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Averaging of nonautonomous damped wave equations with singularly oscillating external forces ☆

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Abstract

We consider, for $\rho \in [0,1]$ and $\varepsilon > 0$ small, the nonautonomous weakly damped wave equation with a singularly oscillating external force

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(t) + \varepsilon^{-\rho} g_1(t/\varepsilon),$$

together with the *averaged* equation

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(t).$$

Under suitable assumptions on the nonlinearity and the external force, we prove the uniform (with respect to ε) boundedness of the attractors $\mathcal{A}^\varepsilon$ in the weak energy space. If $\rho < 1$, we establish the convergence of the attractor $\mathcal{A}^\varepsilon$ of the first equation to the attractor $\mathcal{A}^0$ of the second one, as $\varepsilon \to 0^+$. On the other hand, if $\rho = 1$, this convergence may fail. When $\mathcal{A}^0$ is exponential, then the convergence rate of $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ is controlled by $M\varepsilon^\eta$, for some $M \geqslant 0$ and some $\eta = \eta(\rho) \in (0,1)$.

Résumé

Pour tout $\rho \in [0,1]$ et pour $\varepsilon > 0$ suffisamment petit, on considère l'équation des ondes non autonome faiblement amortie avec une force extérieure singulière et oscillatoire

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(t) + \varepsilon^{-\rho} g_1(t/\varepsilon),$$

et le problème *moyenné*

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(t).$$

Avec des hypothèses adéquates sur la nonlinéarité et sur la force, on obtient une borne uniforme (par rapport à ε) pour les attracteurs $\mathcal{A}^\varepsilon$ dans l'espace faible d'énergie. Si $\rho < 1$, on démontre la convergence de l'attracteur $\mathcal{A}^\varepsilon$ de la première équation vers l'attracteur

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$\mathcal{A}^0$ de la deuxième équation lorsque $\varepsilon \to 0^+$. D'autre part, si $\rho = 1$, cette convergence peut ne pas avoir lieu. Quand $\mathcal{A}^0$ est exponentiel, la vitesse de convergence de $\mathcal{A}^\varepsilon$ vers $\mathcal{A}^0$ est bornée par $M\varepsilon^\eta$, pour certains $M \geqslant 0$ et $\eta = \eta(\rho) \in (0, 1)$.

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1. Introduction

Let $\Omega \Subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class C^2. For $\varepsilon \in (0, 1]$, we consider the nonautonomous semilinear wave equation with Dirichlet boundary conditions:

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(x, t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

Here the space variable $x \in \Omega$, the time $t \in \mathbb{R}$, $u = u(x, t)$ is an unknown real function and the Laplace operator Δ acts in the x-space. The damping $\gamma > 0$ and the parameter $\rho \in [0, 1]$ are both fixed. Along with (1.1), we also consider the *averaged* equation:

$$\partial_t^2 u - \Delta u + \gamma \partial_t u = -f(u) + g_0(x, t), \quad u|_{\partial\Omega} = 0, \tag{1.2}$$

without rapid and singular oscillations, which formally corresponds to $\varepsilon = 0$. Actually, the last assertion is somehow justified by the results proved later in this work. Indeed, at least when $\rho > 0$, the fact that (1.2) can be considered as the (formal) limit as $\varepsilon \to 0^+$ of (1.1) is not intuitive: in principle, the blow up of the oscillation amplitude might overcome the averaging effect due to the term t/ε in g_1.

Concerning the (nonlinear) function $f(u)$, we will assume rather standard dissipation and growing conditions (see below), which are satisfied by the well-known physically relevant examples of nonlinearities, such as

$$f(u) = \kappa |u|^{d-2} u,$$

for $2 \leqslant d \leqslant 4$ and $\kappa > 0$ (a model equation of relativistic quantum mechanics), and

$$f(u) = \kappa \sin(u)$$

(a sine-Gordon model of the Josephson junction driven by a current source). We address the reader to the books [19,30] and references therein for more details on the physical models.

For $\varepsilon \in [0, 1]$, the term

$$g^\varepsilon(x, t) := \begin{cases} g_0(x, t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), & \varepsilon > 0, \\ g_0(x, t), & \varepsilon = 0, \end{cases}$$

represents the *external force*. The aim of this work is to study the properties of Eq. (1.1), depending on the small parameter ε, which reflects the rate of fast time oscillation in the term $\varepsilon^{-\rho} g_1(x, t/\varepsilon)$, having the growing amplitude of order $\varepsilon^{-\rho}$. Both $g_0(x, t)$ and $g_1(x, t)$ are supposed to be translation bounded in the space $L_1^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$.

Along the lines of the Bogolyubov averaging principle [3], the first results related to attractors of nonautonomous evolution equations with rapidly, but nonsingularly (i.e., with $\rho = 0$), time oscillating terms of periodic or almost periodic kind, can be found in the papers [20,22,23]. The averaging of global attractors of nonautonomous dissipative wave equations has been studied in [5,12,22,32,35], in presence of nonsingular time oscillations, and in [8,27,31,34], in presence of nonsingular oscillations in space. For the two-dimensional Navier–Stokes system and for parabolic equations with oscillating parameters, similar problems have been considered in [8,9,13–16]. To the best of our knowledge, the more challenging singular case $\rho > 0$ is treated only in [10,11,33].

Under suitable assumptions on the nonlinearity and the forcing terms, the (nonautonomous) equations (1.1) and (1.2) generate strongly continuous processes in the phase space $E = H_0^1(\Omega) \times L_2(\Omega)$, which possess global attractors $\mathcal{A}^\varepsilon$. Our main purpose is to establish a convergence result for such attractors in the limit $\varepsilon \to 0^+$. An analysis of this kind has been already carried out in [33] (see also [5]). Similar (albeit easier) problems, have been considered in the papers [10,11], focused on the homogenization of the global attractors arising from dissipative equations of mathematical physics, where the forcing term generating the oscillation is of the form $\varepsilon^{-\rho} g_1(x/\varepsilon, t)$. The main achievement of [33] is the uniform (with respect to ε) boundedness and the (Hölder) continuity at $\varepsilon = 0$ of the family $\{\mathcal{A}^\varepsilon\}$. More precisely, it is shown that

$$\sup_{\varepsilon\in[0,1]} \|\mathcal{A}^\varepsilon\|_E < \infty, \tag{1.3}$$

provided that g_0, g_1 are translation compact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$, $f(u)$ is sublinear, the parameter $\rho > 0$ is suitably small and the function

$$G_1(t,\tau) = \int_\tau^t g_1(s)\,ds, \quad t \geqslant \tau,$$

is uniformly (with respect to $t \geqslant \tau$, $\tau \in \mathbb{R}$) bounded in $H_0^1(\Omega)$. If, in addition, the attractor $\mathcal{A}^0$ is exponential (which, by the way, is a rather severe constraint), then

$$\lim_{\varepsilon\to 0^+} \bigl\{\mathrm{dist}_E\bigl(\mathcal{A}^\varepsilon, \mathcal{A}^0\bigr)\bigr\} = 0, \tag{1.4}$$

where dist_E is the usual Hausdorff semidistance in E.

In the present work, we improve the results of [33] in several directions; namely:

(i) we assume g_0, g_1 translation compact in $L_1^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$;
(ii) we allow $f(u)$ to have the critical growth of polynomial order three;
(iii) we require a weaker condition on $G_1(t,\tau)$ (see Section 5);
(iv) we take an arbitrary $\rho \in [0,1)$;
(v) we do not require $\mathcal{A}^0$ to be exponential.

