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On arithmetic partitions of $\mathbb{Z}_n$

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ABSTRACT

Generalizing a classical problem in enumerative combinatorics, Mansour and Sun counted the number of subsets of $\mathbb{Z}_n$ without certain separations. Chen, Wang, and Zhang then studied the problem of partitioning $\mathbb{Z}_n$ into arithmetical progressions of a given type under some technical conditions. In this paper, we improve on their main theorems by applying a convolution formula for cyclic multinomial coefficients due to Raney–Mohanty.

1. Introduction

In his solution of *problème des ménages* Kaplansky [1] showed that the number of ways of selecting k elements, no two consecutive, from n objects arrayed on a cycle is $\frac{n}{n-k}\binom{n-k}{k}$. Let $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ be the set of congruence classes modulo n with usual arithmetic. Then Yamamoto [2] (see also [3, p. 222]) proved that if $n \geqslant pk+1$ the number of ways of selecting k elements from $\mathbb{Z}_n$, no two consecutive, is

$$\frac{n}{n-pk}\binom{n-pk}{k}, \tag{1.1}$$

when $i \pm 1, \ldots, i \pm p$ are regarded as consecutive to i.

In the last three decades a lot of generalizations and variations of Kaplansky's problem have been studied by several authors (see, for example, [4–13]). In particular, Konvalina [4] considered the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that $x_i - x_j \neq 2$ for all $1 \leqslant i, j \leqslant k$, and found that the answer is $\frac{n}{n-k}\binom{n-k}{k}$ if $n \geqslant 2k+1$. Hwang [7] then generalized Konvalina's result to the case $x_i - x_j \neq m$

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and deduced that the desired number is given by the same formula if $n \geqslant mk + 1$. Recently, Mansour and Sun [14] gave the following unification of Yamamoto's and Hwang's formulas.

Theorem 1.1 (*Mansour–Sun*). *Let m, n, p, k be positive integers such that $n \geqslant mpk + 1$. Then the number of k-subsets $\{x_1, x_2, \dots, x_k\}$ of $\mathbb{Z}_n$ such that*

$$x_i - x_j \notin \{m, 2m, \dots, pm\} \quad (1 \leqslant i, j \leqslant k), \tag{1.2}$$

is also given by (1.1).

A short proof of Theorem 1.1 was given by Guo [15] by using Rothe's identity. In order to generalize Mansour–Sun's result, Chen, Wang, and Zhang [16] defined an m-AP-block of length k to be a sequence $(x_1, x_2, \dots, x_k)$ of distinct elements in $\mathbb{Z}_n$ such that $x_{i+1} - x_i = m$ for $1 \leqslant i \leqslant k - 1$ and studied the problem of partitioning $\mathbb{Z}_n$ into m-AP-blocks. The type of such a partition is defined to be the type of the multiset of the lengths of the blocks. For example, the following is a 3-AP-partition of $\mathbb{Z}_{20}$ of type $1^4 2^3 3^2 4^1$:

$$(2),\ (4, 7),\ (5, 8),\ (6),\ (9, 12, 15),\ (10),\ (11),\ (13, 16, 19),\ (14, 17, 0, 3),\ (18, 1).$$

We need to emphasize that $(x, x + m, \dots, x + (n - 1)m)$ and $(x + m, x + 2m, \dots, x + (n - 1)m, x)$ and so on are deemed as different m-AP-blocks in $\mathbb{Z}_{mn}$. For example, all the 2-AP-partitions of $\mathbb{Z}_6$ of type 3^2 are

$$\{(i, i + 2, i + 4), (j + 1, j + 3, j + 5)\}_{i,j=0,2,4}.$$

Chen, Wang, and Zhang [16] constructed a bijection between m-AP-partitions and m'-AP-partitions of $\mathbb{Z}_n$ under some technical conditions, and established the following theorem.

