**Proof.** $f(m)$ being an $f(b)$-vertex, let $u$ be the vertex of $M$ such that $uf(m)$ is an $f(b)$-edge and let $xy$ be any $b$-edge in $M_1$. Since the edges $f(x)f(y)$ and $uf(m)$ are parallel, we see that $u \in [f(m),f(y)] \cup [f(m),f(x)] \subset f(M_1)$. So, there is $t \in M_1$, with $f(t) = u$, and it is then clear that $tm$ is a $b$-edge of $M_1$.

Suppose there is an $(f(b),f(c))$-cycle in $M$. Let $x,y$ be, respectively, a $b$-vertex and a $c$-vertex of $M_1$. Taking any vertex $z$ of the $(f(b),f(c))$-cycle, let $m = m(f(x),f(y),z) \in [f(x),f(y)] \subset f(M_1)$. We have $m = f(m_1)$, with $m_1 \in M_1$. By Corollary 1.14(3), $m$ is an $(f(b),f(c))$-vertex and, by, Theorem 1.16, there is an $(f(b),f(c))$-cycle at $m$. So, $m_1$ is a $(b,c)$-vertex and, since $f(M_1)$ is convex, it is easy to see that there is a $(b,c)$-cycle at $m_1$. □

**Proposition 2.7.** Let $f: M_1 \rightarrow M$ be an isometric embedding of the median graphs $M, M_1$ such that $f(M_1)$ is a convex subgraph of $M$; let $\alpha, \beta \in P_{M_1}$. Then $\phi(f) = f'$ induces an isomorphism of the sites ($P_{M_1}, \leq M_1, x_1, E_{M_1, x_1}$) and ($f(P_{M_1}), \leq \phi, E$), where $\leq$ and $E$ correspond to the order and graph induced on $f(P_{M_1})$ by ($P_M, \leq M, \phi(x_1)$, $E_{M, \phi(x_1)}$).

**Proof.** We already know that $f$ is a morphism of $\mathbb{S}$. If $a < M_1, x_1, b$, then there is no $(a,b)$-cycle in $M_1$ and, so, no $(f(b),f(c))$-cycle in $M$. Since $a$ separates $x_1$ from $b$ in $M_1$, then, using the lemma, we see easily that $f(a)$ separates $x_1$ from $f(b)$ and $f(a) < M, \phi(x_1), f(b)$. Similarly, we find that $ab \in E_{M, x_1}$ implies $f(a)f(b) \in E_{M, f(x_1)}$. □

The lemma and the proposition are false if $f(M_1) \subset M$ is not convex.

As a particular case of Proposition 2.7, let $\alpha \beta$ be an $a$-edge of $M$ and $M_1$ the $a$-component containing $\alpha$. With $f$ the canonical embedding of $M_1$ in $M$, we know that $f(M_1) = M_1$ is a convex subgraph of $M$ (Theorem 1.13). Since any $b \in f(P_{M_1})$ does not separate $\alpha$ and $a$, it is easy to show that $f(P_{M_1}) = \{b \in P_M: \text{there exists an (a,b)-cycle or } a < M, x b\}$. But adding the edge $\alpha \beta$ to a geodesic between $\alpha$ and
a vertex $x$ of $M_1$ gives a geodesic between $x$ and $x$, by Theorem 1.13. So, for $b,c \in \mathcal{U}(M_1)$, it is clear that $b <_{M,x} c$ iff $b <_{M,a} c$ and $b,c \in E_{M,a}$ iff $bc \in E_{M,a}$. This proves the following corollary.

**Corollary 2.8.** If $x_1$ is an $a$-edge of $M$ and $M_1$ the $a$-component containing $x_1$, then $\mathcal{M}(M_1,x_1)$ is isomorphic to $(Q, \leq, E)$, where $Q = \{ b \in P_M : \text{there exists an (a,b)-cycle or } a <_{M,a} b \}$ with $\leq$ and $E$ corresponding to the partial order and graph induced by $\leq_{M,a}$ and $E_{M,a}$ on $Q$.