Then, within (i)–(v), we obtain the conclusions (1.3)–(1.4). The case $\rho = 1$ deserves a particular attention: indeed, although we can prove (1.3), at least in the subcritical case, we provide an example showing that (1.4) may fail, even in the simplest situation where $f \equiv 0$. Finally, if the attractor $\mathcal{A}^0$ is exponential, we have, as in [33], the Hölder continuity property at $\varepsilon = 0$,

$$\mathrm{dist}_E\bigl(\mathcal{A}^\varepsilon, \mathcal{A}^0\bigr) \leqslant M\varepsilon^\eta,$$

for some $M \geqslant 0$ and some $\eta \in (0,1)$.

1.1. Plan of the paper

In Section 2, we introduce the assumptions on the nonlinearity and the forcing term. In Section 3, we recall some basic results on the existence of the uniform attractors $\mathcal{A}^\varepsilon$ associated, for every given $\varepsilon \in [0,1]$, to (1.1) or (1.2). Then, Section 4 is devoted to the analysis of a linear wave equation in presence of an oscillating external force. In Section 5, a uniform (with respect to ε) bound for the attractors $\mathcal{A}^\varepsilon$ is established. The main result is stated and proved in Section 6; namely, the convergence $\mathcal{A}^\varepsilon \to \mathcal{A}^0$ as $\varepsilon \to 0^+$. Finally, in Section 7, we prove the Hölder continuity of $\mathcal{A}^\varepsilon$ at $\varepsilon = 0$ when $\mathcal{A}^0$ is exponential.

1.2. Notations

For $\tau \in \mathbb{R}$, we set $\mathbb{R}_\tau = [\tau, +\infty)$. Throughout the paper, $C > 0$ will stand for a *generic* constant, independent of ε, g_0, g_1 and of the choice of the initial time $\tau \in \mathbb{R}$. In the sequel, we will omit the dependence on the space variable x.

Given a normed space X, we usually denote the norm in X by $\|\cdot\|_X$, and we indicate by

$$\mathrm{dist}_X(B_1, B_2) := \sup_{b_1\in B_1} \inf_{b_2\in B_2} \|b_1 - b_2\|_X,$$

the Hausdorff semidistance in X from a set B_1 to a set B_2.

For $\sigma \in \mathbb{R}$, we consider the scale of Hilbert spaces $H^\sigma := D(A^{\sigma/2})$ endowed with the inner product and norm:

$$\langle u, v\rangle_\sigma := \bigl\langle A^{\sigma/2}u, A^{\sigma/2}v\bigr\rangle_{L_2}, \qquad |u|_\sigma := \bigl\|A^{\sigma/2}u\bigr\|_{L_2},$$

corresponding to the (strictly) positive self-adjoint operator $A = -\Delta$ acting on $L_2(\Omega)$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ (we agree to omit the index σ whenever $\sigma = 0$). The symbol $|\cdot|$ will also be used for the absolute value. Besides, we call $\lambda_1 > 0$ the first eigenvalue of A. Clearly, we have the equalities:

$$H^{-1} = H^{-1}(\Omega), \qquad H = H^0 = L_2(\Omega), \qquad H^1 = H_0^1(\Omega), \qquad H^2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and the generalized Poincaré inequality

$$|u|_{\sigma+1} \geqslant \lambda_1^{1/2} |u|_\sigma, \quad \forall u \in H^{\sigma+1}.$$

We also recall that $H^\sigma \subset H^\sigma(\Omega)$, for $\sigma \geqslant 0$ (see, e.g., [26]). Then, we introduce the *energy spaces*:

$$E^\sigma := H^{\sigma+1} \times H^\sigma, \qquad \|(u,p)\|_{E^\sigma}^2 := |u|_{\sigma+1}^2 + |p|_\sigma^2.$$

Sometimes, we will employ the (equivalent) norm on E^σ:

$$\|(u,p)\|_\sigma^2 := |u|_{\sigma+1}^2 + |p + \alpha u|_\sigma^2 - r|u|_\sigma^2, \tag{1.5}$$

where $\alpha > 0$ is arbitrary and $r < \lambda_1$.

2. Basic assumptions

2.1. Assumptions on f

Let $f \in C^1(\mathbb{R})$, with $f(0) = 0$, be such that

$$|f'(a)| \leqslant C(|a|^{d-2} + 1), \quad \forall a \in \mathbb{R}, \tag{2.1}$$

with

$$2 \leqslant d \leqslant 4.$$

We also have to impose some dissipation conditions. For $d > 2$, we assume that

$$f(a)a \geqslant \nu |a|^d - C, \quad \forall a \in \mathbb{R}, \tag{2.2}$$

where ν is a (possibly small) positive constant. For $d = 2$ (the linear growth condition), in place of (2.2), we require that

$$\liminf_{|a|\to\infty} \frac{f(a)}{a} > -\lambda_1. \tag{2.3}$$

Condition (2.2) is slightly more restrictive than in [2] (see also [8,19,30]). We point out that the inequality (2.1) for $d = 4$ is known to be a *critical growth condition* for the nonlinear function f (see [25]). Indeed, for $d > 4$, the initial-value problem (1.1) may not have unique solutions. It readily follows from (2.1) that

$$|f(a)| \leqslant C(|a|^{d-1} + 1), \quad \forall a \in \mathbb{R}. \tag{2.4}$$

In this paper, we will mostly consider the subcritical case $2 < d < 4$. The critical case $d = 4$ can also be treated, paying the price of adding some extra conditions for the external force (see below). All the results that we will state hold for the much simpler case $d = 2$ as well. Setting

$$F(a) = \int_0^a f(s)\,ds,$$

the inequalities (2.1) and (2.2) imply that

$$\nu_1 |a|^d - C \leqslant F(a), \qquad |F(a)| \leqslant C(|a|^d + 1), \quad \forall a \in \mathbb{R}, \tag{2.5}$$

$$f(a)a \geqslant \nu_2 F(a) - C, \quad \forall a \in \mathbb{R}, \tag{2.6}$$

for some $\nu_1, \nu_2 > 0$ (without loss of generality, we assume $\nu_1, \nu_2 \leqslant 1$). For further use, we also introduce the functional:

$$\mathcal{F}(u) = \int_\Omega F(u(x))\,dx, \quad u \in H^1.$$

2.2. Assumptions on the external force

The functions $g_0(t)$ and $g_1(t)$, as anticipated in the introduction, are assumed to be translation bounded in $L_1^{\mathrm{loc}}(\mathbb{R};H)$. More precisely, $g_0, g_1 \in L_1^b(\mathbb{R};H)$, with

$$\|g_0\|_{L_1^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \left|g_0(s)\right| ds = M_0, \tag{2.7}$$

$$\|g_1\|_{L_1^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \left|g_1(s)\right| ds = M_1, \tag{2.8}$$

for some $M_0, M_1 \geqslant 0$. A straightforward consequence of (2.8) is

$$\int_t^{t+1} \left|g_1(s/\varepsilon)\right| ds = \varepsilon \int_{t/\varepsilon}^{(t+1)/\varepsilon} \left|g_1(s)\right| ds \leqslant \varepsilon(1+1/\varepsilon)M_1 \leqslant 2M_1, \quad \forall t \in \mathbb{R},$$

so that

$$\left\|g_1(\cdot/\varepsilon)\right\|_{L_1^b} \leqslant 2M_1, \quad \forall \varepsilon \in (0,1].$$

Hence, for $\varepsilon > 0$,

$$\left\|g^\varepsilon\right\|_{L_1^b} = \left\|g_0 + \varepsilon^{-\rho} g_1(\cdot/\varepsilon)\right\|_{L_1^b} \leqslant \|g_0\|_{L_1^b} + \varepsilon^{-\rho}\left\|g_1(\cdot/\varepsilon)\right\|_{L_1^b} \leqslant M_0 + 2M_1\varepsilon^{-\rho}.$$

In summary, we learned that

$$\left\|g^\varepsilon\right\|_{L_1^b} \leqslant Q_\varepsilon, \tag{2.9}$$

having set

$$Q_\varepsilon = \begin{cases} M_0 + 2M_1\varepsilon^{-\rho}, & \varepsilon > 0, \\ M_0, & \varepsilon = 0. \end{cases} \tag{2.10}$$

Note that the norm of the external force $g^\varepsilon(t)$ in the space L_1^b can grow with a rate of order $\varepsilon^{-\rho}$ as $\varepsilon \to 0^+$.

