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# A Quasistatic Contact Problem for an Elastoplastic Rod

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We consider a mathematical model which describes the quasistatic contact of an elastoplastic rod with an obstacle. It is based on the Prandtl–Reuss flow law and unilateral conditions imposed on the velocity. Two weak formulations are presented and existence and uniqueness results established. The proofs are based on approximate problems with viscous regularization, which have merit on their own and may be used as the basis for convergent numerical algorithms for the problem.


## 1. INTRODUCTION

We consider a mathematical model which describes the quasistatic contact of an elastoplastic rod with an obstacle. It consists of a coupled system which contains a force balance-equation for the stress field and a variational inequality for the strain field. We establish the existence and uniqueness of the stress field and the existence of the velocity field.

Problems of contact, dynamic or quasistatic, of beams and rods have been investigated recently in [3–5, 8, 12, 13, 18] (see also references there). In these publications the rods or beams were assumed to be elastic or viscoelastic. Initial and boundary problems for plastic materials were considered in [11, 16, 20], but only for the classical displacement-traction boundary conditions. Here, we consider the quasistatic problem with plasticity and a unilateral velocity boundary condition. The dynamic contact or impact problem will be considered in the sequel.



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Plastic deformations are manifested in two or three dimensions. Our purpose in investigating the one-dimensional problem is to obtain deeper understanding of the mathematical structure of such problems and to gain insight into the possible types of behavior of the solutions. The one-dimensional problem can be thought of as only an approximation for compressible materials. Nevertheless, it has merit of its own.

The model is constructed in Section 2. It is based on the Prandtl–Reuss plastic flow rule and a velocity contact condition. In Section 3 we present two variational formulations of the problem and state our existence and uniqueness results. The first formulation is in terms of velocity and stress; the second one is in terms of stress only. A sequence of approximate problems with viscous regularization is described in Section 4. The existence and uniqueness of the solutions to these problems is established using the theory of evolution equations and convex analysis. In Section 5 we establish the a priori estimates on the approximate solutions that are needed to pass to the regularization parameter's limit. Thus our main results are established.

The regularized elastoviscoplastic problem can be considered as a basis for a convergent algorithm for numerical simulations of the model. Such solutions may be useful for testing computer codes designed for two- or three-dimensional elastoplastic contact problems.

## 2. THE MODEL

In this section we construct a model for the process of contact of an elastoplastic rod with an obstacle. The physical setting, depicted in Fig. 1, and the process are as follows. An elastoplastic rod occupies the reference configuration $0 \leq x \leq 1$ and is clamped at its left. The right end is free to move, but its movement is restricted by an obstacle situated at $x = 1$.

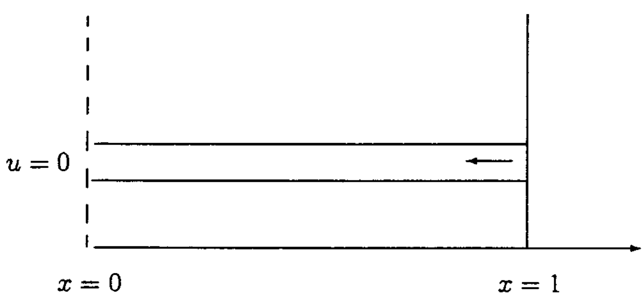


FIG. 1. The setting of the problem.

Once the free end moves to the left, the obstacle follows it and prevents any subsequent motion of the free end to the right. We may consider the obstacle as a semi-rigid wall which moves with the end to the left, but opposes any motion to the right. Since we deal with small displacements, we assume that the rigid wall is permanently positioned at $x = 1$; in the case of large displacements one has to take the wall's motion into account, which complicates the problem considerably by transforming it into a free boundary problem.

Let $u = u(x,t)$ represent the displacement field and $\sigma = \sigma(x,t)$ represent the stress field. We assume that the process is quasistatic; then at each time $t$ $(0 \le t \le T)$, the stress field satisfies the equilibrium equation

$$\sigma_x + f = 0,$$

where $f = f(x,t)$ denotes the (linear) density of applied forces, and the subscript $x$ represents "$\partial/\partial x$." To describe the elastoplastic behavior, we need the elasticity set $K$ and a flow law (see, e.g., Duvaut and Lions [7] or Maugin [15]). We set the *elasticity set* as

$$K = \{\tau \in \mathbb{R} : \sigma_* \le \tau \le \sigma^*\}, \tag{2.1}$$

where $\sigma_*$ and $\sigma^*$ are two constants representing the lower and upper plastic thresholds, such that $\sigma_* < 0 < \sigma^*$. $K$ may be described alternatively as

$$F(\tau) \le 0,$$

where $F$ is the piecewise linear function

$$F(s) = \begin{cases} \sigma_* - s & \text{if } s \le 0, \\ s - \sigma^* & \text{if } s \ge 0. \end{cases} \tag{2.2}$$

The normality law, which relates the rate of strain to the rate of stress, is assumed to be the Prandtl–Reuss flow law

$$\sigma \in K, \qquad \dot{u}_x = A\dot{\sigma} + \lambda. \tag{2.3}$$

Here and below, a dot above a variable represents the time derivative. $\lambda$ represents the plastic flow rate and $A$ is a positive constant representing the elastic properties of the material. Furthermore, we have

$$\lambda = 0$$
$$\text{if } \sigma_* < \sigma < \sigma^*, \text{ or if } \sigma = \sigma^* \text{ and } \dot{\sigma} < 0, \text{ or if } \sigma = \sigma_* \text{ and } \dot{\sigma} > 0;$$
$$\lambda \ge 0 \qquad \text{if } \sigma = \sigma^* \text{ and } \dot{\sigma} = 0;$$
$$\lambda \le 0 \qquad \text{if } \sigma = \sigma^* \text{ and } \dot{\sigma} = 0.$$

These may be written concisely as a variational inequality,

$$\sigma \in K, \qquad \lambda(\tau - \sigma) \geq 0 \qquad \forall \tau \in K. \tag{2.4}$$

Then, the constitutive law (2.3) together with (2.4) may be written as

$$A\dot{\sigma} + \partial\psi_K(\sigma) \ni \dot{u}_x,$$

where $\psi_K$ represents the indicator function of $K$ and $\partial\psi_K$ is its subdifferential.