**Lemma 2.9.** Let $\mathcal{M}(M,x) = (P_M, \leq_{M,a}, E_{M,a})$; if $A$ is a stable initial set of $P_M$, then there is a vertex $x$ in $M$ such that any geodesic between $x$ and $x$ consists of exactly one $b$-edge for each $b \in A$. Moreover, such a vertex is unique.

**Proof.** (by induction on the number $n$ of vertices of $M$). The statement is obvious if $M$ has one or two vertices. Suppose the result is true for median graphs with fewer than $n$ vertices. Let $a$ be a minimal element of $A$. There is a unique $a$-edge $x_1$ adjacent to $x$. Let $M_1$ be the $a$-component containing $x_1$. With the notations of Corollary 2.8, we have that $\mathcal{M}(M_1,x_1)$ is isomorphic to $(Q, \leq, E)$ and $A \setminus \{a\}$ is a stable initial set of $Q$. Since $|M_1| < n$, there is, by the induction hypothesis, a geodesic in $M_1$ between $x_1$ and a vertex $x$ having exactly one $b$-edge for each $b \in A \setminus \{a\}$ and we just add the edge $a \in A$ to this geodesic.

If $x_1, x_2$ are two such vertices, then a geodesic between $x_1$ and $x_2$ going through $m = m(x, x_1, x_2)$ will contain two parallel edges, which contradicts Theorem 1.12.

**Proposition 2.10.** $(M, a)$ being a pointed median graph, let $t_{M,a} : (M, a) \to \mathcal{M}(\mathcal{M}(M, a)) = (\mathcal{U}(P_M, \leq_{M,a}, E_{M,a}),\emptyset)$ be given by $t_{M,a}(x) =$ set of all parallelism classes on a geodesic between $x$ and $x$, then $t_{M,a}$ is an invertible morphism of $\mathcal{M}_a$. Moreover, for any morphism $f : (M, a) \to (M_1, a_1)$, we have $\mathcal{M}_a(f) \circ t_{M,a} = t_{M_1,a_1} \circ t_{M,a}(f)$; so, $t$ is a natural isomorphism between the functors $\mathcal{M}_a$ and $\mathcal{M}_b$.

**Proof.** That $t_{M,a}(x)$ is an initial set of $P_M$ is immediate. To show that $t_{M,a}(x)$ is stable, let $ab \in E_{M,a}$ and $a,b \in t_{M,a}(x)$; then on a geodesic from $x$ to $x$ there is an $a$-edge and a $b$-edge; if the $a$-edge comes first, we have $a <_{M,a} b$ contradicting $ab \in E_{M,a}$. The preceding lemma says that $t_{M,a}$ is surjective. Finally, for vertices $x, y$ of $M$, considering $m = m(x, x, y)$ and the fact that the length of a geodesic is the number of classes appearing on it, it is clear that $d_M(x, y) = |t_{M,a}(x) \Delta t_{M,a}(y)|$, which is precisely the distance in the graph $\mathcal{M}(P_M, \leq_{M,a}, E_{M,a})$.

**Proposition 2.11.** For a site $T = (Q, \leq, E)$, let $v_T : T \to \mathcal{M}_a(T)$, given by $v_T(a) = \{ UV : U, V \in \mathcal{U}(T) \text{ and } U \Delta V = \{a\} \}$, then $v_T$ is an invertible morphism of $\mathcal{M}_a$ and, for every morphism $f : T \to T_1$, we have $\mathcal{M}_a(f) \circ v_T = v_{T_1} \circ v_T(f)$. Consequently, $v$ is a natural isomorphism between the functors $\mathcal{M}_a$ and $\mathcal{M}_b$. 