Throughout this paper, we will always assume (2.1)–(2.3) and (2.7)–(2.8).

We conclude the section recalling a generalized Gronwall lemma which will be needed in the sequel (see [4,28]).

Lemma 2.1. *Let ζ and φ_1, φ_2 be nonnegative locally summable functions on $\mathbb{R}_\tau$ satisfying, for some $\beta > 0$, the differential inequality*:

$$\frac{d}{dt}\zeta + 2\beta\zeta \leqslant \varphi_1 + \varphi_2\zeta^{1/2},$$

for a.e. $t \in \mathbb{R}_\tau$. Then,

$$\zeta(t) \leqslant 2\zeta(\tau)\mathrm{e}^{-2\beta(t-\tau)} + \int_\tau^t \varphi_1(s)\mathrm{e}^{-2\beta(t-s)}\, ds + \left(\int_\tau^t \varphi_2(s)\mathrm{e}^{-\beta(t-s)}\, ds\right)^2,$$

for all $t \in \mathbb{R}_\tau$. Moreover, the inequality

$$\sup_{t\geqslant\tau} \int_\tau^t \varphi(s)\mathrm{e}^{-\beta(t-s)}\, ds \leqslant \frac{1}{1-\mathrm{e}^{-\beta}} \sup_{t\geqslant\tau} \int_t^{t+1} \varphi(s)\, ds \tag{2.11}$$

holds for every nonnegative locally summable function φ on $\mathbb{R}_\tau$ and every $\beta > 0$.

3. Global attractors of nonautonomous damped wave equations

3.1. Well-posedness and dissipativity

We rewrite (1.1)–(1.2) in the unitary abstract form:

$$\partial_t^2 u + Au + \gamma \partial_t u = -f(u) + g^\varepsilon(t), \tag{3.1}$$

and we supplement this equation with the initial conditions given at an arbitrary initial time $\tau \in \mathbb{R}$

$$u|_{t=\tau} = u_\tau, \qquad \partial_t u|_{t=\tau} = p_\tau, \tag{3.2}$$

where $u_\tau \in H^1$ and $p_\tau \in H$ are known data. Then, for every fixed $\varepsilon \in [0,1]$, the initial-value problem (3.1)–(3.2) has a unique solution $u(t)$ such that

$$u \in C_b\big(\mathbb{R}_\tau; H^1\big), \qquad \partial_t u \in C_b(\mathbb{R}_\tau; H),$$

where C_b is the space of bounded continuous functions (see e.g., [2,8,19,25,30]). Adopting, here and in the sequel, the notation:

$$y_\tau = (u_\tau, p_\tau) \quad \text{and} \quad y(t) = \big(u(t), \partial_t u(t)\big),$$

where $u(t)$ is the solution to (3.1)–(3.2), we have that $y \in C_b(\mathbb{R}_\tau; E)$ and $y(\tau) = y_\tau$. Moreover, $y(t)$ satisfies the basic *a priori* estimate:

$$\big\|y(t)\big\|_E^2 \leqslant C\|y_\tau\|_E^d e^{-\beta(t-\tau)} + C\big(1 + Q_\varepsilon^2\big), \quad \forall t \geqslant \tau, \tag{3.3}$$

with Q_ε given by (2.10), for some $\beta > 0$ independent of Q_ε, $\tau \in \mathbb{R}$ and $y_\tau \in E$.

We sketch the proof of (3.3), borrowed from [8] with minor modifications (see also [2,30]). Recalling (1.5), consider the real function:

$$\zeta(t) = \big\|y(t)\big\|^2 + 2\mathcal{F}\big(u(t)\big) + C,$$

where the parameters α and r occurring in the norm $\|y(t)\|$ are defined below. Due to the first inequality in (2.5), it is apparent that $\zeta(t) \geqslant 0$ (for all solutions $u(t)$), provided that we choose C large enough.

Lemma 3.1. *There exist $\alpha > 0$ and $r > 0$ such that the following inequality holds*:

$$\zeta(t) \leqslant 2\zeta(\tau)e^{-\beta(t-\tau)} + C\big(1 + Q_\varepsilon^2\big), \quad \forall t \geqslant \tau, \tag{3.4}$$

where $\beta > 0$ is independent of Q_ε, τ and y_τ.

Proof. Choose $\alpha > 0$ small enough such that

$$\alpha \leqslant \gamma - \alpha, \qquad \alpha(\gamma - \alpha) < \lambda_1, \tag{3.5}$$

and take $r = \alpha(\gamma - \alpha)$. The function $q = \partial_t u + \alpha u$ satisfies:

$$\partial_t q + (\gamma - \alpha)q + Au - \alpha(\gamma - \alpha)u + f(u) = g^\varepsilon.$$

Using the identity,

$$\big\langle f(u), q\big\rangle = \frac{d}{dt}\mathcal{F}(u) + \alpha\big\langle f(u), u\big\rangle,$$

and multiplying the above equation by q, we find the equality:

$$\frac{1}{2}\frac{d}{dt}\zeta + (\gamma - \alpha)|q|^2 + \alpha\big(|u|_1^2 - r|u|^2\big) + \alpha\big\langle f(u), u\big\rangle = \big\langle g^\varepsilon, q\big\rangle.$$

Thus, in light of (3.5),

$$\frac{d}{dt}\zeta + 2\alpha\|y\|^2 + 2\alpha\big\langle f(u), u\big\rangle \leqslant C\big|g^\varepsilon\big|\|y\|.$$

Besides, from (2.6),

$$\langle f(u), u \rangle \geqslant \nu_2 \mathcal{F}(u) - C.$$

Hence,

$$\frac{d}{dt}\zeta + \alpha\nu_2\zeta \leqslant C + C\left|g^\varepsilon\right|\|y\|.$$

Since, from (2.5), $\mathcal{F}(u) \geqslant -C$, setting $\beta = \alpha\nu_2/2$, we finally obtain:

$$\frac{d}{dt}\zeta + 2\beta\zeta \leqslant C\big(1 + \left|g^\varepsilon\right| + \left|g^\varepsilon\right|\zeta^{1/2}\big).$$