We could consider the nonhomogeneous case as well. All we need to assume is that $A \in L^\infty(0,1)$ and $A(x) \geq \alpha$ a.e. $x \in (0,1)$ for some $\alpha > 0$. But, for the sake of simplicity, we consider only the homogeneous case to avoid technical complications.

To complete the statement of the problem, we have to prescribe the initial displacement $u_0(x)$, the initial stress $\sigma_0(x)$, and the boundary conditions.

Let $T > 0$, and set

$$\Omega_T = \{(x,t) : 0 < x < 1, 0 < t < T\}.$$

The classical formulation of the *elastoplastic quasistatic contact problem* is as follows. Find a pair $\{u, \sigma\}$ such that

$$A\dot{\sigma} + \partial\psi_K(\sigma) \ni \dot{u}_x \qquad \text{in } \Omega_T, \tag{2.5}$$

$$\sigma_x + f = 0 \qquad \text{in } \Omega_T, \tag{2.6}$$

$$u(0,t) = 0, \qquad t \in [0,T], \tag{2.7}$$

$$\dot{u}(1,t) \leq 0, \;\; \sigma(1,t) \leq 0, \;\; \sigma(1,t)\dot{u}(1,t) = 0, t \in [0,T], \tag{2.8}$$

$$u(x,0) = u_0(x), \qquad \sigma(x,0) = \sigma_0(x), x \in (0,1). \tag{2.9}$$

Here, (2.8) are the contact conditions at $x = 1$, similar to those used in [7, 10]. The condition $\dot{u}(1,t) \leq 0$ represents the fact that the right end of the rod is restricted to move only to the left; the condition $\sigma(1,t) \leq 0$ means that the reaction of the wall is toward the rod. Finally, the condition $\sigma(1,t)\dot{u}(1,t) = 0$ represents a *complementarity condition*: Either $\sigma(1,t) = 0$ when the end $x = 1$ is away from the wall or $\dot{u}(1,t) = 0$ when the end is still in contact, at time $t$.

*Remark* 2.1. We recall that the usual contact condition, the so-called Signorini condition, is

$$u(1,t) \leq 0, \qquad \sigma(1,t) \leq 0, \;\; \sigma(1,t)u(1,t) = 0. \tag{2.10}$$

It is stated in terms of the displacement of the rod's end, not in terms of its velocity. The interpretation in this case is that the rigid wall does not move, so contact holds when $u(1,t) = 0$, and $u_t > 0$ is possible when the end is not in contact.

*Remark* 2.2. As mentioned in the Introduction, a one-dimensional elastoplastic problem has mainly mathematical interest as it is well known experimentally that plastic flow is almost always observed to preserve volume, i.e., it is incompressible. Therefore, the elasticity set $K$ is usually described in terms of the deviator $\sigma^D$ of the stress tensor $\sigma$ which makes sense only in two or three dimensions. In our formulation, being one-dimensional, the material is necessarily compressible. Nevertheless, the problem has merit on its own, in addition to being a step toward our understanding of multidimensional contact problems for elastoplastic bodies.

## 3. VARIATIONAL FORMULATION AND STATEMENT OF THE MAIN RESULT

We restate problem (2.5)–(2.9) as a variational inequality. To this end, let $(\cdot,\cdot)$ denote the inner product on the space $L^2(0,1)$ and let $|\cdot|_{L^2(0,1)}$ denote the associated norm. We use standard notation for Sobolev spaces (see, e.g., [1] or [14]) and in addition we use the notation

$$U_0 = \{v \in H^1(0,1) : v(0) = 0, v(1) \le 0\}, \tag{3.1}$$

$$\Sigma_0 = \{\tau \in H^1(0,1) : \tau(1) \le 0\}, \tag{3.2}$$

which are the sets of time independent admissible test functions,

$$\Sigma(t) = \{\tau \in H^1(0,1) : \tau_x + f = 0 \text{ a.e. in } (0,1), \tau(1) \le 0\}, \tag{3.3}$$

where $t \in [0,T]$, which is the set of admissible stresses and

$$\mathscr{K} = \{\tau \in L^2(0,1) : \tau(x) \in K \text{ a.e. } x \in (0,1)\}. \tag{3.4}$$

We proceed to construct variational formulations for the problem. If $u$ and $\sigma$ are sufficiently regular and satisfy (2.6)–(2.8), then, for each $t \in [0,T]$,

$$\langle \tau - \sigma(t), \dot{u}_x(t)\rangle + \langle \tau_x - \sigma_x(t), \dot{u}(t)\rangle \ge 0 \qquad \text{for all } \tau \in \Sigma_0, \tag{3.5}$$

$$\langle \tau - \sigma(t), \dot{u}_x(t)\rangle \ge 0 \qquad \text{for all } \tau \in \Sigma(t). \tag{3.6}$$

Moreover, we deduce from (2.5) and (3.4) that for each $t \in [0, T]$

$$\langle A\dot{\sigma}(t), \tau - \sigma(t) \rangle \geq 0 \qquad \text{for all } \tau \in \mathscr{K}. \tag{3.7}$$

Let $v = \dot{u}$. We obtain from (3.5)–(3.7) the following two variational formulations for problem (2.5)–(2.9). The first is a velocity-stress formulation:

*Problem* $P_1$. Find $\{v, \sigma\}$ such that

$$\sigma(t) \in \mathscr{K} \cap \Sigma(t), \qquad t \in [0, T], \tag{3.8}$$

$$\langle A\dot{\sigma}(t), \tau - \sigma(t) \rangle + \langle v(t), \tau_x - \sigma_x(t) \rangle \geq 0$$
$$\text{for all } \tau \in \mathscr{K} \cap \Sigma_0, \text{ a.e. } t \in (0, T), \tag{3.9}$$

$$\sigma(0) = \sigma_0. \tag{3.10}$$

The second, which is obtained by the elimination of the velocity field, is the stress formulation:

*Problem* $P_2$. Find $\sigma$ such that

$$\sigma(t) \in \mathscr{K} \cap \Sigma(t), \qquad t \in [0, T], \qquad \sigma(0) = \sigma_0, \tag{3.11}$$

and

$$\langle A\dot{\sigma}(t), \tau - \sigma(t) \rangle \geq 0 \qquad \text{for all } \tau \in \mathscr{K} \cap \Sigma_0, \text{ a.e. } t \in (0, T). \tag{3.12}$$

Our main concern is the existence of solutions to Problems $P_1$ and $P_2$, which will be studied below.

