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Uniform asymptotics of the finite-time ruin probability for all times ☆

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ABSTRACT

In this paper, we study the asymptotics of the finite-time ruin probability for a generalized renewal risk model with independent strong subexponential claim sizes and widely lower orthant dependent inter-occurrence times. This widely lower orthant dependence structure can include some common negative dependent random variables (r.v.s) such as negatively lower orthant dependent r.v.s, some positive dependent r.v.s and some other dependent r.v.s. For this generalized renewal risk model, we show that the asymptotics of the finite-time ruin probability hold uniformly for all times in the sense that the magnitude of the initial reserve of some insurance portfolio does not influence the length of the operating time. Further, the uniform asymptotics of the random-time ruin probability in the above model have been obtained.

1. Introduction

The main objective of this contribution is the investigation the uniform asymptotic behavior of the finite-time ruin probability in risk models with heavy-tailed claim sizes. The study in Tang [16] is the first paper which considered this topic presenting results on the uniform asymptotics of the finite-time ruin probability for all times under the conditions that the common distribution of the claim sizes belongs to the consistently-varying-tailed distribution class (see definition in Section 1.3 below) and the inter-occurrence times satisfy certain moment condition (see Theorem 2.A below). In order to extend the scope of the claim sizes, Leipus and Šiaulys [11,12], Jiang [7] and Kočetova et al. [9] considered the strong subexponential claim sizes (see definition in Section 1.3 below), which properly include the consistently-varying claim sizes. They obtained the asymptotics of the finite-time ruin probability, which are uniform for time belonging to $[f(x), \infty)$, here $f(x)$ is an arbitrary infinitely increasing function.

In many papers in the literatures claim sizes and inter-occurrence times are both independent and identically distributed (i.i.d.) r.v.s. Recently, Yang et al. [22] considered the strong subexponential claim sizes and negatively lower orthant dependent inter-occurrence times. The technical conditions (see Theorem 2.C below) imposed therein are to a large extend restrictive, for instance being not satisfied for instance by claim sizes with common strong subexponential Weibull distribution.

Motivated by the aforementioned contributions this paper will discuss a more generalized renewal risk model with independent strong subexponential claim sizes and widely lower orthant dependent inter-arrival times (see definition in Section 1.2 below), and give the uniform asymptotics of the finite-time ruin probability for all times without the technical conditions imposed in Yang et al. [22]. The method what we will use is different from that in the literatures mentioned

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above, and the obtained results extend and improve the corresponding results of the above-mentioned literatures. Furthermore, the corresponding results of the random-time ruin probability can also be obtained.

In order to state the motivation of this paper and present the main results, in this section we will introduce the standard renewal risk model, the concepts and basic properties of some widely dependent r.v.s and some related heavy-tailed distribution classes, respectively.

1.1. Standard renewal risk model

A risk model is called the standard renewal risk model if the following conditions are satisfied:

Assumption H_1. The claim sizes $\{Z_i:\ i \geqslant 1\}$ are i.i.d. r.v.s, with common distribution function B on $[0,\infty)$ and finite mean $b > 0$.

Assumption H_2. The inter-occurrence times $\{\theta_i:\ i \geqslant 1\}$ are i.i.d. r.v.s, with common distribution function A on $[0,\infty)$ and finite mean $\lambda^{-1} > 0$.

Assumption H_3. The sequence $\{Z_i:\ i \geqslant 1\}$ is independent of $\{\theta_i:\ i \geqslant 1\}$.

In a special situation, if A has an exponential distribution function, this standard renewal risk model is referred to as Lundberg risk model.

Denote $T_n = \sum_{i=1}^n \theta_i$, $n \geqslant 1$, then $\{T_n:\ n \geqslant 1\}$ constitute a renewal counting process

$$N(t) = \max\{n \geqslant 1:\ T_n \leqslant t\},$$

with a mean function $\lambda(t) = EN(t)$, $t \geqslant 0$. It is well known that

$$\lambda(t) \sim \lambda t \quad \text{as } t \to \infty,$$

where for two positive functions $f(\cdot)$ and $g(\cdot)$, $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

Denote the initial reserve by $x > 0$, and premium intensity by $c > 0$, then the finite-time ruin probability for some time $t > 0$ is

$$\psi(x,t) \equiv P\left(\sup_{0\leqslant n\leqslant N(t)} \sum_{i=1}^n (Z_i - c\theta_i) > x\right).$$

Let τ be a nonnegative r.v., then the random-time ruin probability is

$$\psi(x,\tau) \equiv P\left(\sup_{0\leqslant n\leqslant N(\tau)} \sum_{i=1}^n (Z_i - c\theta_i) > x\right).$$

In order for the ultimate ruin not to be certain, it is natural to assume that the safety loading condition holds, that is

$$\mu \equiv c\lambda^{-1} - b > 0. \tag{1.1}$$

1.2. Wide dependence structures

In this paper, we will consider the widely lower orthant dependent inter-occurrence times. Therefore, in this subsection, we will present some wide dependence structures introduced in Wang et al. [18].

Definition 1.1. For the r.v.s $\{X_i:\ i \geqslant 1\}$, if there exists a finite real sequence $\{g_U(n):\ n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty,\infty)$, $1 \leqslant i \leqslant n$,

$$P\left(\bigcap_{i=1}^n \{X_i > x_i\}\right) \leqslant g_U(n) \prod_{i=1}^n P(X_i > x_i), \tag{1.2}$$

then we say that the r.v.s $\{X_i:\ i \geqslant 1\}$ are widely upper orthant dependent (WUOD); if there exists a finite real sequence $\{g_L(n):\ n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty,\infty)$, $1 \leqslant i \leqslant n$,

$$P\left(\bigcap_{i=1}^n \{X_i \leqslant x_i\}\right) \leqslant g_L(n) \prod_{i=1}^n P(X_i \leqslant x_i), \tag{1.3}$$

then we say that the r.v.s $\{X_i:\ i \geqslant 1\}$ are widely lower orthant dependent (WLOD); if they are both WUOD and WLOD, then we say that the r.v.s $\{X_i:\ i \geqslant 1\}$ are widely orthant dependent (WOD). WUOD, WLOD and WOD r.v.s are called by a joint name wide dependent (WD) r.v.s, and $g_U(n)$, $g_L(n)$, $n \geqslant 1$, are called dominating coefficients.

Next, we introduce some basic properties of WD r.v.s (see Proposition 1.1 of Wang et al. [18]).

Proposition 1.1.

(1) *Let $\{\xi_n\colon n \geqslant 1\}$ be WLOD (WUOD) with dominating coefficients $g_U(n), n \geqslant 1$ ($g_L(n), n \geqslant 1$). If $\{f_n(\cdot)\colon n \geqslant 1\}$ are nondecreasing, then $\{f_n(\xi_n)\colon n \geqslant 1\}$ are still WLOD (WUOD) with dominating coefficients $g_U(n), n \geqslant 1$ ($g_L(n), n \geqslant 1$); if $\{f_n(\cdot)\colon n \geqslant 1\}$ are nonincreasing, then $\{f_n(\xi_n)\colon n \geqslant 1\}$ are WUOD (WLOD) with dominating coefficients $g_U(n), n \geqslant 1$ ($g_L(n), n \geqslant 1$).*

(2) *If $\{\xi_n\colon n \geqslant 1\}$ are nonnegative and WUOD with dominating coefficients $g_U(n), n \geqslant 1$, then for each $n \geqslant 1$,*

$$E\prod_{i=1}^{n}\xi_i \leqslant g_U(n)\prod_{i=1}^{n}E\xi_i.$$

In particular, if $\{\xi_n\colon n \geqslant 1\}$ are WUOD with dominating coefficients $g_U(n), n \geqslant 1$, then for each $n \geqslant 1$ and any $s > 0$,

$$E\exp\left\{s\sum_{i=1}^{n}\xi_i\right\} \leqslant g_U(n)\prod_{i=1}^{n}E\exp\{s\xi_i\}.$$

For examples of WD r.v.s with various dominating coefficients, we refer the reader to Wang et al. [18]. These examples show that WD r.v.s contain some common negatively dependent r.v.s, some positively dependent r.v.s and some others.