Theorem 1.2 (*Chen–Wang–Zhang*). *Let $m, n, k_1, k_2, \dots, k_r$ and $i_2, \dots, i_r$ be positive integers such that $1 < i_2 < \dots < i_r$ and*

$$k_1 > (k_2 + \dots + k_r)\left((m - 1)(i_r - 1) - 1\right). \tag{1.3}$$

Then the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of type $1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$ does not depend on m, and is given by the cyclic multinomial coefficient

$$\frac{n}{k_1 + \dots + k_r}\binom{k_1 + \dots + k_r}{k_1, \dots, k_r}. \tag{1.4}$$

If we specialize the type to $1^{n-(p+1)k}(p + 1)^k$, then the condition (1.3) becomes $n \geqslant mpk + 1$. Furthermore, if $(x_1, x_1 + m, \dots, x_1 + pm), \dots, (x_k, x_k + m, \dots, x_k + pm)$ are the k blocks of length $p + 1$ in an m-AP-partition of $\mathbb{Z}_n$ of type $1^{n-(p+1)k}(p + 1)^k$, then the set $\{x_1, \dots, x_k\}$ satisfies (1.2), and vice versa. Therefore Theorem 1.2 implies Theorem 1.1.

In this paper we shall improve and complete Theorem 1.2 by establishing the following two theorems.

Theorem 1.3. *Let m, n be positive integers, and let $k_1, k_2, \dots, k_r$ be nonnegative integers such that $n = k_1 + 2k_2 + \dots + rk_r$. Let $d = \gcd(m, n)$. If*

$$\Delta := n - d(n - k_1 - \dots - k_r) > 0, \tag{1.5}$$

then the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of type $1^{k_1} 2^{k_2} \cdots r^{k_r}$ is given by (1.4).

It is not hard to see that the condition (1.5) is weaker than (1.3), i.e., the condition (1.3) implies that (1.5). In other words, for fixed n and a given type, there are in general many more m's satisfying (1.5) than satisfying (1.3). For example, by Theorem 1.3, the numbers of m-AP-partitions of $\mathbb{Z}_{120}$ of type $1^{89} 2^3 3^2 5^1 7^2$ are all equal for

$$m = 1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 17, 19, 21, 22, 23, 25, 26, 27, 28, 29, 31,$$
$$33, 34, 35, 37, 38, 39, 41, 43, 44, 46, 47, 49, 51, 52, 53, 55, 57, 58, 59,$$

i.e., for $d = 1, 2, 3, 4, 5$. However, Theorem 1.2 only asserts that these numbers for $m = 1, 2, 3$ are equal.

Theorem 1.4. *Let $k_1, k_2, \dots, k_r, m, n, d$ and Δ be given as in Theorem 1.3. Then the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ is given by*

$$\begin{cases} \dfrac{n}{k_1+\cdots+k_r}\dbinom{k_1+\cdots+k_r}{k_1,\dots,k_r} + \dfrac{n(-1)^{k_2+\cdots+k_r}}{k_2+\cdots+k_r}\dbinom{k_2+\cdots+k_r}{k_2,\dots,k_r}, & \text{if } \Delta = 0, \\ \dfrac{n}{k_1+\cdots+k_r}\dbinom{k_1+\cdots+k_r}{k_1,\dots,k_r} + \clubsuit(-1)^{k_2+\cdots+k_r}\dbinom{k_2+\cdots+k_r}{k_2,\dots,k_r}, & \text{if } \Delta = -d, \end{cases}$$

where

$$\clubsuit = \begin{cases} n, & \text{if } k_2 = 0, \\ n\left(1 - \dfrac{n(1-d^{-1})k_2}{(k_2+\cdots+k_r)(k_2+\cdots+k_r-1)}\right), & \text{if } k_2 > 0. \end{cases}$$

When the type in Theorem 1.4 is $1^{n-(p+1)k}(p+1)^k$ again, then $\Delta = n - mpk$. To assure that there is an m-AP-block of length $p+1$ in $\mathbb{Z}_n$, we need to assume that $n > pm$, which is equivalent to $k \geqslant 2$ if $\Delta = 0$ and $pk > p+1$ if $\Delta = -m$. As mentioned after Theorem 1.2, each family of km-AP-blocks in $\mathbb{Z}_n$ is in one-to-one correspondence with a k-subset of $\mathbb{Z}_n$ satisfying (1.2), we derive the following two results, which can be viewed as complements to Theorem 1.1.