Once the velocity field $v$ has been found from (3.8)–(3.10), the displacement field $u$ is obtained from

$$u(x, t) = \int_0^t v(x, s)\, ds + u_0(x), \tag{3.13}$$

where $u_0$ is the initial displacement.

In the study of this evolution problem, we suppose that the data satisfy

$$f \in H^1(0, T; L^2(0, 1)) \qquad \text{and} \qquad \sigma_0 \in \Sigma(0), \tag{3.14}$$

and that $u_0 \in H^1(0, 1)$ and $u_0(0) = 0$. Moreover, we also assume the following compatibility condition which is similar to the one used in [11, 16, 20]. There exists a function $\chi \in W^{1,\infty}(0, T; L^\infty(0, 1))$ such that

$$\begin{aligned}
&\text{(a)} \quad \chi_x + f = 0, \text{ in } \Omega_T, \\
&\text{(b)} \quad \chi(1, t) \leq 0,\ \dot{\chi}(1, t) \leq 0,\ t \in [0, T], \\
&\text{(c)} \quad \chi(0) = \sigma_0, \\
&\text{(d)} \quad \exists \delta > 0 \text{ such that } \chi(t) + \tau \in \mathscr{K},\ \forall t \in [0, T], \\
&\qquad\quad \tau \in L^\infty(0, T),\ |\tau|_{L^\infty(0,1)} \leq \delta.
\end{aligned} \tag{3.15}$$

Finally, for the study of the regularized problems we also need

$$u_0 \in U_0. \tag{3.16}$$

Let

$$\mathrm{BD}(0,1) = \{u \in L^1(0,1) : u_x \in M^1(0,1)\}, \tag{3.17}$$

where $M^1(0,1)$ is the space of bounded measures on $(0,1)$. We recall (see, e.g., [21, Chap. 2]) that $\mathrm{BD}(0,1)$ is continuously embedded in $L^p(0,1)$, for $1 \le p \le \infty$ and the embedding is compact for $1 \le p < \infty$. Next, let $L^2_w(0,T;\mathrm{BD}(0,1))$ denote the space of all weak* measurable functions $h : [0,T] \to \mathrm{BD}(0,1)$ such that

$$\int_0^T |h(t)|^2_{\mathrm{BD}(0,1)}\, dt < +\infty.$$

For the properties of $L^2_w$ we refer the reader to [20] or [9, Chap. 20].

Our main result is the following:

THEOREM 3.1. *Under the assumptions* (3.14)–(3.16) *there exists a solution* $\{v, \sigma\}$ *of Problem* $P_1$, *with* $\sigma$ *unique, such that*

$$v \in L^2_w(0,T;\mathrm{BD}(0,1)) \qquad \textit{and} \qquad \sigma \in H^1(0,T;H^1(0,1)).$$

*Moreover*, $\sigma$ *is the unique solution of Problem* $P_2$.

The proof of Theorem 3.1 will be given in Section 5 using a sequence of approximate problems, which are obtained by adding viscosity. This acts as regularization of the problem (3.8)–(3.10) or (3.11) and (3.12). Nevertheless, these elastoviscoplastic problems have some merit on their own and we consider them in the next section.

## 4. THE ELASTOVISCOPLASTIC PROBLEMS

In this section we consider a regularized version of the problem. It is effected by introduction of viscosity (see, e.g., [7] or [15]). We assume that viscosity effects become important only once a $\sigma \notin K$ and thus we replace (2.5) with

$$A\dot{\sigma} + \frac{1}{2\mu}(\sigma - P_K\sigma) = \dot{u}_x, \text{ in } \Omega_T, \tag{4.1}$$

where $\mu > 0$ is the viscosity coefficient and $P_K$ is the projection on $K$. Formally (2.5) may be recovered from (4.1) as $\mu \to 0$, which is the main ingredient of the regularization. The classical formulation of the mechanical elastoviscoplastic problem is to find a pair $\{u_\mu, \sigma_\mu\}$ such that (4.1), (2.6)–(2.9) hold.

We turn to variational formulation of the regularized problem. For each $\mu > 0$ we denote by $G_\mu : \mathbb{R} \to \mathbb{R}$ the function defined by

$$G_\mu(s) = \frac{1}{2\mu}(s - P_K s) = \begin{cases} \frac{1}{2\mu}(s - \sigma_*) & \text{if } s < \sigma_*, \\ 0 & \text{if } \sigma_* < s < \sigma^*, \\ \frac{1}{2\mu}(s - \sigma^*) & \text{if } s > \sigma^*. \end{cases} \quad (4.2)$$

A mixed variational formulation for the mechanical elastoviscoplastic problem is the following: Find a pair $\{u_\mu, \sigma_\mu\}$, for $\mu > 0$, such that

$$A\dot{\sigma}_\mu + G_\mu(\sigma_\mu) = \dot{u}_{\mu x} \qquad \text{in } \Omega_T, \quad (4.3)$$

$$\langle \tau - \sigma_\mu(t), \dot{u}_{\mu x}(t) \rangle \geq 0 \qquad \forall \tau \in \Sigma(t), \text{ a.e. } t \in (0, T), \quad (4.4)$$

$$\dot{u}_\mu(t) \in U_0 \qquad \text{a.e. } t \in (0, T), \quad (4.5)$$

$$\sigma_\mu(t) \in \Sigma(t), \qquad t \in [0, T], \quad (4.6)$$

$$u_\mu(0) = u_0, \qquad \sigma_\mu(0) = \sigma_0. \quad (4.7)$$

If we assume that problem (4.1), (2.6)–(2.9) has a regular solution $\{u_\mu, \sigma_\mu\}$ then by performing integration by parts we obtain problem (4.3)–(4.7) (see also (3.6)).