Now, we recall that when $g_L(n) = g_U(n) \equiv 1$ for any $n \geqslant 2$ in (1.2) and (1.3), the r.v.s $\{X_i\colon i \geqslant 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively. If they are both NUOD and NLOD, then we say that the r.v.s $\{X_i\colon i \geqslant 1\}$ are negatively orthant dependent (NOD) (see, e.g. Ebrahimi and Ghosh [5] or Block et al. [1]).

If both (1.2) and (1.3) hold when $g_L(n) = g_U(n) \equiv M$ for some constant $M > 1$ and for any $n \geqslant 2$, the r.v.s $\{X_i\colon i \geqslant 1\}$ are called extended negatively upper ortant dependent (ENUOD) and extended negatively lower ortant dependent (ENLOD), respectively. If they are both ENUOD and ENLOD, then we say that the r.v.s $\{X_i\colon i \geqslant 1\}$ are extended negatively orthant dependent (ENOD) (see, e.g. Liu [13]).

This paper will consider the dependent inter-occurrence times. For this point, Yang et al. [22] have extended Assumption H_2 in the standard renewal risk model to the following assumption.

Assumption H_2^*. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are NLOD r.v.s with common distribution function A on $[0,\infty)$ and finite mean $\lambda^{-1} > 0$.

Now we extend Assumption H_2^* to the following assumption.

Assumption H_2^{**}. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are WLOD r.v.s with common distribution function A on $[0,\infty)$ and finite mean $\lambda^{-1} > 0$.

We call the renewal risk model satisfying Assumptions H_1, H_2^{**} and H_3 as the generalized renewal risk model in this paper. For this generalized renewal risk model, we will investigate the uniform asymptotics of the finite-time ruin probability for the claim sizes with heavy-tailed distribution functions. In the following subsection, some heavy-tailed distribution classes are introduced.

1.3. Some heavy-tailed distribution classes

Let F be a proper distribution function on $[0,\infty)$, i.e. $F(\infty) = 1$. If $\int_0^\infty e^{\delta x}\,\mathrm{d}F(x) = \infty$ for any $\delta > 0$, then we say that the distribution function F is heavy-tailed. The most important subclass of the heavy-tailed distribution class is the subexponential distribution class, which was introduced by Chistyakov [2]. It is well known that the subexponential distribution class plays an important role in queueing theory, risk theory, infinite divisibility theory and many other fields of probability. By definition, we say that a distribution function F belongs to the subexponential distribution class, denoted by $F \in \mathcal{S}$, if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x) \quad \text{as } x \to \infty,$$

where F^{*2} is the convolution of F with itself and $\overline{F} = 1 - F$ is the tail of F.

It is well known that the class $\mathcal{S}$ is properly contained in another heavy-tailed distribution class. We say that a distribution function F belongs to the long-tailed distribution class, denoted by $F \in \mathcal{L}$, if for any $y \in (-\infty,\infty)$,

$$\overline{F}(x+y) \sim \overline{F}(x) \quad \text{as } x \to \infty.$$

Denote

$$\mathscr{H}(F) = \{h : [0, \infty) \to [0, \infty): \ h(x) \uparrow \infty, \ x^{-1}h(x) \to 0 \text{ and } \overline{F}(x+y) \sim \overline{F}(x) \text{ uniformly for all } |y| \leqslant h(x) \text{ as } x \to \infty\}.$$

The class $\mathcal{L}$ has the following well-known properties which are often used in this paper.

Proposition 1.2.

(1) *A distribution function $F \in \mathcal{L}$ if and only if $\mathscr{H}(F) \neq \emptyset$.*
(2) *If $h_1 \in \mathscr{H}(F)$, $h_2 \leqslant h_1$ and $h_2(x) \uparrow \infty$ as $x \to \infty$, then $h_2 \in \mathscr{H}(F)$.*
(3) *If $h \in \mathscr{H}(F)$, then for any $a > 0$, $ah \in \mathscr{H}(F)$.*
(4) *If $F \in \mathcal{S}$ then for any $h \in \mathscr{H}(F)$,*

$$\lim_{x\to\infty} \int_{h(x)}^{x-h(x)} \overline{F}(x-y)\, \mathrm{d}F(y) \big(\overline{F}(x)\big)^{-1} = 0.$$

Klüppelberg [8] introduced a subclass of the class $\mathcal{S}$. We say that a distribution function $F \in \mathcal{S}^*$, if $\int_0^\infty \overline{F}(y)\,\mathrm{d}y < \infty$ and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y \sim 2\int_0^\infty \overline{F}(y)\,\mathrm{d}y\, \overline{F}(x) \quad \text{as } x \to \infty.$$

Klüppelberg [8] pointed out that if $F \in \mathcal{S}^*$ then $F \in \mathcal{S}$ and $F^I \in \mathcal{S}$, where $\overline{F^I}(x) = \min\{1, \int_x^\infty \overline{F}(y)\,\mathrm{d}y\}$ for $x \geqslant 0$. Some equivalent relations between F and F^I have been given by Wang et al. [21] and Wang and Wang [20], among others.

Korshunov [10] introduced another heavy-tailed distribution class which contains the class $\mathcal{S}^*$ and is contained by the class $\mathcal{S}$. We say that a distribution function F is strongly subexponential, denoted by $F \in \mathcal{S}_*$, if $\int_0^\infty \overline{F}(y)\,\mathrm{d}y < \infty$ and

$$\overline{F_u^{*2}}(x) \sim 2\overline{F_u}(x) \quad \text{as } x \to \infty$$

uniformly for $u \in [1, \infty)$, where

$$\overline{F_u}(x) = \begin{cases} \min\{1, \int_x^{x+u} \overline{F}(y)\,\mathrm{d}y\} & \text{if } x \geqslant 0; \\ 1 & \text{if } x < 0. \end{cases}$$

Korshunov [10] also derived an important property of $\mathcal{S}_*$ (see Lemma 4.3(1) below), which is one of the bases of studying uniform asymptotics of the finite-time ruin probability. In addition, from Lemma 9 of Denisov et al. [4] we know that $\mathcal{S}^* \subset \mathcal{S}_*$. But so far we do not known whether the inclusion is strict. It should be noted that there are many common heavy-tailed distribution functions belonging to the class $\mathcal{S}_*$, such as the Weibull distribution with suitably chosen parameters, the Log-normal distribution and the distributions in the following distribution class (see, e.g. Korshunov [10]).

We say that a distribution function F belongs to the consistently-varying-tailed distribution class, denoted by $F \in \mathcal{C}$, if

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = 1.$$

Cline and Samorodnitsky [3] pointed out that, if $F \in \mathcal{C}$ and $\int_0^\infty \overline{F}(y)\,\mathrm{d}y < \infty$, then $F \in \mathcal{S}^*$.

The rest of this paper is organized as follows: In Section 2 we present the existing results about the uniform asymptotics of the finite-time ruin probability for the renewal risk model, and give the motivation and main results of this paper. In Section 3 the main results will be proved. For this, in Section 4 some preparatory results will be given.