Corollary 1.5. *Let $m, p \geqslant 1$, $k \geqslant 2$ and $n = mpk$. Then the number of k-subsets $\{x_1, x_2, \dots, x_k\}$ of $\mathbb{Z}_n$ such that $x_i - x_j \notin \{m, 2m, \dots, pm\}$ for all $1 \leqslant i, j \leqslant k$, is given by*

$$\frac{n}{n-pk}\binom{n-pk}{k} + (-1)^k\frac{n}{k}. \tag{1.6}$$

Actually the above formula is deduced for $m \geqslant 2$, i.e., $n \geqslant (p+1)k$, but it also holds for $m = 1$ if we take the convention

$$\lim_{x\to 0}\frac{n}{x}\binom{x}{k} = (-1)^{k-1}\frac{n}{k},$$

and so (1.6) is equal to 0 in this case. Here is an example for Corollary 1.5. For $m = p = k = 2$, the number of 2-subsets $\{x_1, x_2\}$ of $\mathbb{Z}_8$ such that $x_1 - x_2, x_2 - x_1 \notin \{2, 4\}$ is equal to

$$\frac{8}{4}\binom{4}{2} + 4 = 16,$$

and the corresponding subsets are $\{i, i+1\}$ and $\{i, i+3\}$, where $i \in \mathbb{Z}_8$.

Corollary 1.6. *Let $m, p, k \geqslant 1$ with $pk > p+1$ and let $n = mpk - m$. Then the number of k-subsets $\{x_1, x_2, \dots, x_k\}$ of $\mathbb{Z}_n$ such that $x_i - x_j \notin \{m, 2m, \dots, pm\}$ for all $1 \leqslant i, j \leqslant k$, is given by*

$$\begin{cases} \dfrac{n}{n-k}\dbinom{n-k}{k} + (-1)^{k-1}n(m-2), & \text{if } p = 1, \\ \dfrac{n}{n-pk}\dbinom{n-pk}{k} + (-1)^k n, & \text{if } p \geqslant 2. \end{cases}$$

Similarly, although the above formula is deduced for $mpk - m \geqslant (p+1)k$, it also holds without this condition. The details are left to the interested reader.

Remark. For $0 < m < n$, let $g_m(n, k)$ denote the number of k-subsets $\{x_1, x_2, \dots, x_k\}$ of $\mathbb{Z}_n$ such that $x_i - x_j \neq m$ for all $1 \leqslant i, j \leqslant k$. Hwang [7, Corollary 2] obtained

$$g_m(n,k) = \sum_{j=0}^{\lfloor d/2\rfloor}(-1)^{nj/d}\binom{d}{j}\frac{n-2nj/d}{n-k-nj/d}\binom{n-k-nj/d}{k-nj/d}, \tag{1.7}$$

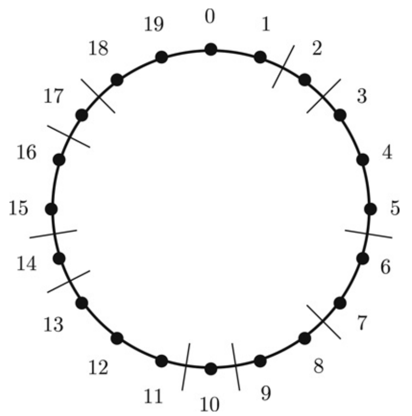

Fig. 1. A 20-cycle dissection of type $1^4 2^3 3^2 4^1$.

where $d = \gcd(m, n)$. Letting $n = mk$ or $n = mk - m$ in (1.7), we are led to the $p = 1$ case of Corollary 1.5 or 1.6. However, since there are two cases in Corollary 1.6, it seems impossible to give a formula like (1.7) to unify Corollaries 1.5 and 1.6 for general p.

We recall and establish some necessary lemmas in Section 2 and prove Theorems 1.3 and 1.4 in Sections 3 and 4, respectively. Our main idea is the following: Lemma 2.4 permits us to reduce the general m-AP-partition problem of $\mathbb{Z}_n$ to the case where m divides n. For the latter we may write the partition number as a multiple sum, which can be computed by applying Raney–Mohanty's identity.