We have the following existence and uniqueness result for the variational problem (4.3)–(4.7):

THEOREM 4.1. *Let* (3.14)–(3.16) *hold. For each* $\mu > 0$ *problem* (4.3)–(4.7) *has a unique solution such that*

$$u_\mu \in H^1(0, T; H^1(0,1)), \qquad \sigma_\mu \in H^1(0, T; H^1(0,1)).$$

The proof of the theorem will be given at the end of this section, after reformulating it and obtaining a number of auxiliary results. We note that (4.4) and (4.6) are equivalent to the nonlinear evolution equation

$$\dot{u}_{\mu x}(t) + \partial\psi_{\Sigma(t)}(\sigma_\mu(t)) \ni 0 \qquad \text{a.e. } t \in (0, T), \quad (4.8)$$

where $\partial\psi_{\Sigma(t)}$ denotes the subdifferential of the indicator function $\psi_{\Sigma(t)}$. Since the set $\Sigma(t)$ depends on time, we shall replace (4.8) by a nonlinear evolution equation associated with a fixed convex set. To this end, we assume that (3.14)–(3.16) hold true and let

$$\Sigma_- = \{\tau \in \mathbb{R} : \tau \le 0\}, \tag{4.9}$$

and

$$V = \{v \in H^1(0,1) : v(0) = 0\}. \tag{4.10}$$

$V$ is a real Hilbert space endowed with the inner product $\langle u, v\rangle_V = \langle u_x, v_x\rangle$, and associated norm $|v|_V^2 = \langle v_x, v_x\rangle$. Using the Riesz representation theorem, we may define $\tilde{\sigma} : [0,T] \to L^2(0,1)$ by

$$\langle \tilde{\sigma}(t), v_x\rangle = \langle f(t), v\rangle \qquad \forall v \in V,\, t \in [0,T]. \tag{4.11}$$

Thus, $\tilde{\sigma}_x(t) + f(t) = 0$, for all $t \in [0,T]$, and then (3.13) yields

$$\tilde{\sigma} \in H^1(0,T; H^1(0,1)). \tag{4.12}$$

Let

$$\bar{\sigma}_\mu = \sigma_\mu - \tilde{\sigma} \qquad \text{and} \qquad \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0). \tag{4.13}$$

We have

LEMMA 4.2. *The pair* $\{u_\mu, \sigma_\mu\}$ *is a solution of* (4.3)–(4.7) *with* $u_\mu \in H^1(0,T; H^1(0,1))$, $\sigma_\mu \in H^1(0,T; H^1(0,1))$ *if and only if* $u_\mu \in H^1(0,T; V)$, $\tilde{\sigma}_\mu \in H^1(0,T; H^1(0,1))$ *and*

$$A\dot{\tilde{\sigma}}_\mu + A\dot{\tilde{\sigma}} + G_\mu(\tilde{\sigma}_\mu + \tilde{\sigma}) = \dot{u}_{\mu x} \qquad \text{in } \Omega_T, \tag{4.14}$$

$$\langle \tau - \tilde{\sigma}_\mu(t), \dot{u}_{\mu x}(t)\rangle \ge 0 \qquad \forall \tau \in \Sigma_- \;\; a.e.\; t \in (0,T), \tag{4.15}$$

$$\dot{u}_\mu(t) \in U_0 \qquad a.e.\; t \in (0,T), \tag{4.16}$$

$$\tilde{\sigma}_\mu(t) \in \Sigma_-, \qquad t \in [0,T], \tag{4.17}$$

$$u_\mu(0) = u_0, \qquad \tilde{\sigma}_\mu(0) = \tilde{\sigma}_0. \tag{4.18}$$

*Proof.* Using (4.11) we obtain that $\tilde{\sigma}_x(t) + f(t) = 0$ on $(0,1)$ and $\tilde{\sigma}(1,t) = 0$ for $t \in [0,T]$. Thus,

$$\tau \in \Sigma(t) \Leftrightarrow \tau - \tilde{\sigma}(t) \in \Sigma_- \qquad \text{for all } t \in [0,T]. \tag{4.19}$$

Lemma 4.2 follows now from (4.12), (4.13), and (4.19).

To solve (4.14)–(4.18) we shall use the fixed point method, similarly to [17, 19], where it was used in the study of contact problems too. To this end, let $\eta \in L^2(\Omega_T)$ and consider the following variational problem.

Find $\{u_{\mu\eta}, \sigma_{\mu\eta}\}$ such that

$$A\dot{\sigma}_{\mu\eta} + \eta = \dot{u}_{\mu\eta x} \qquad \text{in } \Omega_T, \tag{4.20}$$

$$\big\langle \tau - \sigma_{\mu\eta}(t), \dot{u}_{\mu\eta x}(t)\big\rangle \geq 0 \qquad \forall\tau \in \Sigma_- \text{ a.e. } t \in (0,T), \tag{4.21}$$

$$\dot{u}_{\mu\eta}(t) \in U_0 \qquad \text{a.e. } t \in (0,T), \tag{4.22}$$

$$\sigma_{\mu\eta}(t) \in \Sigma_-, \qquad t \in [0,T], \tag{4.23}$$

$$u_{\mu\eta}(0) = u_0, \qquad \overline{\sigma}_{\mu\eta}(0) = \sigma_0. \tag{4.24}$$

We have the following result:

LEMMA 4.3. *The variational problem* (4.20)–(4.24) *has a unique solution for each* $\eta \in L^2(\Omega_T)$. *Moreover,* $u_{\mu\eta} \in H^1(0,T; H^1(0,1))$ *and* $\sigma_{\mu\eta} \in H^1(0,T; H^1(0,1))$.