2. Main results

2.1. Brief review

Tang [16] first investigated the uniform asymptotics of the finite-time ruin probability with heavy-tailed claims. Before giving his result, we first introduce some notation. For any distribution function F on $[0, \infty)$, denote

$$\overline{F_*}(y) = \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x), \quad y > 0 \quad \text{and} \quad J_F^+ = -\lim_{y\to\infty} \log \overline{F_*}(y)/\log y.$$

Theorem 2.A. *(See Theorem 3.1 of Tang [16].) Suppose that Assumptions H_1, H_2, H_3 and (1.1) hold. If $B \in \mathcal{C}$ and*

$$E\theta_1^p < \infty \quad \text{for some } p > J_B^+ + 1, \tag{2.1}$$

then it holds uniformly for all $t \in \Lambda \equiv \{t\colon \lambda(t) > 0\}$,

$$\psi(x,t) \sim \mu^{-1} \int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u \quad \text{as } x \to \infty, \tag{2.2}$$

that is

$$\lim_{x\to\infty} \sup_{t\in\Lambda} \left| \psi(x,t) \left(\mu^{-1} \int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u \right)^{-1} - 1 \right| = 0.$$

Tang [16] also pointed out that for any infinitely increasing function f, for $t \in [f(x), \infty)$, $\lambda(t)$ in (2.2) can be replaced by λt.

It is well known that the class $\mathcal{C}$ unfortunately excludes some heavy-tailed distribution functions such as the Weibull distribution with suitably chosen parameters, the Log-normal distribution and so on. In order to extend the scope of applications, Leipus and Šiaulys [11,12], Jiang [7] and Kočetova et al. [9] have done a series of improvements culminating in the following theorem.

Theorem 2.B. *(See Theorem 4 of Kočetova et al. [9].) Suppose that Assumptions* H_1, H_2, H_3 *and* (1.1) *hold. If* $B \in \mathcal{S}_*$, *then* (2.2) *holds uniformly for all* $t \in [f(x), \infty)$, *where* $f(x)$ *is an arbitrary infinitely increasing function as* $x \to \infty$.

Yang et al. [22] considered the NLOD inter-occurrence times. Under some technical conditions, they extended the range of the operating time.

Theorem 2.C. *(See Theorem 2.1 of Yang et al. [22].) Suppose that Assumptions* H_1, H_2^*, H_3 *and* (1.1) *hold and* $B \in \mathcal{S}_*$. *Let the following conditions be satisfied*

$$\lim_{x\to\infty} \exp\{-\delta\sqrt{x}\}\big(\overline{B}(x)\big)^{-1} = 0 \quad \text{for any } \delta > 0 \quad \text{and} \quad \limsup_{x\to\infty} \overline{B}(x-\sqrt{x})\big(\overline{B}(x)\big)^{-1} \leqslant 1. \tag{2.3}$$

Then for any $t_0 \in \Lambda$, (2.2) *holds uniformly for all* $t \in [t_0, \infty)$.

2.2. Motivations of this paper

Based on Theorems 2.A, 2.B and 2.C, we aim to investigate the following three problems.

Firstly, is it possibly to extend the range of t in Theorem 2.B from $[f(x), \infty)$ to Λ?

Secondly, is it possibly to cancel the condition (2.1) in Theorem 2.A and the condition (2.3) in Theorem 2.C?

Finally, we are interested whether the above results still hold if $\{\theta_i\colon i \geqslant 1\}$ is a sequence of identically distributed and WLOD r.v.s, which is wider and more practical than the independent and NLOD cases.

This paper will answer the above three problems directly.

2.3. Main results

Before presenting the main results of this paper, we first introduce a condition, which will be required only to deal with the uniformity for the time $t \in (\underline{t}, t_0)$, where $\underline{t} = \inf\{t\colon t \in \Lambda\}$ and t_0 is an arbitrary constant in Λ. Particularly, if $\underline{t} \in \Lambda$ then this condition is not needed.

Condition 2.1. For the α in (2.5), there exist $t_0 \in \Lambda \cap (0, \infty)$ and $f \in \mathscr{H}(B)$ such that

$$\big(f(x)\big)^{\alpha}\big(P(\theta_1 \leqslant t_0)\big)^{f(x)}\big(\overline{B}(x)\big)^{-1} \to 0 \quad \text{as } x \to \infty. \tag{2.4}$$

Here, we note that if $B \in \mathcal{C}$ then B satisfies Condition 2.1 naturally. In fact, since $B \in \mathcal{C}$, for any $0 < p < 1$, $f(x) = x^p \in \mathscr{H}(B)$, and there exists $q > 0$ such that $\lim_{x\to\infty} x^q \overline{B}(x) = \infty$. Thus for any $s > 0$, $\lim_{x\to\infty} e^{-sx^p}(\overline{B}(x))^{-1} = 0$ and (2.4) holds. In addition, many strong subexponential distributions, such as lognormal distribution and some heavy-tailed Weibull distributions, do not belong to $\mathcal{C}$, but satisfy the above condition.

Theorem 2.1. *Suppose that Assumptions* H_1, H_2^{**}, H_3 *and* (1.1) *hold. Assume that there exists a finite constant* $\alpha > 0$ *such that*

$$\lim_{n\to\infty} g_L(n) n^{-\alpha} = 0. \tag{2.5}$$

(1) *If $B \in \mathcal{S}_*$, then for any $t_0 \in \Lambda$,*

$$\limsup_{x\to\infty} \sup_{t\in[t_0,\infty)} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u\right)^{-1} \leqslant 1. \tag{2.6}$$

(2) *Furthermore, when $\underline{t}$ does not belong to Λ, assume that Condition* 2.1 *is satisfied. Then*

$$\limsup_{x\to\infty} \sup_{t\in\Lambda} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u\right)^{-1} \leqslant 1. \tag{2.7}$$

In order to obtain the lower limitation, we will require the inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ to satisfy one of the following conditions.

Condition 2.2. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are NLOD r.v.s.

Condition 2.3. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are WOD r.v.s and there exists a positive and nondecreasing function $g(x)$ such that $\lim_{x\to\infty} g(x) = \infty$, $x^{-k}g(x)\overline{\downarrow}$ for some $0 < k < 1$, $E\theta_1 g(\theta_1) < \infty$ and $\max\{g_L(n), g_U(n)\} \leqslant g(n)$ for all $n \geqslant 1$.

Here, $x^{-k}g(x)\overline{\downarrow}$ means that there exists a finite constant $C > 0$ such that

$$x_1^{-k}g(x_1) \geqslant Cx_2^{-k}g(x_2) \quad \text{for all } 0 \leqslant x_1 < x_2 < \infty.$$

Condition 2.4. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are WOD r.v.s with $E\theta_1^p < \infty$ for some $2 \leqslant p < \infty$ and there exists a constant $\alpha > 0$ such that

$$\lim_{n\to\infty} \max\{g_U(n), g_L(n)\}n^{-\alpha} = 0.$$

Condition 2.5. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ are WOD r.v.s with $Ee^{\beta\theta_1} < \infty$ for some $0 < \beta < \infty$ and for any $\gamma > 0$,

$$\lim_{n\to\infty} \max\{g_U(n), g_L(n)\}e^{-\gamma n} = 0.$$

Remark 2.1. It is noted that if the inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ satisfy one of Conditions 2.2–2.5 then the strong law of large numbers holds, i.e.

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^n \theta_i = \lambda^{-1}. \tag{2.8}$$

Indeed,

- for Condition 2.2, (2.8) can be obtained by using Theorem 1 of Matula [14];
- for Condition 2.3, (2.8) can be obtained by using Theorem 1.4 of Wang and Cheng [19];
- for Condition 2.4, (2.8) can be obtained by using the proof of Theorem 1.1 of Wang and Cheng [19];
- for Condition 2.5, (2.8) can be obtained by using the proof of Theorem 1.2 of Wang and Cheng [19].

From the proof of the following results, we find that we use Conditions 2.2–2.5 only to make the inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ satisfying the strong law of large numbers. Thus we can also use the following condition to replace Conditions 2.2–2.5.

Condition 2.6. The inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ satisfy (2.8) and (2.5).

Theorem 2.2. *Suppose that Assumptions H_1, H_2^{**}, H_3 and* (1.1) *hold. Assume that the inter-occurrence times $\{\theta_i\colon i \geqslant 1\}$ satisfy one of Conditions* 2.2–2.5.