2. Some lemmas

A *dissection of an n-cycle* is a 1-AP-partition of $\mathbb{Z}_n$, which can be depicted by inserting a bar between any two consecutive blocks on an n-cycle. For example, Fig. 1 illustrates a 20-cycle dissection of type $1^4 2^3 3^2 4^1$.

It is easy to see that the number of dissections of $\mathbb{Z}_n$ is given by (1.4). Indeed, deleting the segment containing 0 in any dissection of n-cycle of type $1^{k_1} 2^{k_2} \cdots r^{k_r}$ yields a dissection of a $(n-i)$-line of type $1^{k_1} \cdots i^{k_i - 1} \cdots r^{k_r}$ if the segment containing 0 is of length i $(1 \leqslant i \leqslant n)$. So the number of such dissections of n-cycle is equal to

$$i \binom{k_1 + \cdots + (k_i - 1) + \cdots + k_r}{k_1, \ldots, k_i - 1, \ldots, k_r}. \tag{2.1}$$

Summing (2.1) over all i yields the following known result (see [17, Lemma 3.1]).

Lemma 2.1 (*Chen–Lih–Yeh*). *For an n-cycle, the number of dissections of type* $1^{k_1} 2^{k_2} \cdots r^{k_r}$ *is given by the cyclic multinomial coefficient* (1.4).

For any variable x and nonnegative integers $k_1, \ldots, k_r$ define the multinomial coefficient

$$\binom{x}{k_1, k_2, \ldots, k_r} := \frac{x(x-1)\ldots(x - k_1 - \cdots - k_r + 1)}{k_1! k_2! \ldots k_r!}.$$

Note that when $x = k_1 + k_2 + \cdots + k_r$ the above definition coincides with the classical definition of multinomial coefficient and

$$\binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r} = \binom{k_1 + \cdots + k_r}{k_2, \ldots, k_r}.$$

The following convolution formula for multinomial coefficients is due to Raney–Mohanty [18,19]. For other proofs of (2.2), we refer the reader to [20–22].

Lemma 2.2 (*Raney–Mohanty's identity*). *For any variables $x, y, z_1, \dots, z_m$ and nonnegative integers $N_1, \dots, N_m$, there holds*

$$\sum_{\substack{0\leqslant t_i\leqslant N_i\\ i=1,\dots,m}} \frac{x}{x-t_1z_1-\cdots-t_mz_m}\binom{x-t_1z_1-\cdots-t_mz_m}{t_1,\dots,t_m} \times \frac{y}{y-(N_1-t_1)z_1-\cdots-(N_m-t_m)z_m}\binom{y-(N_1-t_1)z_1-\cdots-(N_m-t_m)z_m}{N_1-t_1,\dots,N_m-t_m} = \frac{x+y}{x+y-N_1z_1-\cdots-N_mz_m}\binom{x+y-N_1z_1-\cdots-N_mz_m}{N_1,\dots,N_m}. \tag{2.2}$$

We also need the following elementary arithmetical result (see [23, Theorem 5.32 and Exercise 16 on page 127] or [15]).

Lemma 2.3. *Let m, n be positive integers. If $\gcd(m, n) = d$, then there exists an integer a such that $\gcd(a, n) = 1$ and $am \equiv d \pmod n$.*

The following is our key lemma.

Lemma 2.4. *If $m, n \geqslant 1$ and $\gcd(m, n) = d$, then there is a bijection from the set of m-AP-partitions of $\mathbb{Z}_n$ to the set of d-AP-partitions of $\mathbb{Z}_n$. Moreover this bijection keeps the type of partitions.*

Proof. By Lemma 2.3, there exists an inversible element $a \in \mathbb{Z}_n$ such that $am = d$. Let a^{-1} be the inverse of a. For any subset B of $\mathbb{Z}_n$ and $x \in \mathbb{Z}_n$, let $xB = \{xb: b \in B\}$. If $\{B_1, B_2, \dots, B_s\}$ is an m-AP-partition of $\mathbb{Z}_n$, then $\{aB_1, aB_2, \dots, aB_s\}$ is a d-AP-partition of $\mathbb{Z}_n$. Conversely, if $\{C_1, C_2, \dots, C_s\}$ is a d-AP-partition of $\mathbb{Z}_n$, then $\{a^{-1}C_1, a^{-1}C_2, \dots, a^{-1}C_s\}$ is an m-AP-partition of $\mathbb{Z}_n$. Obviously, this correspondence keeps the type of partitions. This proves the lemma. □