Applying Lemma 2.1, and keeping in mind (2.9), we derive (3.4). $\square$

In light of the equivalence of the norms in E, it is clear that (3.3) immediately follows from (3.4). Indeed,

$$\zeta(t) \geqslant \left\|y(t)\right\|^2 - C \geqslant C\big(\left\|y(t)\right\|_E^2 - 1\big),$$

whereas the second inequality in (2.5) and the embedding $H^1 \subset L_d(\Omega)$, $2 < d \leqslant 4$, yield

$$\mathcal{F}(u) \leqslant C\big(|u_\tau|_1^d + 1\big) \leqslant C\big(\left\|y(\tau)\right\|_E^d + 1\big),$$

so that

$$\zeta(\tau) \leqslant C\big(\left\|y(\tau)\right\|_E^d + 1\big).$$

3.2. *The dynamical processes and their attractors*

Our next step is to consider, for every $\varepsilon \in [0, 1]$, the *dynamical process* in the *weak energy space E*

$$\big\{U_\varepsilon(t, \tau)\big\}, \quad t \geqslant \tau,\ \tau \in \mathbb{R},$$

corresponding to problem (3.1)–(3.2). The mappings $U_\varepsilon(t, \tau) : E \to E$ act by the formula:

$$U_\varepsilon(t, \tau)y_\tau = y(t).$$

The *a priori* estimate (3.3), along with (2.9), imply that the process $\{U_\varepsilon(t, \tau)\}$ has a *uniformly* (with respect to $\tau \in \mathbb{R}$) *absorbing set*

$$B^\varepsilon = \big\{y \in E \mid \|y\|_E \leqslant C(1 + Q_\varepsilon)\big\}, \tag{3.6}$$

which, for a fixed ε, is bounded in E. That is, for any bounded set $B \subset E$ of initial data, there is a time $T = T(B, \varepsilon)$ such that

$$U_\varepsilon(t, \tau)B \subseteq B^\varepsilon, \quad \forall\tau \in \mathbb{R},\ \forall t \geqslant \tau + T.$$

Note that the diameter of the absorbing set B^ε grows up to infinity as $\varepsilon \to 0^+$.

We recall that a dynamical process $\{U(t, \tau)\}$ acting on a Banach space X is called *uniformly* (with respect to $\tau \in \mathbb{R}$) *asymptotically compact* if there exists a compact set $P \Subset X$ which is *uniformly attracting*. Namely, for any bounded set $B \subset X$ of initial data,

$$\operatorname{dist}_X\big(U(t, \tau)B, P\big) \to 0 \quad \text{as } t - \tau \to +\infty.$$

Remark 3.2. Assuming the existence of a bounded absorbing set B_0 for a general process $\{U(t, \tau)\}$ in X, a sufficient condition in order to have uniform asymptotic compactness is that the process admits the decomposition $U(t, \tau) = W_1(t, \tau) + W_2(t, \tau)$, where the maps W_1, W_2 (not necessarily processes) satisfy:

$$\left\|W_1(t, \tau)y_\tau\right\|_X \leqslant C\mathrm{e}^{-\beta(t-\tau)},$$
$$\left\|W_2(t, \tau)y_\tau\right\|_Y \leqslant \mathcal{Q}(t - \tau),$$

for all initial data $y_\tau \in B_0$, where $Y \Subset X$, $\beta > 0$ and $\mathcal{Q}$ is some increasing positive function.

Definition 3.3. A closed set $\mathcal{A} \subset X$ is called the *uniform* (with respect to $\tau \in \mathbb{R}$) *global attractor* of the process $\{U(t,\tau)\}$ acting on X if $\mathcal{A}$ is a minimal uniformly attracting set. The minimality property means that $\mathcal{A}$ belongs to any closed uniformly attracting set of the process $\{U(t,\tau)\}$.

In light of a general result from [6–8,21], any asymptotically compact process possesses a (compact) global attractor. For our particular case, we can state the following proposition.

Proposition 3.4. *For any fixed $\varepsilon \in [0,1]$ and every $d < 4$, the process $\{U_\varepsilon(t,\tau)\}$ in E is uniformly asymptotically compact and, therefore, it has the global attractor $\mathcal{A}^\varepsilon$.*

The proof of this assertion can be found in [8] (see also [2,30]).

Remark 3.5. In fact, using the techniques of [1,17,18], it is not hard to prove that the above result is also true for the critical case $d = 4$, provided that the functions $g_0(t)$ and $g_1(t)$ belong to the more regular space $L_1^b(\mathbb{R}; H^\delta)$, for $\delta > 0$; that is,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \{ |g_0(s)|_\delta + |g_1(s)|_\delta \} \, ds < \infty.$$

Remark 3.6. The obvious embedding $\mathcal{A}^\varepsilon \subset B^\varepsilon$, together with (3.6), entail that

$$\|\mathcal{A}^\varepsilon\|_E := \sup\{\|y\|_E \mid y \in \mathcal{A}^\varepsilon\} \leqslant C(1 + Q_\varepsilon).$$

In fact, it is easy to construct an example of Eq. (3.1) with an external force of the form $g^\varepsilon(t)$, with $\rho > 0$, such that

$$\|\mathcal{A}^\varepsilon\|_E \geqslant \varepsilon^{-\rho}, \quad \forall \varepsilon > 0.$$

Thus, the size of the global attractor $\mathcal{A}^\varepsilon$ of Eq. (3.1) with singularly oscillating terms can grow to infinity as the oscillating rate $1/\varepsilon \to +\infty$. Later in this work, we will state some further conditions allowing us to establish the uniform (with respect to $\varepsilon \in [0,1]$) boundedness of $\mathcal{A}^\varepsilon$ in the space E.

3.3. The structure of the global attractor

We describe the structure of the global attractors $\mathcal{A}^\varepsilon$ in the case where both $g_0(t)$ and $g_1(t)$ are *translation compact* in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$. By definition, this means that the sets

$$\mathcal{T}_{g_0} := \{ g_0(t+\tau) \mid \tau \in \mathbb{R} \} \quad \text{and} \quad \mathcal{T}_{g_1} := \{ g_1(t+\tau) \mid \tau \in \mathbb{R} \}$$

are precompact in the space $L_1(-T,T;H)$, for each $T > 0$ (several translation compactness criteria can be found in [8]). It is then apparent that the function $g^\varepsilon(t)$ is also translation compact in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$, for every fixed $\varepsilon \in [0,1]$. Let $\mathcal{H}(g^\varepsilon)$ denote the *hull* of the function $g^\varepsilon(t)$ in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$, defined by:

$$\mathcal{H}(g^\varepsilon) := \left\{ \begin{array}{l} \hat{g} \in L_1^{\mathrm{loc}}(\mathbb{R}; H) \mid \exists \{\tau_n\} \subset \mathbb{R} \text{ such that } g^\varepsilon(t+\tau_n) \to \hat{g}(t) \\ \text{strongly in } L_1(-T,T;H) \text{ as } n \to \infty \text{ for each } T > 0 \end{array} \right\}.$$

It is easy to show (cf. [8]) that if $\hat{g} \in \mathcal{H}(g^\varepsilon)$ and $\varepsilon > 0$, then

$$\hat{g}^\varepsilon(t) = \hat{g}_0(t) + \varepsilon^{-\rho} \hat{g}_1(t/\varepsilon),$$

for some $\hat{g}_0 \in \mathcal{H}(g_0)$ and $\hat{g}_1 \in \mathcal{H}(g_1)$ (the hulls of g_0 and g_1, respectively). Besides,

$$\|\hat{g}\|_{L_1^b} \leqslant \|g^\varepsilon\|_{L_1^b}.$$

Thus, in view of (2.9), we have that

$$\|\hat{g}\|_{L_1^b} \leqslant Q_\varepsilon, \quad \forall \hat{g} \in \mathcal{H}(g^\varepsilon),$$

and the equation

$$\partial_t^2 \hat{u} + A\hat{u} + \gamma \partial_t \hat{u} = -f(\hat{u}) + \hat{g}(t) \tag{3.7}$$

generates a dynamical process in E, that we denote by $\{U_{\hat{g}}(t,\tau)\}$. In which case, inequality (3.3) reads:

$$\left\| U_{\hat{g}}(t,\tau) y_\tau \right\|_E^2 \leqslant C \|y_\tau\|_E^d \mathrm{e}^{-\beta(t-\tau)} + C\left(1 + Q_\varepsilon^2\right), \quad \forall t \geqslant \tau.$$