*Proof.* First, we note that (3.14), (4.19), and (4.13) imply that $\tilde{\sigma}_0 \in \Sigma_-$. Also, $\Sigma_-$ is a closed convex subset of $L^2(0,1)$. So, it follows from the standard theory of evolution equations involving maximal monotone operators (see, e.g., [6]) that there exists $\sigma_{\mu\eta} \in H^1(0,T; L^2(0,1))$, such that

$$\sigma_{\mu\eta}(t) \in \Sigma_- \qquad \text{for all } t \in [0,T], \tag{4.25}$$

$$\langle A\dot{\sigma}_{\mu\eta}, \tau - \sigma_{\mu\eta}\rangle + \langle \eta, \tau - \sigma_{\mu\eta}\rangle \geq 0 \qquad \forall\tau \in \Sigma_-, \text{ a.e. on } (0,T), \tag{4.26}$$

$$\sigma_{\mu\eta}(0) = \tilde{\sigma}_0. \tag{4.27}$$

Then (4.25) implies that $\sigma_{\mu\eta} \in H^1(0,T; L^2(0,1))$. Let $v_{\mu\eta} \in L^2(0,T;V)$ be given by

$$v_{\mu\eta}(x,t) = \int_0^x \big(A\dot{\sigma}_{\mu\eta}(y,t) + \eta(y,t)\big)\, dy, \tag{4.28}$$

for $x \in (0,1)$ and a.e. $t \in (0,T)$. Then, by (4.26), we have

$$\langle \tau - \sigma_{\mu\eta}, v_{\mu\eta x}\rangle \geq 0 \qquad \text{for all } \tau \in \Sigma_-, \text{ a.e. on } (0,T). \tag{4.29}$$

We claim that $v_{\mu\eta}(t) \in U_0$ a.e. on $(0,T)$. To this end, let $t \in [0,T]$ such that

$$\big\langle \tau - \sigma_{\mu\eta}(t), v_{\mu\eta x}(t)\big\rangle \geq 0 \qquad \text{for all } \tau \in \Sigma_-. \tag{4.30}$$

We note that $U_0$ is a closed convex subset of $V$ and we let $P$ be the projection map on $U_0$. Assume that $v_{\mu\eta}(t) \notin U_0$. Then

$$\left\langle Pv_{\mu\eta}(t) - v_{\mu\eta}(t), v\right\rangle_V \geq \left\langle Pv_{\mu\eta}(t) - v_{\mu\eta}(t), Pv_{\mu\eta}(t)\right\rangle_V$$
$$\geq \left\langle Pv_{\mu\eta}(t) - v_{\mu\eta}(t), v_{\mu\eta}(t)\right\rangle_V,$$

for all $v \in U_0$. Therefore, there exists $\alpha \in \mathbb{R}$ such that

$$\left\langle Pv_{\mu\eta}(t) - v_{\mu\eta}(t), v\right\rangle_V > \alpha > \left\langle Pv_{\mu\eta}(t) - v_{\mu\eta}(t), v_{\mu\eta}(t)\right\rangle_V, \quad (4.31)$$

for all $v \in U_0$. Let $\tilde{\tau}(t) \in L^2(0,1)$ be given by

$$\tilde{\tau}(t) = \left(Pv_{\mu\eta}(t) - v_{\mu\eta}(t)\right)_x.$$

We deduce from (4.31) that

$$\left\langle \tilde{\tau}(t), v_x\right\rangle_V > \alpha > \left\langle \tilde{\tau}(t), v_{\mu\eta x}(t)\right\rangle_V, \quad (4.32)$$

for all $v \in U_0$. Now, we choose $v = 0$ in (4.32) and obtain

$$\alpha < 0. \quad (4.33)$$

Suppose now that there exists $v \in U_0$ such that

$$\left\langle \tilde{\tau}(t), v_x\right\rangle_V < 0. \quad (4.34)$$

Then, using (4.32) and noting that $\lambda v \in U_0$ for all $\lambda \geq 0$, it follows that

$$\lambda\left\langle \tilde{\tau}(t), v_x\right\rangle_V > \alpha,$$

for all $\lambda \geq 0$, and passing to the limit $\lambda \to \infty$ we obtain that $\alpha = -\infty$ which contradicts (4.31). So, $\langle \tilde{\tau}(t), v_x \rangle_V \geq 0$ for all $v \in U_0$, and thus $\tilde{\tau}(t) \in \Sigma_-$. Now, using (4.30), (4.32), and (4.33) yield

$$\left\langle \sigma_{\mu\eta}(t), v_{\mu\eta x}(t)\right\rangle_V < 0. \quad (4.35)$$

Moreover, it follows from (4.30) that if $\tau = 2\sigma_{\mu\eta}(t)$ then

$$\left\langle \sigma_{\mu\eta}(t), v_{\mu\eta x}(t)\right\rangle_V \geq 0. \quad (4.36)$$

Since (4.35) and (4.36) are incompatible we conclude that $v_{\mu\eta}(t) \in U_0$. Now, we choose

$$u_{\mu\eta}(t) = \int_0^t v_{\mu\eta}(s)\, ds + u_0,$$

for $t \in [0, T]$, and then Lemma 4.3 follows from (4.25) and (4.27)–(4.29).

Let us now consider the operator $\Lambda_\mu : L^2(\Omega_T) \to L^2(\Omega_T)$ defined by

$$\Lambda_\mu \eta = A\dot{\tilde{\sigma}} + G_\mu\big(\sigma_{\mu\eta} + \tilde{\sigma}\big), \tag{4.37}$$

where, for every $\eta \in L^2(\Omega_T)$, $\{u_{\mu\eta}, \sigma_{\mu\eta}\}$ denotes the associated solution of the variational problem (4.20)–(4.24). We have

LEMMA 4.4. *The operator $\Lambda_\mu$ has a unique fixed point $\eta_\mu \in L^2(\Omega_T)$.*

*Proof.* Let $\eta_1, \eta_2 \in L^2(\Omega_T)$. It follows from (4.26) and (4.27) that

$$\left|\sigma_{\mu\eta_1}(t) - \sigma_{\mu\eta_2}(t)\right|^2_{L^2(0,1)} \le C\int_0^t |\eta_1(s) - \eta_2(s)|^2_{L^2(0,1)}\,ds, \tag{4.38}$$

for all $t \in [0,T]$, where $C > 0$ depends only on $A$. Using (4.37) and (4.2) yields

$$\left|\Lambda_\mu\eta_1(t) - \Lambda_\mu\eta_2(t)\right|^2_{L^2(0,1)} \le \left|\sigma_{\mu\eta_1}(t) - \sigma_{\mu\eta_2}(t)\right|^2_{L^2(0,1)}, \tag{4.39}$$

for all $t \in [0,T]$. Then, it follows from (4.38) and (4.39) that

$$\left|\Lambda_\mu\eta_1(t) - \Lambda_\mu\eta_2(t)\right|^2_{L^2(0,1)} \le \frac{C}{\mu^2}\int_0^t |\eta_1(s) - \eta_2(s)|^2_{L^2(0,1)}\,ds, \tag{4.40}$$

for all $t \in [0,T]$. This shows that a power $\Lambda_\mu^p$ of $\Lambda_\mu$ is a contraction mapping on $L^2(\Omega_T)$.