It follows from Lemma 2.4 that if there exists an m-AP-partition of $\mathbb{Z}_n$ of a given type $1^{k_1}2^{k_2}\cdots r^{k_r}$ ($k_r > 0$) then $\gcd(m, n)r \leqslant n$.

3. Proof of Theorem 1.3

By Lemma 2.4, it suffices to consider the case where m divides n, i.e., $d = m$. Let $n = mn_1$ and divide $\mathbb{Z}_n$ into m subsets of the same cardinality n_1:

$$\mathbb{Z}_{n,j} = \{mi + j: i = 0, \dots, n_1 - 1\}, \quad 0 \leqslant j \leqslant m - 1.$$

Hence $\mathbb{Z}_n = \biguplus_{j=0}^{m-1}\mathbb{Z}_{n,j}$. Let $\mathcal{B} = \{B_1, B_2 \dots, B_s\}$ be an m-AP-partition of $\mathbb{Z}_n$ of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ ($r \leqslant n_1$). Then $B_{i,j} = \mathbb{Z}_{n,j} \cap B_i$ is equal to $\emptyset$ or B_i for $1 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant m-1$. Furthermore, since the transformation $x \mapsto (x-j)/m$ maps each m-AP-block $B_{i,j}$ of $\mathbb{Z}_{n,j}$ ($0 \leqslant j \leqslant m-1$) to a 1-AP-block $B'_{i,j}$ of $\mathbb{Z}_{n_1}$, each m-AP-partition $\mathcal{B}_j = \{B_{1,j}, \dots, B_{s,j}\}$ corresponds bijectively to a 1-AP-partition $\mathcal{B}'_j$ of $\mathbb{Z}_{n_1}$ with the same type. Thus, we have established a bijection between the set of m-AP-partitions of $\mathbb{Z}_n$ and the set of m-tuples of 1-AP-partitions of $\mathbb{Z}_{n_1}$: $\mathcal{B} \leftrightarrow (\mathcal{B}'_0, \dots, \mathcal{B}'_{m-1})$.

Now assume that the m-AP-partition $\mathcal{B}$ is of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ ($r \leqslant n_1$), and the corresponding 1-AP-partition $\mathcal{B}'_j$ is of type $1^{k_{1,j}}2^{k_{2,j}}\cdots r^{k_{r,j}}$ ($0 \leqslant j \leqslant m-1$). Clearly,

$$\begin{cases} k_{2,0} + k_{2,1} + \cdots + k_{2,m-1} = k_2,\\ k_{3,0} + k_{3,1} + \cdots + k_{3,m-1} = k_3,\\ \cdots\\ k_{r,0} + k_{r,1} + \cdots + k_{r,m-1} = k_r. \end{cases} \tag{3.1}$$

By Lemma 2.1 and noticing that $n_1 = k_{1,j} + 2k_{2,j} + \cdots + rk_{r,j}$, the number of 1-AP-partitions of $\mathbb{Z}_{n_1}$ of type $1^{k_{1,j}}2^{k_{2,j}}\cdots r^{k_{r,j}}$ is equal to

$$\frac{n_1}{k_{1,j}+k_{2,j}+\cdots+k_{r,j}}\binom{k_{1,j}+k_{2,j}+\cdots+k_{r,j}}{k_{1,j},k_{2,j},\ldots,k_{r,j}} = \frac{n_1}{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}\binom{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}{k_{2,j},\ldots,k_{r,j}}.$$