At this point, we consider the family of processes $\{U_{\hat{g}}(t,\tau)\}$, $\hat{g} \in \mathcal{H}(g^\varepsilon)$. As shown in [8], this family is $(E \times \mathcal{H}(g^\varepsilon), E)$-continuous. Namely, for every fixed pair of times (t,τ), with $t \geqslant \tau$ and $\tau \in \mathbb{R}$, we have that

$$U_{\hat{g}_n}(t,\tau) y_{\tau n} \to U_{\hat{g}}(t,\tau) y_\tau \quad \text{in } E,$$

whenever $y_{\tau n} \to y_\tau$ in E and $\hat{g}_n \to \hat{g}$ in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$. For the (single) process $\{U_{\hat{g}}(t,\tau)\}$, with external force $\hat{g} \in \mathcal{H}(g^\varepsilon)$, we consider the *kernel* $\mathcal{K}_{\hat{g}}$ of the wave equation with *this* external force. Recall that the kernel $\mathcal{K}_{\hat{g}}$ in E is the family of all solutions $\hat{y}(t) = (\hat{u}(t), \partial_t \hat{u}(t))$ to the equation which are defined on the entire time axis $\{t \in \mathbb{R}\}$ (such solutions are called *complete trajectories*) and bounded in E. Formally, this conditions reads: If $\hat{y} \in C_b(\mathbb{R}; E)$ and

$$U_{\hat{g}}(t,\tau)\hat{y}(\tau) = \hat{y}(t), \quad \forall t \geqslant \tau,\ \forall \tau \in \mathbb{R},$$

then $\hat{y} \in \mathcal{K}_{\hat{g}}$. The set $\mathcal{K}_{\hat{g}}(t) = \{\hat{y}(t) \mid \hat{y} \in \mathcal{K}_{\hat{g}}\} \subset E$ is called the *kernel section* at time t.

The following fact is proved in [8].

Proposition 3.7. *Assume that $g_0(t)$ and $g_1(t)$ are translation compact in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$. Then, for every $\varepsilon \in [0,1]$, the global attractor $\mathcal{A}^\varepsilon$ of Eq.* (3.1) *has the form*:

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{g} \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\hat{g}}(0). \tag{3.8}$$

Moreover, for every $\hat{g} \in \mathcal{H}(g^\varepsilon)$, the kernel $\mathcal{K}_{\hat{g}}$ is nonempty.

Remark 3.8. Actually, although for simplicity we assumed the translation compactness of $g_0(t)$ and $g_1(t)$ in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$, the conclusion of Proposition 3.7 holds under weaker conditions. Indeed, in light of the results of the recent paper [29], it is enough to require that the sets $\mathcal{T}_{g_0}$ and $\mathcal{T}_{g_1}$ are compact in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$ with respect to *whatever* metrizable topology. In which case, it suffices to replace $\mathcal{H}(g^\varepsilon)$ in (3.1) with the set,

$$\left\{\hat{g} \in L_1^{\mathrm{loc}}(\mathbb{R}; H) \mid \exists \{\tau_n\} \subset \mathbb{R} \text{ such that } g^\varepsilon(t + \tau_n) \to \hat{g}(t)\right\},$$

where now the convergence takes place in the given metric.

4. On the linear wave equation with oscillating external force

We consider the linear damped wave equation with null initial data given at an initial time $\tau \in \mathbb{R}$

$$\partial_t^2 v + Av + \gamma \partial_t v = k(t), \quad v|_{t=\tau} = 0, \quad \partial_t v|_{t=\tau} = 0.$$

The following lemma is classical.

Lemma 4.1. *If $k \in L_1^{\mathrm{loc}}(\mathbb{R}; H^\sigma)$, for some $\sigma \in \mathbb{R}$, then the above problem has a unique solution $v(t)$ such that*

$$v \in C\left(\mathbb{R}_\tau; H^{\sigma+1}\right), \qquad \partial_t v \in C\left(\mathbb{R}_\tau; H^\sigma\right).$$

Moreover, the inequality

$$\left|v(t)\right|_{\sigma+1} + \left|\partial_t v(t)\right|_\sigma \leqslant C \int_\tau^t \mathrm{e}^{-\beta(t-s)} \left|k(s)\right|_\sigma ds$$

holds for every $t \geqslant \tau$, for some $\beta > 0$, independent of the initial time $\tau \in \mathbb{R}$.

Proof. Reasoning as in the proof of Lemma 3.1, we easily find the inequality

$$\frac{d}{dt}\|\eta\|_\sigma^2 + 2\beta\|\eta\|_\sigma^2 \leqslant C|k|_\sigma\|\eta\|_\sigma,$$

for some $\beta > 0$, where $\eta(t) = (v(t), \partial_t v(t))$. The conclusion is drawn from Lemma 2.1. □

Setting

$$K(t,\tau) = \int_\tau^t k(s)\,ds, \quad t \geqslant \tau,\ \tau \in \mathbb{R},$$

the main result of the section reads as follows.

Proposition 4.2. *Let $k \in L_1^{\mathrm{loc}}(\mathbb{R}; H^{\sigma_0})$, for some $\sigma_0 \in \mathbb{R}$, and assume that*

$$\sup_{t\geqslant\tau,\,\tau\in\mathbb{R}} \left\{ \left|K(t,\tau)\right|_{\sigma-1} + \int_t^{t+1} \left|K(s,\tau)\right|_\sigma ds \right\} \leqslant \ell, \tag{4.1}$$

for some $\ell \geqslant 0$. Then the solution $v(t)$ to the problem

$$\partial_t^2 v + Av + \gamma\partial_t v = k(t/\varepsilon), \quad v|_{t=\tau} = 0, \quad \partial_t v|_{t=\tau} = 0,$$

with $\varepsilon \in (0,1]$, satisfies the inequality

$$\left|v(t)\right|_\sigma + \left|\partial_t v(t)\right|_{\sigma-1} \leqslant C\ell\varepsilon, \quad \forall t \geqslant \tau,$$

where C is independent of k.

Proof. Without loss of generality, we may assume $\tau = 0$. Denoting:

$$V(t) = \int_0^t v(s)\,ds,$$

we have, for any $t \geqslant 0$,

$$\partial_t V(t) = v(t) = \int_0^t \partial_t v(s)\,ds,$$

$$\partial_{tt}^2 V(t) = \partial_t v(t) = \int_0^t \partial_{tt}^2 v(s)\,ds,$$

as $v(0) = 0$ and $\partial_t v(0) = 0$. Integrating the wave equation in time, we see that the function $V(t)$ solves the problem:

$$\partial_t^2 V + AV + \gamma\partial_t V = K_\varepsilon(t), \quad V|_{t=0} = 0, \quad \partial_t V|_{t=0} = 0, \tag{4.2}$$

with external force

$$K_\varepsilon(t) = \int_0^t k(s/\varepsilon)\,ds.$$

Since

$$K_\varepsilon(t) = \varepsilon\int_0^{t/\varepsilon} k(s)\,ds = \varepsilon K(t/\varepsilon, 0),$$

it follows from (4.1) that

$$\sup_{t\geqslant 0}\bigl|K_\varepsilon(t)\bigr|_{\sigma-1}\leqslant \ell\varepsilon, \tag{4.3}$$

and

$$\sup_{t\geqslant 0}\int_t^{t+1}\bigl|K_\varepsilon(s)\bigr|_\sigma\,ds\leqslant 2\ell\varepsilon.$$

Indeed, (4.3) is straightforward, whereas

$$\int_t^{t+1}\bigl|K_\varepsilon(s)\bigr|_\sigma\,ds=\varepsilon^2\int_{t/\varepsilon}^{(t+1)/\varepsilon}\bigl|K(s,0)\bigr|_\sigma\,ds\leqslant\varepsilon^2(1+1/\varepsilon)\sup_{t\geqslant 0}\left\{\int_t^{t+1}\bigl|K(s,0)\bigr|_\sigma\,ds\right\}\leqslant 2\ell\varepsilon.$$