(1) *If $B \in \mathcal{L}$, then for any $t_0 \in \Lambda$,*

$$\liminf_{x\to\infty} \inf_{t\in[t_0,\infty)} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u\right)^{-1} \geqslant 1. \tag{2.9}$$

(2) *Furthermore, if* $B \in \mathcal{S}_*$ *then*

$$\liminf_{x\to\infty} \inf_{t\in\Lambda} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u\right)^{-1} \geqslant 1. \tag{2.10}$$

From Theorems 2.1 and 2.2, the following two results can be obtained immediately.

Corollary 2.1. *Suppose that Assumptions* H_1, H_2^{**}, H_3 *and* (1.1) *hold. Assume that the inter-occurrence times* $\{\theta_i\colon i \geqslant 1\}$ *satisfy* (2.5) *and one of Conditions* 2.2–2.5.

(1) *If* $B \in \mathcal{S}_*$, *then for any* $t_0 \in \Lambda$, (2.2) *holds uniformly for all* $t \in (t_0, \infty)$.
(2) *Furthermore, if Condition* 2.1 *is satisfied, then* (2.2) *holds uniformly for all* $t \in \Lambda$.

Before giving the last result of the paper, we introduce another assumption.

Assumption H_4. Let τ be a nonnegative r.v., and independent of the sequences $\{Z_i\colon i \geqslant 1\}$ and $\{\theta_i\colon i \geqslant 1\}$.

Corollary 2.2. *Suppose that Assumptions* H_1, H_2^{**}, H_3, H_4 *and* (1.1) *hold. Assume that the inter-occurrence times* $\{\theta_i\colon i \geqslant 1\}$ *satisfy one of Conditions* 2.2–2.5.

(1) *If* $B \in \mathcal{S}_*$, *then for any* $t_0 \in \Lambda$, *it holds uniformly for all* τ *such that* $P(\tau \mathbf{1}_{\{\tau \geqslant t_0\}} \in \Lambda) > 0$,

$$\psi(x,\tau) \sim \mu^{-1} E \int_x^{x+\mu\lambda(\tau)} \overline{B}(u)\,\mathrm{d}u \quad \text{as } x \to \infty. \tag{2.11}$$

(2) *In addition, if Condition* 2.1 *is satisfied, then* (2.11) *holds uniformly for all* τ *such that* $P(\tau \in \Lambda) > 0$.

Remark 2.2. We note that we do not know, whether or not Condition 2.1 can be cancelled in Theorem 2.1(2), and also in Corollaries 2.1(2) and 2.2(2). Of course, Condition 2.1 is only applied in discussing the case of $t \in [\underline{t}, t_0]$, where $\underline{t} = \inf\{t\colon \lambda(t) > 0\}$. Because of the arbitrariness of $t_0 \in \Lambda$, the case of $t \in [\underline{t}, t_0]$ is hardly significant in practice. But we still hope that there will be a perfect result in mathematics.

3. Proofs of theorems

We prove Theorems 2.1 and 2.2 in Sections 3.1 and 3.2, respectively.

3.1. Proof of Theorem 2.1

(1) For any $0 < \epsilon < 1$, from Lemma 4.2 below, we know that there exists $t_1 \geqslant t_0$ such that for any $t \geqslant t_1$

$$\lambda(t) \geqslant 1 \quad \text{and} \quad \left(EN^2(t)\right)^{\frac{1}{2}} \leqslant (1+\epsilon)\lambda(t). \tag{3.1}$$

According to $B \in \mathcal{S}_* \subset \mathcal{L}$ and Proposition 1.2(1), we get $\mathscr{H}(B) \neq \emptyset$. Arbitrarily choosing $f \in \mathscr{H}(B)$, we divide $[t_0, \infty)$ by f and t_1 into three intervals $[t_0, t_1)$, $[t_1, f(x))$ and $[f(x), \infty)$.

Now we deal with the supremums on the three intervals, respectively.

1° Case of $t \in [t_1, f(x))$

By (1.1), we can choose sufficiently small $\delta \in (0,1)$, such that

$$-\mu_1 \equiv E\left(Z_1 - c\lambda^{-1}(1-\delta)\right) = b - c\lambda^{-1}(1-\delta) < 0.$$

Set $\xi_i = Z_i - c\lambda^{-1}(1-\delta)$, $i \geqslant 1$, and $\eta = c\max_{n\geqslant 0}\sum_{i=1}^n (\lambda^{-1}(1-\delta) - \theta_i)$. Taking a function $h \in \mathscr{H}(B)$, we get that

$$\begin{aligned}\psi(x,t) &\leqslant P\left(\max_{0\leqslant n\leqslant N(t)} \sum_{i=1}^n \xi_i + \eta > x\right)\\ &= \sum_{m=0}^\infty P\left(\max_{0\leqslant n\leqslant m} \sum_{i=1}^n \xi_i + \eta > x, N(t) = m\right)\end{aligned}$$

$$= \sum_{m=0}^{\infty}\left(\int_0^{h(x)} + \int_{h(x)}^{x-h(x)} + \int_{x-h(x)}^{x}\right) P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n}\xi_i > x-y\right) \mathrm{d}P\big(\eta \leqslant y, N(t)=m\big) + P(\eta > x)$$

$$\equiv \sum_{j=1}^{3}\psi_j(x,t) + \psi_4(x). \tag{3.2}$$

For $\psi_1(x,t)$, by $\mu_1 > 0$, $B \in \mathcal{S}_*$ and Lemma 4.3(1) below, for the above ϵ, there exists $x_1 > 0$ such that for any $x \geqslant x_1$, it holds uniformly for all $t \in \Lambda$ that

$$\begin{aligned}
\psi_1(x,t) &\leqslant \sum_{m=0}^{\infty} P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n}\xi_i > x-h(x)\right) P\big(N(t)=m\big) \\
&\leqslant (1+\epsilon)\mu_1^{-1}\sum_{m=0}^{\infty}\int_{x-h(x)}^{x-h(x)+\mu_1 m} \overline{B}(u)\,\mathrm{d}u\, P\big(N(t)=m\big) \\
&\leqslant (1+\epsilon)\lambda(t)\overline{B}\big(x-h(x)\big) \\
&\leqslant (1+\epsilon)^2\lambda(t)\overline{B}(x).
\end{aligned} \tag{3.3}$$

Next we treat $\psi_4(x)$. Denote $X_i = \lambda^{-1}(1-\delta) - \theta_i,\ i \geqslant 1$, then they satisfy the conditions of Lemma 4.4 below. So using Lemma 4.4 and (3.1), for the above ϵ, there exists $x_2 \geqslant x_1$ such that for any $x \geqslant x_2$, we have

$$\psi_4(x) \leqslant \epsilon\overline{B}(x)\lambda(t) \tag{3.4}$$

holds uniformly for all $t \in [t_1, \infty)$.