For $m, n \geqslant 1$ let $f_{m,n}(k_1, \ldots, k_r)$ be the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of type $1^{k_1}2^{k_2}\cdots r^{k_r}$. Then

$$f_{m,n}(k_1,\ldots,k_r) = \sum_{(k_{i,j})}\prod_{j=0}^{m-1}\frac{n_1}{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}\binom{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}{k_{2,j},\ldots,k_{r,j}}, \quad (3.2)$$

where the summation is over all matrices $(k_{i,j})_{\substack{2\leqslant i\leqslant r\\ 0\leqslant j\leqslant m-1}}$ of nonnegative integral coefficients $k_{i,j}$ satisfying (3.1) and

$$\begin{cases} n_1 - k_{2,0} - \cdots - (r-1)k_{r,0} > 0, \\ n_1 - k_{2,1} - \cdots - (r-1)k_{r,1} > 0, \\ \cdots \\ n_1 - k_{2,m-1} - \cdots - (r-1)k_{r,m-1} > 0. \end{cases} \quad (3.3)$$

Recall that

$$\Delta = n - m(n-k_1-\cdots-k_r) = mn_1 - m(k_2+\cdots+(r-1)k_r).$$

If $\Delta > 0$, then we have $n_1 > k_2 + \cdots + (r-1)k_r$, and thus all nonnegative integral solutions to (3.1) also satisfy (3.3) as $k_i \geqslant k_{i,j}$ $(2 \leqslant i \leqslant r, 0 \leqslant j \leqslant m-1)$.

It remains to prove that the right-hand side of (3.2) is equal to (1.4), namely

$$\frac{mn_1}{mn_1-k_2-\cdots-(r-1)k_r}\binom{mn_1-k_2-\cdots-(r-1)k_r}{k_2,\ldots,k_r}. \quad (3.4)$$

We proceed by induction on $m \geqslant 1$. This is equivalent to repeatedly applying Raney–Mohanty's identity (2.2). The case $m = 1$ is obviously true. Suppose that the formula is true for $m - 1$ with $m \geqslant 2$ and let $k_{i,0} + k_{i,1} + \cdots + k_{i,m-2} = k_i'$ be fixed for $i = 2, \ldots, r$. Then

$$\sum_{\substack{k_{i,0},\ldots,k_{i,m-2}\\ i=2,\ldots,r}}\prod_{j=0}^{m-2}\frac{n_1}{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}\binom{n_1-k_{2,j}-\cdots-(r-1)k_{r,j}}{k_{2,j},\ldots,k_{r,j}} = \frac{(m-1)n_1}{(m-1)n_1-k_2'-\cdots-(r-1)k_r'}\binom{(m-1)n_1-k_2'-\cdots-(r-1)k_r'}{k_2',\ldots,k_r'}.$$

Plugging this into (3.2) yields

$$f_{m,n}(k_1,\ldots,k_r) = \sum_{\substack{k_i'+k_{i,m-1}=k_i\\ i=2,\ldots,r}}\frac{(m-1)n_1}{(m-1)n_1-k_2'-\cdots-(r-1)k_r'}\binom{(m-1)n_1-k_2'-\cdots-(r-1)k_r'}{k_2',\ldots,k_r'} \times \frac{n_1}{n_1-k_{2,m-1}-\cdots-(r-1)k_{r,m-1}}\binom{n_1-k_{2,m-1}-\cdots-(r-1)k_{r,m-1}}{k_{2,m-1},\ldots,k_{r,m-1}}$$

which is (3.4) by applying Raney–Mohanty's identity (2.2).

4. Proof of Theorem 1.4

For the case $\Delta = 0$ or $\Delta = -m$, the number $f_{m,n}(k_1, \ldots, k_r)$ is again given by (3.2). However, we will meet with $n_1 - k_{2,j} - \cdots - (r-1)k_{r,j} \leqslant 0$ for some $0 \leqslant j \leqslant m-1$ in some nonnegative integral solutions $(k_{i,j})$ to (3.1). It is convenient here to consider a more general form of (3.2) as follows. For any variable x, let $f_{m,n}(x; k_1, \ldots, k_r)$ be the following expression

$$\sum_{(k_{i,j})} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \cdots - (r-1)k_{r,j}} \binom{x - k_{2,j} - \cdots - (r-1)k_{r,j}}{k_{2,j}, \ldots, k_{r,j}},$$

where $(k_{i,j})$ ranges over the same integral matrices as (3.2).