Accordingly, by (2.11),

$$\int_0^t \mathrm{e}^{-\beta(t-s)}\bigl|K_\varepsilon(s)\bigr|_\sigma\,ds\leqslant C\ell\varepsilon,$$

and applying Lemma 4.1 to $V(t)$, we obtain:

$$\bigl|V(t)\bigr|_{\sigma+1}+\bigl|\partial_t V(t)\bigr|_\sigma\leqslant C\int_0^t \mathrm{e}^{-\beta(t-s)}\bigl|K_\varepsilon(s)\bigr|_\sigma\,ds\leqslant C\ell\varepsilon.$$

In particular,

$$\bigl|v(t)\bigr|_\sigma=\bigl|\partial_t V(t)\bigr|_\sigma\leqslant C\ell\varepsilon.$$

Besides, on account of (4.2),

$$\bigl|\partial_t v(t)\bigr|_{\sigma-1}=\bigl|\partial_t^2 V(t)\bigr|_{\sigma-1}\leqslant\bigl|AV(t)\bigr|_{\sigma-1}+\gamma\bigl|\partial_t V(t)\bigr|_{\sigma-1}+\bigl|K_\varepsilon(t)\bigr|_{\sigma-1}.$$

But

$$\bigl|AV(t)\bigr|_{\sigma-1}=\bigl|V(t)\bigr|_{\sigma+1}\leqslant C\ell\varepsilon,$$
$$\bigl|\partial_t V(t)\bigr|_{\sigma-1}\leqslant C\bigl|\partial_t V(t)\bigr|_\sigma\leqslant C\ell\varepsilon,$$

while, from (4.3),

$$\bigl|K_\varepsilon(t)\bigr|_{\sigma-1}\leqslant\ell\varepsilon.$$

Therefore, the desired estimate follows. □

Remark 4.3. Condition (4.1) is satisfied, for instance, if $k\in L_\infty(\mathbb{R};H^{\sigma-1})\cap L_1^{\mathrm{loc}}(\mathbb{R};H^\sigma)$ is a time periodic function of period $T>0$ having zero mean, that is,

$$\int_0^T k(s)\,ds=0.$$

Other examples of quasiperiodic and almost periodic in time functions satisfying (4.1) can be found in [6,8].

Remark 4.4. A sufficient condition in order for (4.1) to hold is to require that

$$\sup_{t\in\mathbb{R}}\bigl|K(t)\bigr|_\sigma\leqslant\frac{\ell\lambda_1^{1/2}}{2+2\lambda_1^{1/2}},$$

having set

$$K(t)=\int_0^t k(s)\,ds,\quad t\in\mathbb{R}.$$

Indeed, just note that $K(t,\tau)=K(t)-K(\tau)$ and use the Poincaré inequality.

Remark 4.5. If we assume only,

$$\sup_{t\geqslant\tau,\,\tau\in\mathbb{R}}\int_t^{t+1}\left|K(s,\tau)\right|_\sigma ds\leqslant\ell,$$

then, recasting the proof of the proposition, it is immediate to verify that $v(t)$ satisfies the weaker inequality:

$$\left|v(t)\right|_\sigma+\int_t^{t+1}\left|\partial_t v(s)\right|_{\sigma-1}ds\leqslant C\ell\varepsilon,\quad\forall t\geqslant\tau.$$

5. Uniform boundedness of the global attractors $\mathcal{A}^\varepsilon$

We now provide some conditions ensuring the uniform (with respect to $\varepsilon\in[0,1]$) boundedness of the global attractors $\mathcal{A}^\varepsilon$ of the nonautonomous wave equation (3.1) constructed in Section 3. These conditions relate to the function g_1 which introduces singular oscillations in the external force $g_0(t)+\varepsilon^{-\rho}g_1(t/\varepsilon)$ of the equation.

Setting

$$G_1(t,\tau)=\int_\tau^t g_1(s)\,ds,\quad t\geqslant\tau,$$

our main assumption reads:

$$\sup_{t\geqslant\tau,\,\tau\in\mathbb{R}}\left\{\left|G_1(t,\tau)\right|_{\vartheta-1}+\int_t^{t+1}\left|G_1(s,\tau)\right|_\vartheta ds\right\}\leqslant\ell,\tag{5.1}$$

for some $\ell\geqslant 0$, where

$$\vartheta=\vartheta(d)=\begin{cases}1 & \text{if } 2\leqslant d\leqslant 3,\\ 3(1-2/d) & \text{if } 3<d<4,\\ 3/2+\delta\quad(\delta>0) & \text{if } d=4.\end{cases}$$

Theorem 5.1. *Let $G_1(t,\tau)$ satisfy* (5.1). *In addition, for the critical case $d=4$, assume $\rho<1$. Then, the global attractors $\mathcal{A}^\varepsilon$ of the dissipative wave equations* (3.1) *with external force $g^\varepsilon(t)$ are uniformly* (*with respect to $\varepsilon\in[0,1]$*) *bounded in the weak energy space E, that is,*

$$\sup_{\varepsilon\in[0,1]}\left\|\mathcal{A}^\varepsilon\right\|_E<\infty.$$

We carry out the proof for $d>2$, leaving the much easier case $d=2$ to the reader.

Proof of the case $d<4$. As usual, let $y(t)=(u(t),\partial_t u(t))$, $t\geqslant\tau$, be the solution to (3.1) with initial data $y(\tau)=y_\tau=(u_\tau,p_\tau)\in E$. We consider the auxiliary linear wave equation with null initial data:

$$\partial_t^2 v+Av+\gamma\partial_t v=\varepsilon^{-\rho}g_1(t/\varepsilon),\quad v|_{t=\tau}=0,\quad\partial_t v|_{t=\tau}=0.\tag{5.2}$$

On account of Lemma 4.1 and Proposition 4.2, this problem admits a unique solution $\eta(t)=(v(t),\partial_t v(t))$ satisfying the inequality:

$$\left|v(t)\right|_\vartheta+\left|\partial_t v(t)\right|_{\vartheta-1}\leqslant C\ell\varepsilon^{1-\rho},\quad\forall t\geqslant\tau.\tag{5.3}$$

We now define the function

$$w(t) = u(t) - v(t),$$

which clearly satisfies the equation

$$\partial_t^2 w + Aw + \gamma \partial_t w = -f(w) - \big(f(w+v) - f(w)\big) + g_0(t),$$

with initial conditions:

$$w|_{t=\tau} = u_\tau, \quad \partial_t w|_{t=\tau} = p_\tau.$$

Calling

$$\omega(t) = \big(w(t), \partial_t w(t)\big),$$

arguing exactly as in Lemma 3.1, we obtain the inequality:

$$\frac{d}{dt}\xi + 2\alpha\|\omega\|^2 + 2\alpha\langle f(w), w\rangle \leqslant C|g_0|\|\omega\| - \langle f(w+v) - f(w), q\rangle,$$

where we set

$$\xi(t) = \big\|\omega(t)\big\|^2 + 2\mathcal{F}\big(w(t)\big) + C,$$

with C large enough such that $\xi(t) \geqslant 0$, and

$$q(t) = \partial_t w(t) + \alpha w(t).$$

Here, $\|\omega\|$ is given by (1.5), α satisfies (3.5) and $r = \alpha(\gamma - \alpha) < \lambda_1$. Exploiting (2.5)–(2.6), we readily see that

$$2\alpha\langle f(w), w\rangle \geqslant \alpha\nu_2\mathcal{F}(w) + \alpha\nu_1\nu_2\|w\|_{L_d}^d - C,$$

and

$$C|g_0|\,\|\omega\| \leqslant C|g_0|\big(1 + \xi^{1/2}\big).$$

Moreover, from the Cauchy inequality,

$$-\langle f(w+v) - f(w), q\rangle \leqslant \alpha\|\omega\|^2 + C\big|f(w+v) - f(w)\big|^2.$$