For $\psi_2(x,t)$, by Lemma 4.3(1), $B \in \mathcal{S}_* \subset \mathcal{L}$ and integration by parts, for the above ϵ, there exists $x_3 \geqslant x_2$ such that for any $x \geqslant x_3$, it holds uniformly for all $t \in \Lambda$ that

$$\begin{aligned}
\psi_2(x,t) &\leqslant (1+\epsilon)\sum_{m=0}^{\infty}\int_{h(x)}^{x-h(x)}\mu_1^{-1}\int_{x-y}^{x-y+\mu_1 m}\overline{B}(u)\,\mathrm{d}u\,\mathrm{d}P\big(\eta \leqslant y, N(t)=m\big) \\
&\leqslant (1+\epsilon)\sum_{m=0}^{\infty} m\int_{h(x)}^{x-h(x)}\overline{B}(x-y)\,\mathrm{d}P\big(\eta \leqslant y, N(t)=m\big) \\
&= -(1+\epsilon)\sum_{m=0}^{\infty} m\int_{h(x)}^{x-h(x)}\overline{B}(x-y)\,\mathrm{d}P\big(\eta > y, N(t)=m\big) \\
&= (1+\epsilon)\sum_{m=0}^{\infty} m\Bigg(\overline{B}\big(x-h(x)\big)P\big(\eta > h(x), N(t)=m\big) - \overline{B}\big(h(x)\big)P\big(\eta > x-h(x), N(t)=m\big) \\
&\quad + \int_{h(x)}^{x-h(x)} P\big(\eta > x-y, N(t)=m\big)\,\mathrm{d}B(y)\Bigg) \\
&\equiv \sum_{j=1}^{3}\psi_{2j}(x,t).
\end{aligned} \tag{3.5}$$

We now deal with $\psi_{21}(x,t)$ and $\psi_{23}(x,t)$, respectively, but don't handle $\psi_{22}(x,t)$, which will be estimated combining with $\psi_3(x,t)$. Using $B \in \mathcal{S}_* \subset \mathcal{L}$, Hölder inequality, (3.1) and Lemma 4.4, for the above ϵ, there exists $x_4 \geqslant x_3$ such that for any $x \geqslant x_4$, it holds uniformly for all $t \in [t_1, \infty)$ that

$$\begin{aligned}
\psi_{21}(x,t) &\leqslant (1+\epsilon)^2\overline{B}(x)EN(t)\mathbf{1}_{\{\eta > h(x)\}} \\
&\leqslant (1+\epsilon)^2\overline{B}(x)\big(EN^2(t)\big)^{\frac{1}{2}}\big(P\big(\eta > h(x)\big)\big)^{\frac{1}{2}} \\
&\leqslant (1+\epsilon)^3\overline{B}(x)\lambda(t)\big(P\big(\eta > h(x)\big)\big)^{\frac{1}{2}} \\
&\leqslant \epsilon\overline{B}(x)\lambda(t).
\end{aligned} \tag{3.6}$$

Again using Hölder inequality, by Lemma 4.4, (3.1) and Proposition 1.2(4), for the above ϵ, there exists $x_5 \geqslant x_4$ such that for any $x \geqslant x_5$, we have

$$\begin{aligned}\psi_{23}(x,t) &= (1+\epsilon)\int_{h(x)}^{x-h(x)} EN(t)\mathbf{1}_{\{\eta>x-y\}}\,\mathrm{d}B(y)\\ &\leqslant (1+\epsilon)\big(EN^2(t)\big)^{\frac{1}{2}}\int_{h(x)}^{x-h(x)}\big(P(\eta>x-y)\big)^{\frac{1}{2}}\,\mathrm{d}B(y)\\ &\leqslant \epsilon(1+\epsilon)^2\lambda(t)\int_{h(x)}^{x-h(x)}\overline{B}(x-y)\,\mathrm{d}B(y)\\ &\leqslant \epsilon\lambda(t)\overline{B}(x) \end{aligned} \tag{3.7}$$

holds uniformly for all $t \in [t_1, \infty)$.

For $\psi_3(x,t)$, also using the integration by parts, we obtain that

$$\begin{aligned}\psi_3(x,t) &\leqslant \sum_{m=0}^{\infty}\Bigg(P\bigg(\sup_{0\leqslant n\leqslant m}\sum_{i=1}^{n}\xi_i > h(x)\bigg)P\big(\eta > x-h(x), N(t)=m\big)\\ &\quad + \int_0^{h(x)} P\big(\eta > x-y, N(t)=m\big)\,\mathrm{d}P\bigg(\sup_{0\leqslant n\leqslant m}\sum_{i=1}^{n}\xi_i \leqslant y\bigg)\Bigg)\\ &\equiv \psi_{31}(x,t)+\psi_{32}(x,t). \end{aligned} \tag{3.8}$$

For the above ϵ, again by Lemma 4.3(1) and $B \in \mathcal{S}_* \subset \mathcal{L}$, we know that there exists $x_6 \geqslant x_5$ such that for any $x \geqslant x_6$,

$$\begin{aligned}\psi_{31}(x,t) &\leqslant (1+\epsilon)\sum_{m=0}^{\infty}\mu_1^{-1}\int_{h(x)}^{h(x)+\mu_1 m}\overline{B}(u)\,\mathrm{d}u\; P\big(\eta > x-h(x), N(t)=m\big)\\ &\leqslant -\psi_{22}(x,t)\end{aligned}$$

holds uniformly for all $t \in \Lambda$. Thus it holds uniformly for all $t \in \Lambda$ that

$$\psi_{31}(x,t)+\psi_{22}(x,t) \leqslant 0. \tag{3.9}$$

Since $h \in \mathscr{H}(B)$, by Lemma 4.4 and (3.1), for the above ϵ, there exists $x_7 \geqslant x_6$ such that for any $x \geqslant x_7$,

$$\begin{aligned}\psi_{32}(x,t) &\leqslant \sum_{m=0}^{\infty} P\big(\eta > x-h(x), N(t)=m\big)\\ &= P\big(\eta > x-h(x)\big)\\ &\leqslant \epsilon\overline{B}(x)\lambda(t)\end{aligned} \tag{3.10}$$

holds uniformly for all $t \in [t_1, \infty)$.

According to (3.2)–(3.10), we know that, for the above ϵ, when $x \geqslant x_7$,

$$\sup_{t\in[t_1,\infty)}\psi(x,t)\big(\overline{B}(x)\lambda(t)\big)^{-1} \leqslant (1+\epsilon)^2+4\epsilon.$$

Since the arbitrariness of ϵ and by Lemma 4.5 below, we can get that

$$\limsup_{x\to\infty}\sup_{t\in[t_1,f(x))}\psi(x,t)\bigg(\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\bigg)^{-1} \leqslant 1. \tag{3.11}$$

2° Case of $t \in [t_0, t_1)$

From the proof of 1°, we know that, for the above $0<\varepsilon<1$, when $x \geqslant x_3$, (3.3) and (3.5) hold uniformly for all $t \in \Lambda$. Thus, it is sufficient to prove (3.6), (3.7), (3.10) and (3.4) hold uniformly for all $t \in [t_0, t_1)$. We first prove (3.6). From the proof of 1°, we know that, for the above $0<\varepsilon<1$, there exists $x_4 \geqslant x_3$ such that for any $x \geqslant x_4$,

$$\begin{aligned}\psi_{21}(x,t) &\leqslant (1+\epsilon)^2\big(EN^2(t)\big)^{\frac{1}{2}}\big(\lambda(t)\big)^{-1}\big(P\big(\eta > h(x)\big)\big)^{\frac{1}{2}}\overline{B}(x)\lambda(t)\\ &\leqslant (1+\epsilon)^2\big(EN^2(t_1)\big)^{\frac{1}{2}}\big(\lambda(t_0)\big)^{-1}\big(P\big(\eta > h(x)\big)\big)^{\frac{1}{2}}\overline{B}(x)\lambda(t)\\ &\leqslant \epsilon\overline{B}(x)\lambda(t)\end{aligned}$$

holds uniformly for all $t \in [t_0, t_1)$. Similar to the proof above, (3.7), (3.10) and (3.4) also hold uniformly for all $t \in [t_0, t_1)$. Hence, we have

$$\sup_{t\in[t_0,t_1)} \psi(x,t)\big(\overline{B}(x)\lambda(t)\big)^{-1} \leqslant (1+\epsilon)^2 + 4\epsilon.$$

By the arbitrariness of ϵ and Lemma 4.5, we obtain

$$\limsup_{x\to\infty}\sup_{t\in[t_0,t_1)} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\right)^{-1} \leqslant 1. \tag{3.12}$$

3° Case of $t \in [f(x),\infty)$

Applying Lemmas 4.4 and 4.6(1), along the line of the proof of Proposition 2.1 of Leipus and Šiaulys [12], we have

$$\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\right)^{-1} \leqslant 1. \tag{3.13}$$

By (3.11)–(3.13), we obtain (2.6) immediately.