Now we have:

*Proof of Theorem* 4.1. Let $\eta_\mu \in L^2(\Omega_T)$ be the fixed point of $\Lambda_\mu$ and let $u_\mu \in H^1(0,T,H^1(0,1))$, $\tilde{\sigma}_\mu \in H^1(0,T,H^1(0,1))$ be the functions given by Lemma 4.3 for $\eta = \eta_\mu$. It follows that $\{u_\mu, \tilde{\sigma}_\mu\}$ is a solution of (4.14)–(4.18) and, using Lemma 4.2, we obtain the existence part in Theorem 4.1.

The uniqueness part of Theorem 4.1 follows from the uniqueness of the fixed point of the operator $\Lambda_\mu$.

## 5. PROOF OF THE MAIN RESULT

In this section we prove Theorem 3.1. We suppose that (3.14) and (3.15) hold and that $u_0$ satisfies (3.16). For each viscosity constant $\mu > 0$ let $\{u_\mu, \sigma_\mu\}$ denote the solution of the elastoviscoplastic problem (4.3)–(4.7), let $v_\mu = \dot{u}_\mu$, and let $\mathscr{G}_\mu : L^2(0,1) \to \mathbb{R}_+$ be the function

$$\mathscr{G}_\mu(\tau) = \frac{1}{2\mu}|\tau - P_K\tau|^2_{L^2(0,1)}. \tag{5.1}$$

In the sequel $C$ will denote a strictly positive constant which does not depend on $\mu$ and whose value may change from place to place.

We have:

LEMMA 5.1. *The sequence* $\{\sigma_\mu\}$ *is bounded in* $L^\infty(0,T;L^2(0,1))$.

*Proof.* Using (4.3) and (4.4) we obtain

$$\langle A\dot{\sigma}_\mu + G_\mu(\sigma_\mu), \tau - \sigma_\mu\rangle \geq 0 \qquad \forall \tau \in \Sigma(t) \text{ a.e. } t \in (0,T), \tag{5.2}$$

and from (3.15) we obtain

$$\chi(t) \in \mathscr{K} \cap \Sigma(t) \qquad \forall t \in [0,T]. \tag{5.3}$$

Let $\bar{\sigma}_\mu = \sigma_\mu - \chi$. Using (5.2) and (5.3) yields

$$\langle A\dot{\bar{\sigma}}_\mu, \bar{\sigma}_\mu\rangle + \langle G_\mu(\sigma_\mu), \bar{\sigma}_\mu\rangle \leq -\langle A\dot{\chi}, \bar{\sigma}_\mu\rangle \qquad \text{a.e. on } (0,T). \tag{5.4}$$

We note that the function $G_\mu$, defined in (4.2), is the derivative of the convex function $s \to (1/2\mu)|s - P_K s|^2$. So, using (5.1) and the subgradient inequality we find

$$\mathscr{G}_\mu(\tau) - \mathscr{G}_\mu(\sigma) \geq \langle G_\mu(\sigma), \tau - \sigma\rangle \qquad \forall \sigma, \tau \in L^2(0,1). \tag{5.5}$$

Using (5.3) we obtain $\mathscr{G}_\mu(\chi) = 0$ for all $t \in [0,T]$. Then, (5.5) implies

$$\langle G_\mu(\sigma_\mu), \bar{\sigma}_\mu\rangle \geq \mathscr{G}_\mu(\sigma_\mu), \qquad t \in [0,T]. \tag{5.6}$$

It follows from (5.4) and (5.6) that

$$\langle A\dot{\bar{\sigma}}_\mu, \bar{\sigma}\rangle + \mathscr{G}_\mu(\sigma_\mu) \leq -\langle A\dot{\chi}, \bar{\sigma}_\mu\rangle \qquad \text{a.e. on } (0,T). \tag{5.7}$$

From (4.7) and (3.15)(c) we have $\bar{\sigma}_\mu(0) = 0$ and since $\mathscr{G}_\mu(\sigma_\mu) \geq 0$, the previous inequality yields

$$\begin{aligned} |\bar{\sigma}_\mu(t)|^2_{L^2(0,1)} &\leq C\int_0^t \langle A\dot{\chi}, \bar{\sigma}_\mu\rangle\, ds \\ &\leq C\int_0^t |A\dot{\chi}|_{L^2(0,1)} |\bar{\sigma}_\mu|_{L^2(0,1)}\, ds, \end{aligned} \tag{5.8}$$

for all $t \in [0,T]$. Lemma 5.1 now follows from (5.8) and the regularity of $\chi$.

LEMMA 5.2. *The sequence* $\{G_\mu(\sigma_\mu)\}$ *is bounded in* $L^1(0,T;L^1(0,1))$.