Hence, setting $\beta_1 = \alpha\nu_2/4$ and $\nu_3 = \alpha\nu_1\nu_2$, we end up with

$$\frac{d}{dt}\xi + 2\beta_1\xi + \nu_3\|w\|_{L_d}^d \leqslant C\big(1 + |g_0| + |g_0|\xi^{1/2}\big) + C\big|f(w+v) - f(w)\big|^2. \tag{5.4}$$

We claim that

$$C\big|f(w+v) - f(w)\big|^2 \leqslant \nu_3\|w\|_{L_d}^d + C\big(1 + \ell^{2d/(4-d)}\big). \tag{5.5}$$

Indeed, from (2.1),

$$\big|f(a+b) - f(a)\big| \leqslant C\big(1 + |a|^{d-2} + |b|^{d-2}\big)|b|, \quad \forall a, b \in \mathbb{R}.$$

Therefore, we have the estimate:

$$\big|f(w+v) - f(w)\big|^2 \leqslant C\bigg(\int_\Omega \big|w(x,\cdot)\big|^{2(d-2)}\big|v(x,\cdot)\big|^2\,dx + \|v\|_{L_{2(d-1)}}^{2(d-1)} + |v|^2\bigg).$$

Setting

$$p_1 = \frac{d}{2(d-2)} > 1, \qquad p_2 = \frac{d}{4-d} > 1,$$

and observing that $0 < 2(d-2) < d$, using the Hölder inequality with exponents p_1 and p_2 and the Young inequality, we get:

$$\int_\Omega \big|w(x,\cdot)\big|^{2(d-2)}\big|v(x,\cdot)\big|^2\,dx \leqslant \|w\|_{L_d}^{2(d-2)}\|v\|_{L_{2d/(4-d)}}^2 \leqslant \nu_3\|w\|_{L_d}^d + C\|v\|_{L_{2d/(4-d)}}^{2d/(4-d)},$$

as $0 < 2(d-2) < d$. Besides,

$$\|v\|_{L_{2(d-1)}}^{2(d-1)} + |v|^2 \leqslant C\big(\|v\|_{L_{2d/(4-d)}}^{2(d-1)} + \|v\|_{L_{2d/(4-d)}}^2\big),$$

as $2 < 2d/(4-d)$ and $2(d-1) < 2d/(4-d)$. Combining the above estimates, we obtain:

$$\big|f(w+v) - f(w)\big|^2 \leqslant \nu_3 \|w\|_{L_d}^d + C\big(1 + \|v\|_{L_{2d/(4-d)}}^{2d/(4-d)}\big). \tag{5.6}$$

The Sobolev embedding theorem entails that

$$H^\vartheta \subset H^{3(1-2/d)} \subset L_{2d/(4-d)}(\Omega).$$

Consequently, by (5.3),

$$\|v\|_{L_{2d/(4-d)}} \leqslant C\ell,$$

and plugging this inequality into (5.6), the claim (5.5) is proved.

Collecting (5.4)–(5.5), we draw the differential inequality:

$$\frac{d}{dt}\xi + 2\beta_1 \xi \leqslant C\big(1 + \ell^{2d/(4-d)} + |g_0| + |g_0|\xi^{1/2}\big).$$

Then, recalling (2.7), Lemma 2.1 yields

$$\xi(t) \leqslant 2\xi(\tau)\mathrm{e}^{-\beta_1(t-\tau)} + C\big(1 + \ell^{2d/(4-d)} + M_0^2\big), \quad \forall t \geqslant \tau.$$

Since $\omega(\tau) = y(\tau) = y_\tau$, arguing as in the final part of the proof of Lemma 3.1, we obtain the estimate, similar to (3.3) but *independent* of ε,

$$\big\|\omega(t)\big\|_E^2 \leqslant C\|y_\tau\|_E^d \mathrm{e}^{-\beta_1(t-\tau)} + C\big(1 + \ell^{2d/(4-d)} + M_0^2\big),$$

for all $t \geqslant \tau$, $\tau \in \mathbb{R}$.

To complete the proof, we note that, from (5.3) and the embedding $E^{\vartheta-1} \subseteq E$,

$$\big\|\eta(t)\big\|_E^2 \leqslant \big\|\eta(t)\big\|_{E^{\vartheta-1}}^2 = \big|v(t)\big|_\vartheta^2 + \big|\partial_t v(t)\big|_{\vartheta-1}^2 \leqslant C\ell^2, \quad \forall t \geqslant \tau.$$

Thus, $y(t) = \omega(t) + \eta(t)$ fulfills the inequality:

$$\big\|y(t)\big\|_E^2 \leqslant C\|y_\tau\|_E^d \mathrm{e}^{-\beta_1(t-\tau)} + C\big(1 + \ell^{2d/(4-d)} + M_0^2\big), \tag{5.7}$$

for all $t \geqslant \tau$, $\tau \in \mathbb{R}$. This means that the dynamical process $\{U_\varepsilon(t,\tau)\}$ possesses the absorbing set,

$$B_\star = \big\{\|y\|_E \leqslant C\big(1 + \ell^{d/(4-d)} + M_0\big)\big\},$$

for all $\varepsilon \in [0,1]$. As a byproduct,

$$\mathcal{A}^\varepsilon \subset B_\star, \quad \forall \varepsilon \in [0,1],$$

and the desired conclusion is established for the subcritical case $d < 4$. □

Remark 5.2. In fact, we proved a further result for the case $2 \leqslant d < 3$. Namely, if the function $G_1(t,\tau)$ satisfies (5.1) with

$$\vartheta = 3(1 - 2/d) < 1,$$

then the global attractors $\mathcal{A}^\varepsilon$ are uniformly bounded in the space $E^{\vartheta-1}$ that is,

$$\sup_{\varepsilon\in[0,1]} \big\|\mathcal{A}^\varepsilon\big\|_{E^{\vartheta-1}} < \infty.$$

Proof of the case $d = 4$. Here, contrary to the previous situation, we are no longer in a position to obtain an estimate like (5.5). Instead, since now $\vartheta = 3/2 + \delta$, using the embedding $H^\vartheta \subset L_\infty(\Omega)$ (see [26,30]), inequality (5.3) (here, $\rho < 1$) implies that

$$\|v\|_{L_\infty} \leqslant C\ell\varepsilon^{1-\rho},$$

which, together with the obvious estimate,

$$\left|f(w+v)-f(w)\right|^2 \leqslant C\|w\|_{L_4}^4\|v\|_{L_\infty}^2 + C\left(\|v\|_{L_\infty}^6 + \|v\|_{L_\infty}^2\right),$$

yield the control

$$C\left|f(w+v)-f(w)\right|^2 \leqslant C_0\varepsilon^{2(1-\rho)}\|w\|_{L_4}^4 + C\left(1+\ell^6\right),$$

for some $C_0 = C\ell^2$. Hence, in light of (5.4),

$$\frac{d}{dt}\xi + 2\beta_1\xi + \left(\nu_3 - C_0\varepsilon^{2(1-\rho)}\right)\|w\|_{L_4}^4 \leqslant C\left(1+\ell^6+|g_0|+|g_0|\xi^{1/2}\right),$$