(2) In order to prove (2.7), using (2.6), it is sufficient to prove that for some t_0 in Condition 2.1,

$$\limsup_{x\to\infty}\sup_{t\in(\underline{t},t_0)} \psi(x,t)\left(\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\right)^{-1} \leqslant 1. \tag{3.14}$$

We will use the line of the proof of Proposition 4.3 of Tang [16]. Let $d = c\lambda^{-1}$, using the function f in Condition 2.1, for any $t \in (\underline{t}, t_0)$ we derive

$$\begin{aligned}\psi(x,t) &\leqslant P\left(\sum_{i=1}^{N(t)}(Z_i - d) > x - dN(t)\right)\\ &\leqslant P\left(\max_{1\leqslant n\leqslant N(t)}\sum_{i=1}^{n}(Z_i-d) > x - df(x)\right) + P\big(N(t) > f(x)\big)\\ &\equiv \psi_5(x,t) + \psi_6(x,t).\end{aligned} \tag{3.15}$$

By Lemma 4.3(1), Proposition 1.2(3) and Lemma 4.5, for any $0 < \epsilon < 1$, there exists $x_8 > 0$ such that for any $x \geqslant x_8$,

$$\begin{aligned}\psi_5(x,t) &= \sum_{m=0}^{\infty} P\left(\max_{1\leqslant n\leqslant m}\sum_{i=1}^{n}(Z_i-d) > x - df(x)\right)P\big(N(t)=m\big)\\ &\leqslant (1+\epsilon)\mu^{-1}\sum_{m=0}^{\infty}\int_{x-df(x)}^{x-df(x)+\mu m}\overline{B}(u)\,\mathrm{d}u\,P\big(N(t)=m\big)\\ &\leqslant (1+\epsilon)\overline{B}\big(x-df(x)\big)\lambda(t)\\ &\leqslant (1+\epsilon)^2\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\end{aligned} \tag{3.16}$$

holds uniformly for $t \in (\underline{t}, t_0)$.

Now we deal with $\psi_6(x,t)$. Let $[x]$ be the integer part of x. Applying Markov inequality and the definition of WLOD r.v.s, we have

$$\begin{aligned}\psi_6(x,t) &\leqslant \big(f(x)\big)^{-1} EN(t)\mathbf{1}_{N(t)>f(x)}\\ &\leqslant \big(f(x)\big)^{-1}\sum_{n=[f(x)]}^{\infty} nP\left(\sum_{i=1}^{n}\theta_i\leqslant t\right)\\ &\leqslant \big(f(x)\big)^{-1}\sum_{n=[f(x)]}^{\infty} nP(\theta_1\leqslant t,\dots,\theta_n\leqslant t)\\ &\leqslant \big(f(x)\big)^{-1}\lambda(t)\sum_{n=[f(x)]}^{\infty} ng_L(n)\big(P(\theta_1\leqslant t)\big)^{n-1}.\end{aligned}$$

By (2.5), for a sufficiently small t_0, there exists a positive constant C such that when x is sufficiently large,

$$\begin{aligned}\psi_6(x,t) &\leqslant \big(f(x)\big)^{-1}\lambda(t)\sum_{n=[f(x)]}^{\infty} n^{1+\beta}\big(P(\theta_1\leqslant t)\big)^{n-1}\\ &\leqslant C\lambda(t)\big(f(x)\big)^{\beta}\big(P(\theta_1\leqslant t)\big)^{[f(x)]-1}.\end{aligned}$$

By Condition 2.1 and Lemma 4.5, for the above ϵ, there exists $x_9\geqslant x_8$, when $x\geqslant x_9$, it is holds uniformly for all $t\in(\underline{t},t_0)$ that,

$$\psi_6(x,t)\leqslant \epsilon\lambda(t)\overline{B}(x)\leqslant \epsilon(1+\epsilon)\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u. \tag{3.17}$$

It immediately follows from (3.16), (3.17) and the arbitrariness of ϵ that (3.14) holds. Thus we get that (2.7) holds. This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2

(1) We prove this result for two cases $t\in[t_0,f(x))$ and $t\in[f(x),\infty)$, where $f\in\mathscr{H}(B)$.

1° Case of $t\in[f(x),\infty)$

It follows from (2.8) and Anscombe's Theorem (see, e.g. Lemma 2.5.3 of Embrechts et al. [6]) that

$$N(t)\sim\lambda t\quad \text{as } t\to\infty \text{ a.s.} \tag{3.18}$$

Thus for an arbitrary constant $0<\kappa<1$, we have

$$\lim_{t\to\infty} P\big(N(t)\geqslant(1-\kappa)\lambda t\big)=1. \tag{3.19}$$

For any a $0<\delta<1$ and $l\geqslant 1$, write $\xi_i\equiv Z_i-c\lambda^{-1}(1+\delta)$, $\eta_i\equiv\lambda^{-1}(1+\delta)-\theta_i$, $i\geqslant 1$ and $-\mu_2\equiv E\xi_1=b-c\lambda^{-1}(1+\delta)$. According to (1.1), we have $\mu_2>\mu>0$. By $B\in\mathcal{L}$ and Lemma 4.3(2), we get for any $0<\epsilon<1$, there exists $x_1>0$ such that for any $x\geqslant x_1$,

$$\begin{aligned}\psi(x,t) &\geqslant \sum_{m=0}^{\infty}P\left(\inf_{n\geqslant 1}\sum_{i=1}^{n}\eta_i>-l,\ N(t)=m\right)P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n}\xi_i>x+l\right)\\ &\geqslant (1-\epsilon)\sum_{m=0}^{\infty}P\left(\inf_{n\geqslant 1}\sum_{i=1}^{n}\eta_i>-l,\ N(t)=m\right)\mu_2^{-1}\int_x^{x+\mu_2 m}\overline{B}(u)\,\mathrm{d}u\\ &\geqslant (1-\epsilon)\sum_{m\geqslant(1-\kappa)\lambda t}P\left(\inf_{n\geqslant 1}\sum_{i=1}^{n}\eta_i>-l, N(t)=m\right)\mu_2^{-1}\int_x^{x+\mu m}\overline{B}(u)\,\mathrm{d}u\\ &\geqslant (1-\epsilon)\mu_2^{-1}P\left(\inf_{n\geqslant 1}\sum_{i=1}^{n}\eta_i>-l,\ N(t)\geqslant(1-\kappa)\lambda t\right)\int_x^{x+\mu(1-\kappa)\lambda t}\overline{B}(u)\,\mathrm{d}u\end{aligned}$$

holds uniformly for $t\in[f(x),\infty)$.

Since $E\eta_1 = \delta\lambda^{-1} > 0$ and by (2.8), we get

$$\lim_{l\to\infty} P\left(\inf_{n\geqslant 1}\sum_{i=1}^{n}\eta_i > -l\right) = \lim_{l\to\infty} P\left(\sup_{n\geqslant 1}\sum_{i=1}^{n}(-\eta_i) < l\right) = 1. \tag{3.20}$$

According to (3.19) and (3.20), for the above ϵ and any $0 < \kappa < 1$, there exists $x_2 \geqslant x_1$ such that for any $x \geqslant x_2$ and any $l \geqslant x_2$,

$$\psi(x,t) \geqslant (1-\epsilon)^2\mu_2^{-1}\int_x^{x+\mu(1-\kappa)\lambda t}\overline{B}(u)\,\mathrm{d}u$$

holds uniformly for $t \in [f(x),\infty)$.

By (4.8) of Tang [16] or (28) of Leipus and Šiaulys [12], we obtain that for the above ϵ and $0<\beta<1$, there exists $x_3 \geqslant x_2$ such that for $x \geqslant x_3$,

$$\psi(x,t) \geqslant (1-\epsilon)^3\mu_2^{-1}(1-2\kappa)(1-\kappa)^{-1}\int_x^{x+\mu\lambda t}\overline{B}(u)\,\mathrm{d}u \tag{3.21}$$

holds uniformly for $t \in [f(x),\infty)$.