*Proof.* Using (5.4) and Lemma 5.1 yields

$$\int_0^T \langle G_\mu(\sigma_\mu), \bar{\sigma}_\mu\rangle\, dt \leq C. \tag{5.9}$$

Let $\tau \in L^\infty(0,1)$ such that $|\tau|_{L^\infty(0,1)} \le \delta$. Using (3.15)(d) we have $\chi(t) + \tau \in \mathscr{K}$, which implies $\mathscr{G}_\mu(\chi + \tau) = 0$ for all $t \in [0,T]$. Moreover, from (5.5) we obtain

$$\left\langle G_\mu(\sigma_\mu), \chi + \tau - \sigma_\mu \right\rangle \le \mathscr{G}_\mu(\chi + \tau) - \mathscr{G}_\mu(\sigma_\mu) \le 0,$$

for all $t \in [0,T]$. Hence,

$$\left\langle G_\mu(\sigma_\mu), \tau \right\rangle \le \left\langle G_\mu(\sigma_\mu), (\bar{\sigma}_\mu) \right\rangle,$$

for all $t \in [0,T]$. Using this inequality yields

$$\left| G_\mu(\sigma_\mu) \right|_{L^1(0,1)} = \frac{1}{\delta} \sup \left\langle G_\mu(\sigma_\mu), \tau \right\rangle \le \left\langle G_\mu(\sigma_\mu), \bar{\sigma}_\mu \right\rangle, \tag{5.10}$$

for all $t \in [0,T]$; here the supremum is taken over the set

$$\left\{ \tau \in L^\infty(0,1), |\tau|_{L^\infty(0,1)} \le \delta \right\}.$$

Lemma 5.2 follows now from (5.9) and (5.10).

LEMMA 5.3. *The sequence $\{\dot{\sigma}_\mu\}$ is bounded in $L^2(\Omega_T)$.*

*Proof.* Using (3.15) and (4.16), after integration by parts we obtain

$$\langle \dot{\chi}, \dot{u}_{\mu x} \rangle \ge \langle \dot{f}, \dot{u}_\mu \rangle \qquad \text{a.e. on } (0,T). \tag{5.11}$$

Moreover, it follows from (4.15), after some manipulations, that

$$\langle \dot{\bar{\sigma}}_\mu, \dot{u}_{\mu x} \rangle = 0 \qquad \text{a.e. on } (0,T),$$

and, recalling (4.13) and (4.11), we obtain

$$\langle \dot{\sigma}_\mu, \dot{u}_{\mu x} \rangle = \langle \dot{f}, \dot{u}_\mu \rangle \qquad \text{a.e. on } (0,T). \tag{5.12}$$

We now multiply (4.3) by $\dot{\bar{\sigma}}_\mu = \dot{\sigma}_\mu - \dot{\chi}$, integrate over on $(0,1)$, and obtain

$$\langle A\dot{\sigma}_\mu, \dot{\bar{\sigma}}_\mu \rangle + \langle G_\mu(\sigma_\mu), \dot{\bar{\sigma}}_\mu \rangle = \langle \mu_{\mu x}, \dot{\sigma}_\mu - \dot{\chi} \rangle \qquad \text{a.e. on } (0,T).$$

Using (5.11) and (5.12) leads to

$$\langle A\dot{\sigma}_\mu, \dot{\bar{\sigma}}_\mu \rangle + \langle G_\mu(\sigma_\mu), \dot{\bar{\sigma}} \rangle \le 0 \qquad \text{a.e. on } (0,T).$$

This implies

$$\langle A\dot{\bar{\sigma}}_\mu, \dot{\bar{\sigma}}_\mu \rangle + \left\langle G_\mu(\sigma_\mu), \dot{\sigma}_\mu \right\rangle \le \left\langle G_\mu(\sigma_\mu), \dot{\chi} \right\rangle - \langle A\dot{\chi}, \dot{\bar{\sigma}}_\mu \rangle,$$

a.e. on $(0, T)$. Integration on $[0, T]$ and Lemma 5.2 yield

$$A|\dot{\bar{\sigma}}|^2_{L^2(\Omega_T)} + \int_0^T \langle G_\mu(\sigma_\mu), \dot{\sigma}_\mu \rangle \, dt \le C_1 + C_2 |\dot{\bar{\sigma}}_\mu|_{L^2(\Omega_T)}. \tag{5.13}$$

Now,

$$\begin{aligned} \int_0^T \langle G_\mu(\sigma_\mu), \dot{\sigma}_\mu \rangle \, dt &= \mathscr{G}_\mu\big(\sigma_\mu(T)\big) - \mathscr{G}_\mu\big(\sigma_\mu(0)\big) \\ &\ge -\mathscr{G}_\mu(\sigma_0) \\ &= 0, \end{aligned}$$

where we used (3.15)(c), (d). Hence, we deduce from (5.13) that $\{\dot{\bar{\sigma}}_\mu\}$ is a bounded sequence in $L^2(\Omega_T)$ which proves Lemma 5.3.

LEMMA 5.4. *The sequence* $\{v_\mu\}$ *is bounded in* $L^2(0, T; \mathrm{BD}(0,1))$.

*Proof.* Using (5.4) and Lemmas 5.1 and 5.3 we deduce that the sequence $\{\langle G_\mu(\sigma_\mu), \bar{\sigma}_\mu \rangle\}$ is bounded in $L^2(0, T; \mathbb{R})$. It follows from (5.10) that the sequence $\{G_\mu(\sigma_\mu)\}$ is bounded in $L^2(0, T; L^1(0,1))$. Thus, we obtain from (4.3) and Lemma 5.3 that $\{v_{\mu x}\}$ is a bounded sequence in $L^2(0, T; L^1(0,1))$. Now, since we have a homogeneous Dirichlet boundary condition at $x = 1$, the lemma follows from the result on the equivalence of norms in, e.g., [21, Chap. 2].

We now prove Theorem 3.1.

*Existence*. Using Lemmas 5.1 and 5.3 we deduce that there exists $\sigma \in H^1(0, T, L^2(0,1))$ such that, for a subsequence still denoted by $\{\sigma_\mu\}$, we have

$$\sigma_\mu \to \sigma \qquad \text{weak* in } L^\infty\big(0, T, L^2(0,1)\big), \tag{5.14}$$

$$\dot{\sigma}_\mu \to \dot{\sigma} \qquad \text{weakly in } L^2(\Omega_T), \tag{5.15}$$

when $\mu \le 0$. We note that since $\mathrm{BD}(0,1)$ is the dual of the Banach space $X$ (see, e.g., [21, Chap. 2]), then $L^2_w(0, T; \mathrm{BD}(0,1))$ is the dual of $L^2(0, T; X)$. Thus, we deduce, using Lemma 5.4, that there exists $v \in L^2_w(0, T; \mathrm{BD}(0,1))$ such that, for a subsequence still denoted by $\{v_\mu\}$, we have

$$v_\mu \to v \qquad \text{weak* in } L^2_w\big(0, T; \mathrm{BD}(0,1)\big). \tag{5.16}$$