Let M be the set of all nonnegative integral matrices $(k_{i,j})_{\substack{2\leqslant i\leqslant r\\ 0\leqslant j\leqslant m-1}}$ satisfying (3.1), and let S be the set of all $(k_{i,j})$ in M such that (3.3) does not hold. Then

$$\begin{aligned} f_{m,n}(x; k_1, \ldots, k_r) &= \sum_{(k_{i,j})\in M} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \cdots - (r-1)k_{r,j}} \binom{x - k_{2,j} - \cdots - (r-1)k_{r,j}}{k_{2,j}, \ldots, k_{r,j}} \\ &\quad - \sum_{(k_{i,j})\in S} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \cdots - (r-1)k_{r,j}} \binom{x - k_{2,j} - \cdots - (r-1)k_{r,j}}{k_{2,j}, \ldots, k_{r,j}}. \end{aligned} \tag{4.1}$$

When $\Delta = 0$, we have $n_1 = k_2 + 2k_3 + \cdots + (r-1)k_r$, and S reduces to

$$S_1 := \left\{(k_{i,j}):\ \text{for some } j_0 \text{ and all } i,\ \text{we have } k_{i,j_0} = k_i \text{ and } k_{i,j} = 0 \text{ if } j \neq j_0\right\}.$$

So the second summation on the right-hand side of (4.1) becomes

$$\frac{mx}{x - n_1} \binom{x - n_1}{k_2, \ldots, k_r},$$

while the first summation can be summed by using Raney–Mohanty's identity. It follows that

$$\begin{aligned} f_{m,n}(x; k_1, \ldots, k_r) &= \frac{mx}{mx - k_2 - \cdots - (r-1)k_r} \binom{mx - k_2 - \cdots - (r-1)k_r}{k_2, \ldots, k_r} - \frac{mx}{x - n_1} \binom{x - n_1}{k_2, \ldots, k_r}. \end{aligned} \tag{4.2}$$

Letting $x = n_1$ in (4.2) and noticing the following fact

$$\lim_{z\to 0} \frac{1}{z} \binom{z}{a_1, \ldots, a_s} = \frac{(-1)^{a_1+\cdots+a_s-1}}{a_1 + \cdots + a_s} \binom{a_1 + \cdots + a_s}{a_1, \ldots, a_s}, \tag{4.3}$$

one obtains the first formula in Theorem 1.4.

When $\Delta = -m$, we have $n_1 = k_2 + 2k_3 + \cdots + (r-1)k_r - 1$. If $k_2 = 0$, then $S = S_1$, while if $k_2 > 0$, then

$$\begin{aligned} S = S_1 \cup \{(k_{i,j}):\ &\text{for some } j_0 \neq j_1,\ \text{we have } k_{2,j_0} = k_2 - 1, k_{2,j_1} = 1, \\ &k_{i,j_0} = k_i (2 < i \leqslant r) \text{ and } k_{i,j} = 0 \text{ otherwise}\}. \end{aligned}$$

It follows that

$$\begin{aligned} f_{m,n}(x; k_1, \ldots, k_r) &= \frac{mx}{mx - k_2 - \cdots - (r-1)k_r} \binom{mx - k_2 - \cdots - (r-1)k_r}{k_2, \ldots, k_r} \\ &\quad - \frac{mx}{x - n_1 - 1} \binom{x - n_1 - 1}{k_2, \ldots, k_r} - \chi(k_2 > 0) \frac{m(m-1)x^2}{x - n_1} \binom{x - n_1}{k_2 - 1, k_3, \ldots, k_r}. \end{aligned} \tag{4.4}$$

Letting $x = n_1$ in (4.4) and using (4.3) and

$$\binom{-1}{a_1, \dots, a_s} = (-1)^{a_1+\cdots+a_s} \binom{a_1+\cdots+a_s}{a_1, \dots, a_s},$$

we obtain the second formula in Theorem 1.4.

Remark. It is also possible to compute $f_{m,n}(k_1, \dots, k_r)$ for the case $\Delta = -2\gcd(m, n)$ or $\Delta = -3\gcd(m, n)$. But the result is more complicated and is omitted here.

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