2° Case of $t \in [t_0, f(x))$

For any $\gamma > \lambda$, the above $0<\epsilon<1$ and $t_1 \geqslant t_0$ in (3.1), by Hölder inequality, (3.1) and Lemma 4.2, there exists $x_4 \geqslant x_3$ such that for any $x \geqslant x_4$, it holds uniformly for $t \in [t_1, f(x))$ that

$$\begin{aligned} EN(t)\mathbf{1}_{\{N(t)>\gamma f(x)\}} &\leqslant \big(EN^2(t)\big)^{\frac12}\big(P\big(N(t)>\gamma f(x)\big)\big)^{\frac12} \\ &\leqslant (1+\epsilon)\lambda(t)\big(P\big(N\big(f(x)\big)>\gamma f(x)\big)\big)^{\frac12} \\ &\leqslant \epsilon\lambda(t). \end{aligned}$$

Again by Hölder inequality, for the above γ and ϵ, there exists $x_5 \geqslant x_4$ such that for any $x \geqslant x_5$, it holds uniformly for $t \in [t_0, t_1)$ that

$$\begin{aligned} EN(t)\mathbf{1}_{\{N(t)>\gamma f(x)\}} &\leqslant \big(EN^2(t)\big)^{\frac12}\big(P\big(N(t)>\gamma f(x)\big)\big)^{\frac12} \\ &\leqslant \big(EN^2(t_1)\big)^{\frac12}\big(P\big(N(t_1)>\gamma f(x)\big)\big)^{\frac12}\lambda(t)\big(\lambda(t_0)\big)^{-1} \\ &\leqslant \epsilon\lambda(t). \end{aligned}$$

Hence, for the above γ and ϵ, for any $x \geqslant x_5$, it holds uniformly for $t \in [t_0, f(x))$ that

$$EN(t)\mathbf{1}_{\{N(t)>\gamma f(x)\}} \leqslant \epsilon\lambda(t). \tag{3.22}$$

Let $d = c\lambda^{-1}$, by $B \in \mathcal{L}$, Lemma 4.3(2), $f \in \mathscr{H}(B)$, Proposition 1.2(3), (3.22) and Lemma 4.5, for the above γ and ϵ, there exists $x_6 \geqslant x_5$ such that for any $x \geqslant x_6$, it holds uniformly for $t \in [t_0, f(x))$ that

$$\begin{aligned} \psi(x,t) &\geqslant P\left(\sum_{i=1}^{N(t)} Z_i - cf(x) > x\right) \\ &\geqslant P\left(\max_{0\leqslant n\leqslant N(t)}\sum_{i=1}^{n}(Z_i-d) > x+cf(x)\right) \\ &= \sum_{m=0}^{\infty} P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n}(Z_i-d) > x+cf(x)\right)P\big(N(t)=m\big) \\ &\geqslant (1-\epsilon)\mu^{-1}\sum_{m=0}^{\infty}\int_{x+cf(x)}^{x+cf(x)+\mu m}\overline{B}(u)\,\mathrm{d}u\,P\big(N(t)=m\big) \\ &\geqslant (1-\epsilon)\sum_{m=0}^{[\gamma f(x)]}\overline{B}\big(x+(\gamma+c)f(x)\big)mP\big(N(t)=m\big) \end{aligned}$$

$$\begin{aligned}
&\geqslant (1-\epsilon)^2\overline{B}(x)EN(t)\mathbf{1}_{\{N(t)\leqslant \gamma f(x)\}}\\
&\geqslant (1-\epsilon)^3\overline{B}(x)\lambda(t)\\
&\geqslant (1-\epsilon)^3\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u.
\end{aligned}\tag{3.23}$$

Combining (3.21) and (3.23) and taking into account the arbitrariness of ϵ and κ, we obtain (2.9) immediately.

(2) Since $B\in\mathcal{S}_*\subset\mathcal{L}$, (3.21) still holds uniformly for $t\in[f(x),\infty)$. Now we deal with the case that $t\in(\underline{t}, f(x))$. Let $d=c\lambda^{-1}$, by Lemma 4.6(2), (3.18), Lemma 4.3(3) and Proposition 1.2(3), for any $0<\epsilon<1$, there exists $x_7\geqslant x_6$ such that for any $x\geqslant x_7$, it holds uniformly that for $t\in(\underline{t}, f(x))$,

$$\begin{aligned}
\psi(x,t)&\geqslant P\left(\sum_{i=1}^{N(t)}Z_i-cf(x)>x\right)\\
&\geqslant P\left(\max_{1\leqslant n\leqslant N(t)}\sum_{i=1}^{n}(Z_i-d)>x+cf(x)\right)\\
&\geqslant (1-\epsilon)\mu^{-1}\int_{x+cf(x)}^{x+cf(x)+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u\\
&\geqslant (1-\epsilon)^2\mu^{-1}\int_x^{x+\mu\lambda(t)}\overline{B}(u)\,\mathrm{d}u,
\end{aligned}$$

which combining with (3.21) and by the arbitrariness of ϵ, we get that (2.10) holds.

This completes the proof of Theorem 2.2.

4. Some auxiliary results

In this section, we give some auxiliary results, which not only have been used to prove the main results of this paper, but also have their independent significance on their own rights. The first lemma is Lemma 3.2 and Corollary 3.1 of Wang and Cheng [19], which will be used in the proof of Lemma 4.2.

Lemma 4.1. *Let X_i, $i\geqslant 1$, be WLOD r.v.s with common distribution function F on $(-\infty,\infty)$ and positive finite mean μ^{-1}. Set $X_1^-=-X_1\mathbf{1}_{\{X_1\leqslant 0\}}$. Assume that $E(X_1^-)^p<\infty$ for some $p\geqslant 2$ and (2.5) holds for some finite constant $\alpha>0$.*

(1) *Then*

$$A(p,t)\equiv\sum_{n=1}^{\infty}n^{p-2}P\left(\sum_{i=1}^{n}X_i\leqslant t\right)\sim (p-1)^{-1}(\mu t)^{p-1}.$$

(2) *If $p>2$, then for any $2\leqslant p'<p$,*

$$A(p',t)=o\big(A(p,t)\big).$$

(3) *If $a_n\sim dn^{p-2}$ as $n\to\infty$, where d is a positive finite constant, then*

$$\begin{aligned}
\sum_{n=1}^{\infty}a_nP\left(\sum_{i=1}^{n}X_i\leqslant t\right)&\sim dA(p,t)\\
&\sim d(p-1)^{-1}(\mu t)^{p-1}.
\end{aligned}$$

Lemma 4.2. *Suppose that the inter-occurrence times $\{\theta_i\colon i\geqslant 1\}$ are WLOD r.v.s and satisfy (2.5). Then for any positive integer r,*

$$E\big(N(t)\big)^r\sim(\lambda t)^r\quad\text{as } t\to\infty.\tag{4.1}$$

Proof. Since $\theta_i, i \geqslant 1$, are nonnegative r.v.s, by (3.4) of Wang and Cheng [19], for any positive integer r,

$$E\big(N(t)\big)^r = E\left(N(t) + 2!\sum_{n_1=2}^{\infty}\binom{n_1-1}{1}\mathbf{1}_{\{\sum_{i=1}^{n_1}\theta_i\leqslant t\}} + \cdots + r!\sum_{n_1=r}^{\infty}\binom{n_1-1}{r-1}\mathbf{1}_{\{\sum_{i=1}^{n_1}\theta_i\leqslant t\}}\right). \tag{4.2}$$