Proof. Condition S3 for sites ensures that \( v_T(a) \neq \emptyset \) and, in fact, that \( v_T \) is a well-defined bijection between \( Q \) and \( P_{\mathbb{N}(T)} \). The proof is then routine. \( \square \)

**Corollary 2.12.** The categories \( \mathcal{M} \) and \( \mathcal{S} \) are equivalent.

**Corollary 2.13.** Every median graph can be constructed from sites by the method of Theorem 2.12.

**Corollary 2.14.** Let \( \sigma(M, \alpha) = (P_M, \leq_{M, \alpha}, E_{M, \alpha}) \) and \( a, b \in P_M \), with \( a \prec_{M, \alpha} b \). Then \( a \) is covered by \( b \) in \( P_M \) iff there is an \((a, b)\)-vertex in \( M \).

**Proof.** If \( a \) is covered by \( b \), then \( W = \{ c \in P_M : c \leq_{M, \alpha} b \} \), \( V = W \setminus \{ b \} \), \( U = V \setminus \{ a \} \) are vertices of the graph \( \mathcal{M}_\alpha(M, \alpha) \) isomorphic to \((M, \alpha)\) and, in that graph, \( V \) corresponds to an \((a, b)\)-vertex. The converse is easy. \( \square \)

Consider \((M, \alpha)\) as the median semilattice produced by the canonical order \( \leq_{\alpha} \) associated with \( \alpha \) (Theorem 2.1). Let \( J \) be the set of join-irreducible elements of this semilattice, that is, nonnull elements covering exactly one other element. \( x \in J \) iff all the geodesics from \( \alpha \) to \( x \) terminate by the same edge. So, \( x \in J \) iff \( t_{M, \alpha}(x) \subset P_M \) has a greatest element \( \phi(x) \). This gives a bijection \( \phi \) between \( J \) and \( P_M \) from which we obtain the following result.

**Corollary 2.15.** If \( \sigma(M, \alpha) = (P_M, \leq_{M, \alpha}, E_{M, \alpha}) \), then \((P_M, \leq_{M, \alpha})\) is order-isomorphic to the set \( J \) of join-irreducible elements of the median semilattice \((M, \alpha)\). Moreover, \( \{x, y\} \subset J \) has an upper bound in \((M, \alpha)\) iff \( \phi(x) \phi(y) \in E_{M, \alpha} \).

This shows that the construction of Theorems 2.2 and 2.3 generalize to median semilattices and sites the classical duality of Birkhoff between distributive lattices and their subset of join-irreducible elements. This can be summarized in the following theorem.

**Theorem 2.16.** If \( M \) is a median semilattice, then \( M \) is isomorphic to the median semilattice of the stable initial sets of the site \((J, \leq, E)\), where \((J, \leq)\) is the subposet of sup-irreducible elements of \( M \) and \((J, E)\) is a graph with \( x \leq E y \) iff \( \{x, y\} \) has no upper bound in \( M \).

The graph \((G, \alpha)\) in Fig. 9 has been drawn in such a way that it can be viewed as the diagram of a median semilattice with smallest element \( \alpha \); the figure then shows the associated site.

Finally, given a pointed median graph \((M, \alpha)\), it is easy to see that the bijection \( t_{M, \alpha} \) of Proposition 2.10 is an isomorphism between the median semilattices \((M, \leq_{\alpha})\)
and $(\mathfrak{M}(P, \leq_{M,a}, E_{M,a}), \leq_{\emptyset})$. But the latter is an initial set of the distributive lattice $(\mathfrak{M}(P, \leq_{M,a}, \emptyset), \leq_{\emptyset})$ obtained by replacing $E_{M,a}$ by $\emptyset$. This is Sholander's theorem ([8]; see also [2]).

**Added in proof.** The authors learned recently that the notion of site is already known in Computer Science under the name 'conflict event structure'. See, for example, Degano, P., De Nicola, R. and Montanari, U., Partial Ordering Descriptions and Observations of nondeterministic processes, in *Lecture Notes in Computer Sciences*, vol. 354, Springer Verlag, 1989.

**References**