We shall prove that $\{v, \sigma\}$ is a solution of Problem $P_1$, (3.8)–(3.10), and $\sigma$ is a solution of Problem $P_2$, (3.11), and (3.12). To this end we note that (5.14) and (5.15) imply

$$\sigma_\mu(t) \to \sigma(t) \qquad \text{weakly in } L^2(0,1) \quad t \in [0, T]. \tag{5.17}$$

Since $\Sigma(t)$ is a convex set in $L^2(0,1)$, it follows from (4.6) and (5.17) that

$$\sigma(t) \in \Sigma(t), \qquad t \in [0,T]. \tag{5.18}$$

Moreover, we have from (5.5)

$$\mathscr{G}_\mu(\sigma) \le \mathscr{G}_\mu(\sigma_\mu) - \langle G_\mu(\sigma), \sigma_\mu - \sigma \rangle, \qquad t \in [0,T],$$

and using (4.2) we deduce

$$\int_0^T \mu \mathscr{G}_\mu(\sigma) \le \mu \int_0^T \mathscr{G}_\mu(\sigma_\mu)\, dt - \int_0^T \langle \sigma - P_K \sigma, \sigma_\mu - \sigma \rangle\, dt. \tag{5.19}$$

Using (5.7) and Lemma 5.1 results in

$$\int_0^T \mathscr{G}_\mu(\sigma_\mu)\, dt \le C. \tag{5.20}$$

It now follows from (5.1), (5.14), (5.19), and (5.20) that

$$\tfrac{1}{2} \int_0^T |\sigma - P_K \sigma|^2\, dt = 0,$$

which implies that $\sigma = P_K \sigma$ a.e. in $(0,1)$ for all $t \in [0,T]$. Using now (3.4) we deduce

$$\sigma(t) \in \mathscr{K}, \qquad t \in [0,T]. \tag{5.21}$$

Thus, from (5.18) and (5.21) we obtain (3.8). Next, we note that

$$\langle \tau - \sigma_\mu, \dot{u}_{\mu x} \rangle + \langle \tau_x - \sigma_{\mu x}, \dot{u}_\mu \rangle \ge 0, \qquad \forall \tau \in \Sigma_0, \tag{5.22}$$

a.e. on $(0,T)$. Indeed, we choose $\tau = \tilde{\sigma}$ in (4.4), where $\tilde{\sigma}$ is given in (4.11), thus

$$\langle f, \dot{u}_\mu \rangle \ge \langle \sigma_\mu, \dot{u}_{\mu x} \rangle,$$

a.e. on $(0,T)$. Using (4.5) and (4.6) yields

$$\langle f, \dot{u}_\mu \rangle \le \langle \sigma_\mu, \dot{u}_{\mu x} \rangle,$$

a.e. on $(0,T)$. Thus, $\langle f, \dot{u}_\mu \rangle = \langle \sigma_\mu, \dot{u}_{\mu x} \rangle$, a.e. on $(0,T)$. Since $\dot{u}_\mu(0,t) = 0$,

$$\sigma_\mu(1,t)\dot{u}_\mu(1,t) = 0 \qquad \text{a.e. on } (0,T). \tag{5.23}$$

We now obtain (5.22) by integration by parts and using (3.2), (4.5), and (5.23). To establish (3.9) we use (4.3), so

$$\langle A\dot{\sigma}_\mu, \tau - \sigma_\mu \rangle + \langle G_\mu(\sigma_\mu), \tau - \sigma_\mu \rangle \ge \langle \dot{u}_{\mu x}, \tau - \sigma_\mu \rangle \qquad \forall \tau \in \Sigma_0,$$

a.e. on $(0, T)$, and, then (5.5) and (5.22) imply

$$\langle A\dot{\sigma}_\mu, \tau - \sigma_\mu \rangle + \mathscr{G}_\mu(\tau) - \mathscr{G}_\mu(\sigma_\mu) + \langle \dot{u}_\mu, \tau_x - \sigma_{\mu x} \rangle \geq 0 \qquad \forall \tau \in \Sigma_0,$$

a.e. on $(0, T)$. Choosing $\tau \in L^2(\Omega_T)$, such that $\tau \in \mathscr{K} \cap \Sigma_0$ a.e. on $(0, T)$, in the previous inequality and noting that $\sigma_{\mu x} = \sigma_x = -f$, and $\dot{u}_\mu = v_\mu$, we obtain

$$\langle A\dot{\sigma}_\mu, \tau - \sigma_\mu \rangle + \langle v_\mu, \tau_x - \sigma_x \rangle \geq 0, \tag{5.24}$$

a.e. on $(0, T)$. We now integrate over $(0, s)$

$$\int_0^s \langle A\dot{\sigma}_\mu, \tau \rangle \, dt + \int_0^s \langle v_\mu, \tau_x - \sigma_x \rangle \, dt \geq \int_0^s \langle A\dot{\sigma}_\mu, \sigma_\mu \rangle \, dt, \tag{5.25}$$

for $s \in [0, T]$. Using (5.14)–(5.16), (5.25), and a lower-semicontinuity argument yield

$$\int_0^s \langle A\dot{\sigma}, \tau \rangle \, dt + \int_0^s \langle v, \tau_x - \sigma_x \rangle \, dt \geq \int_0^s \langle A\dot{\sigma}, \sigma \rangle \, dt, \qquad s \in [0, T],$$

and, after a classical use of Lebesgue points for an $L^1$ function, we obtain (3.9).

Let us finally remark that (3.10) is a consequence of (4.7) and (5.17). Thus, we have established that $\{v, \sigma\}$ is a solution to Problem $P_1$. Moreover, it follows from (3.3), (3.13), and (5.18) that $\sigma \in H^1(0, T; H^1(0, 1))$. Choosing $\tau \in \mathscr{K} \cap \Sigma(t)$ in (3.9) we deduce (3.12). Therefore, $\sigma$ is a solution to Problem $P_2$.

*Uniqueness*. The uniqueness of $\sigma$ in Theorem 3.1 follows from standard arguments of convex analysis.

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