Thus, by (4.2), (2.5) and Lemma 4.1, we have

$$\begin{aligned} E\big(N(t)\big)^r &\sim r\sum_{n=r}^{\infty} n^{(r+1)-2} P\left(\sum_{i=1}^{n}\theta_i \leqslant t\right) \\ &\sim (\lambda t)^r \quad \text{as } t\to\infty. \qquad \square \end{aligned}$$

Lemma 4.3. *Let $\{X_i\colon i\geqslant 1\}$ be i.i.d. r.v.s with common distribution function F and finite mean $EX_1 < 0$.*

(1) (*Theorem of Korshunov* [10]) *If $F\in\mathcal{S}_*$ then it holds uniformly for all $m\geqslant 1$ that*

$$P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n} X_i > x\right) \sim |EX_1|^{-1}\int_x^{x+|EX_1|m}\overline{F}(y)\,\mathrm{d}y \quad \text{as } x\to\infty.$$

(2) (*Lemma* 1 *of Korshunov* [10]) *If $F\in\mathcal{L}$ then*

$$\lim_{x\to\infty}\inf_{m\geqslant 1} P\left(\max_{0\leqslant n\leqslant m}\sum_{i=1}^{n} X_i > x\right)\left(|EX_1|^{-1}\int_x^{x+|EX_1|m}\overline{F}(y)\,\mathrm{d}y\right)^{-1} \geqslant 1.$$

(3) (*Theorem* 5.1 *of Tang* [17]) *Assume that $F\in\mathcal{S}_*$. In addition, suppose that $\{N(t)\colon t\geqslant 0\}$ is a counting process, which is independent of $\{X_i\colon i\geqslant 1\}$ and satisfied $\lambda(t) = EN(t) < \infty$ for any $t\geqslant 0$. If $N(t)(\lambda(t))^{-1} \xrightarrow{P} 1$ as $t\to\infty$ and*

$$\lim_{x\to\infty}\sup_{t\in\Lambda\cap(0,1]}\big(\lambda(t)\big)^{-1} EN(t)\mathbf{1}_{\{N(t)>x\}} = 0, \tag{4.3}$$

then it holds uniformly for $t\in\Lambda$ that

$$P\left(\max_{0\leqslant n\leqslant N(t)}\sum_{i=1}^{n} X_i > x\right) \sim |EX_1|^{-1}\int_x^{x+|EX_1|\lambda(t)}\overline{F}(y)\,\mathrm{d}y \quad \text{as } x\to\infty.$$

In order to deal with the WLOD inter-occurrence times, the following lemma will extend Lemma 3.3 of Leipus and Šiaulys [11] by applying the probability method. The proof of Leipus and Šiaulys [11] is based on Sgibnev [15], which used Banach algebra method.

Lemma 4.4. *Let $\{X_i\colon i\geqslant 1\}$ be WUOD r.v.s with common distribution function F and finite mean $EX_1 < 0$. Assume that $\{X_i\colon i\geqslant 1\}$ satisfy for any $\epsilon > 0$,*

$$\lim_{n\to\infty} g_U(n)e^{-\epsilon n} = 0 \tag{4.4}$$

and $\rho_0 = \sup\{\rho\geqslant 0\colon q(\rho)\equiv\int_{-\infty}^{\infty} e^{\rho y}\,\mathrm{d}F(y) < \infty\}\in(0,\infty)$. Then for any $\rho\in(0,\min\{\rho_0,\rho_1\})$, we have

$$\lim_{x\to\infty} e^{\rho x} P\left(\sup_{n\geqslant 0}\sum_{i=1}^{n} X_i > x\right) = 0, \tag{4.5}$$

where ρ_1 is the Lundberg index of F, that is, if there exists $0<\rho<\infty$ such that $q(\rho)=1$, then set $\rho_1=\rho$; otherwise define $\rho_1=\infty$.

Proof. According to Proposition 1.1(2), we have

$$\begin{aligned} P\left(\sup_{n\geqslant 0}\sum_{i=1}^{n} X_i > x\right) &\leqslant \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i > x\right) \\ &\leqslant e^{-\rho x}\sum_{n=1}^{\infty} g_U(n)\big(q(\rho)\big)^n. \end{aligned} \tag{4.6}$$

For $0 < \rho < \rho_0$, $q(\rho)$ have derivatives of any order and

$$q'(0) = EX_1 < 0, \qquad q''(\rho) = EX_1^2 \exp\{\rho X_1\} > 0,$$

implying that q is convex. If $\rho_1 = \infty$, then for $\rho \in (0, \rho_0)$, $q(\rho) < 1$; if $\rho_1 < \infty$, then for $\rho \in (0, \rho_1)$, $q(\rho) < 1$ still holds. Hence, we can take some $\epsilon > 0$ such that $e^{\epsilon} q(\rho) < 1$. By using (4.6) and (4.4), we know that, for any $\rho \in (0, \min\{\rho_0, \rho_1\})$, there exist $0 < \rho < \rho' < \min\{\rho_0, \rho_1\}$ and $N > 0$ such that

$$\lim_{x\to\infty} e^{\rho x} P\left(\sup_{n\geqslant 0} \sum_{i=1}^{n} X_i > x\right) \leqslant \lim_{x\to\infty} e^{-(\rho'-\rho)x}\left(\sum_{n=1}^{N} g_U(n)\big(q(\rho')\big)^n + \sum_{n=N+1}^{\infty} \big(e^{\epsilon} q(\rho')\big)^n\right) = 0. \qquad \square$$

Lemma 4.5. *Suppose that Assumptions H_1, H_2^{**} and H_3 hold and* (2.5) *is satisfied. If $B \in \mathcal{L}$ and $f \in \mathscr{H}(B)$, then it holds uniformly for $t \in (\underline{t}, f(x))$ that*

$$\mu^{-1} \int_{x}^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u \sim \overline{B}(x)\lambda(t) \quad \text{as } x \to \infty. \tag{4.7}$$

Proof. Clearly, for all $t \in \Lambda$, it holds uniformly that

$$\overline{B}\big(x + \mu\lambda(t)\big)\lambda(t) \leqslant \mu^{-1} \int_{x}^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u \leqslant \overline{B}(x)\lambda(t).$$

For any $0 < \epsilon < 1$, according to $f \in \mathscr{H}(B)$, Lemma 4.2 and Proposition 1.2(3), there exists $x_1 > 0$ such that for any $x \geqslant x_1$, it holds uniformly for $t \in (\underline{t}, f(x))$ that

$$\begin{aligned}\overline{B}\big(x + \mu\lambda(t)\big) &\geqslant \overline{B}\big(x + \mu\lambda\big(f(x)\big)\big)\\ &\geqslant \overline{B}\big(x + (1+\epsilon)\mu\lambda f(x)\big)\\ &\geqslant (1-\epsilon)\overline{B}(x).\end{aligned}$$

So, (4.7) holds uniformly for all $t \in (\underline{t}, f(x))$. $\square$

The following Lemma 4.6(1) and (2) can be easily obtained by Lemma 2.2 of Wang and Cheng [19] and Lemma 2.1 of Wang et al. [18], respectively. The result of (4.9) for the i.i.d. case is due to Theorem 1 of Kočetova et al. [9].

Lemma 4.6. *Suppose that $\{\theta_i\colon\ i \geqslant 1\}$ is a sequence of nonnegative WLOD r.v.s with finite mean $\lambda^{-1} > 0$ and satisfying for any $\epsilon > 0$*

$$\lim_{n\to\infty} g_L(n) e^{-\epsilon n} = 0. \tag{4.8}$$

$\{N(t)\colon\ t \geqslant 0\}$ is a renewal counting process generated by $\{\theta_i\colon\ i \geqslant 1\}$. Then

(1) *for every real number $a > \lambda$, there exists $\delta > 1$ such that*

$$\lim_{t\to\infty} \sum_{k>at} P\big(N(t) \geqslant k\big)\delta^k = 0. \tag{4.9}$$

(2) *$\{N(t)\colon\ t \geqslant 0\}$ satisfies* (4.3).

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