so that, choosing $\varepsilon_0 = \varepsilon_0(\ell,\rho)$ as

$$\varepsilon_0 := \min\left\{1, (\nu_3/C_0)^{1/2(1-\rho)}\right\},$$

we have:

$$\left(\nu_3 - C_0\varepsilon^{2(1-\rho)}\right) \geqslant 0, \quad \forall\varepsilon \leqslant \varepsilon_0,$$

and the inequality

$$\frac{d}{dt}\xi + 2\beta_1\xi \leqslant C\left(1+\ell^6+|g_0|+|g_0|\xi^{1/2}\right)$$

holds for all $\varepsilon \leqslant \varepsilon_0$. In which case, we repeat the argument of the previous proof, so establishing the existence of the absorbing set

$$B_0 = \left\{\|y\|_E \leqslant C\left(1+\ell^3+M_0\right)\right\},$$

for $\varepsilon \leqslant \varepsilon_0$. On the other hand, if $\varepsilon_0 < \varepsilon \leqslant 1$, owing to (3.6), we have also the absorbing set B^{ε_0}. In conclusion, for all $\varepsilon \in [0,1]$, we found the bounded absorbing set,

$$B_\star := B_0 \cup B^{\varepsilon_0},$$

independent of ε. Then, $\mathcal{A}^\varepsilon \subset B_\star$, for all $\varepsilon \in [0,1]$. □

Remark 5.3. In the critical case $d=4$, the radius of the absorbing set $B_\star$ goes to infinity as $\rho \to 1$. Thus, if $d=4$, the boundedness of the global attractors $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0^+$ remains an open problem.

Remark 5.4. Assuming in Theorem 5.1 that $G_1(t,\tau)$ fulfills, in place of (5.1), the weaker inequality:

$$\sup_{t\geqslant\tau,\,\tau\in\mathbb{R}} \int_t^{t+1} \left|G_1(s,\tau)\right|_\vartheta ds \leqslant \ell,$$

we have that, for every $\varepsilon \in [0,1]$,

$$\sup\left\{|u_\tau|_1 \mid (u_\tau, p_\tau) \in \mathcal{A}^\varepsilon\right\} \leqslant C_\ell,$$

and

$$\sup_{(u,\partial_t u)\in\mathcal{K}^\varepsilon} \left\{\left|u(t)\right|_1^2 + \int_t^{t+1} \left|\partial_t u(s)\right|^2 ds\right\} \leqslant C_\ell, \quad \forall t \in \mathbb{R},$$

for some $C_\ell \geqslant 0$ depending on ℓ, where $\mathcal{K}^\varepsilon$ is the kernel of the process $\{U_\varepsilon(t,\tau)\}$.

6. Convergence of the global attractors $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0^+$

The main result of the paper is the following:

Theorem 6.1. *Let $\rho < 1$, and let $g_0(t)$ and $g_1(t)$ be translation compact in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$. Besides, let $G_1(t,\tau)$ satisfy* (5.1). *Then, the global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ with respect to the Hausdorff semidistance in E as $\varepsilon \to 0^+$, that is,*

$$\lim_{\varepsilon \to 0^+} \{\mathrm{dist}_E(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} = 0.$$

The remaining of the section is devoted to the proof of the theorem. In the sequel, the generic constant C may depend on ℓ.

Remark 6.2. As it will be clear from the proof, the result is still true if we replace the term $\varepsilon^{-\rho}$ appearing in the function g^ε with $\varepsilon^{-1}\mu(\varepsilon)$, where $\mu(\varepsilon)$ is any nonnegative function such that $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Our first task is to compare the solutions to (3.1) corresponding to $\varepsilon > 0$ and $\varepsilon = 0$, respectively, starting from the same initial data. To this end, let us denote by $u^\varepsilon(t)$ the solution to (3.1) corresponding to $\varepsilon \in [0,1]$, with initial conditions

$$u^\varepsilon|_{t=\tau} = u_\tau, \qquad \partial_t u^\varepsilon|_{t=\tau} = p_\tau,$$

where $y_\tau = (u_\tau, p_\tau)$ belongs to the absorbing ball $B_\star$ in E found in Section 5. In particular, owing to (5.7) (or (3.3) and (5.7) if $d = 4$), we have the uniform bound:

$$\left|u^\varepsilon(t)\right|_1 \leqslant R_0, \quad \forall t \geqslant \tau. \tag{6.1}$$

Note that $R_0 = R_0(\rho)$ if $d = 4$. For $\varepsilon > 0$, the function,

$$w(t) = u^\varepsilon(t) - u^0(t),$$

satisfies:

$$\partial_t^2 w + Aw + \gamma \partial_t w = -f(u^\varepsilon) + f(u^0) + \varepsilon^{-\rho} g_1(t/\varepsilon), \quad w|_{t=\tau} = 0, \quad \partial_t w|_{t=\tau} = 0.$$

Hence, letting $v(t)$ be the solution to (5.2), the function

$$q(t) = w(t) - v(t)$$

clearly solves the problem:

$$\partial_t^2 q + \gamma \partial_t q + Aq = -\big(f(u^\varepsilon) - f(u^0)\big), \quad q|_{t=\tau} = 0, \quad \partial_t q|_{t=\tau} = 0.$$

Taking the scalar product with $\partial_t q$, we obtain:

$$\frac{d}{dt}\{|q|_1^2 + |\partial_t q|^2\} + 2\gamma|\partial_t q|^2 \leqslant 2\left|f(u^\varepsilon) - f(u^0)\right||\partial_t q| \leqslant 2\gamma|\partial_t q|^2 + C\left|f(u^\varepsilon) - f(u^0)\right|^2.$$

Exploiting (2.1), (6.1), the Hölder inequality and the embedding $L_6(\Omega) \subset H^1$, we see at once that

$$\left|f(u^\varepsilon) - f(u^0)\right| \leqslant C\big(\|u^\varepsilon\|_{L_6}^2 + \|u^0\|_{L_6}^2 + 1\big)\big(\|q\|_{L_6} + \|v\|_{L_6}\big) \leqslant C\big(1 + R_0^2\big)\big(|q|_1 + |v|_1\big).$$

On the other hand, by (5.3) and the fact that $\vartheta \geqslant 1$,

$$\left|v(t)\right|_1 \leqslant C\varepsilon^{1-\rho}, \quad \forall t \geqslant \tau. \tag{6.2}$$

Combining the two estimates,

$$C\left|f(u^\varepsilon) - f(u^0)\right|^2 \leqslant C\big(1 + R_0^4\big)|q|_1^2 + C\big(1 + R_0^4\big)\varepsilon^{2(1-\rho)}.$$

Thus, setting $R_1 = 1 + R_0^4$, we end up with the inequality:

$$\frac{d}{dt}\{|q|_1^2 + |\partial_t q|^2\} \leqslant CR_1\{|q|_1^2 + |\partial_t q|^2\} + CR_1\varepsilon^{2(1-\rho)}.$$

Since $q(\tau)=\partial_t q(\tau)=0$, the Gronwall lemma leads to

$$\left|q(t)\right|_1^2+\left|\partial_t q(t)\right|^2\leqslant \varepsilon^{2(1-\rho)}\mathrm{e}^{CR_1(t-\tau)},\quad \forall t\geqslant\tau.$$

Finally, for the function $w(t)=q(t)+v(t)$, using again (5.3), we have:

$$\left|w(t)\right|_1^2+\left|\partial_t w(t)\right|^2\leqslant C\varepsilon^{2(1-\rho)}\mathrm{e}^{CR_1(t-\tau)},\quad \forall t\geqslant\tau.$$

In conclusion, we proved the following result.

Lemma 6.3. *The deviation,*

$$\omega(t)=y^\varepsilon(t)-y^0(t)=\left(u^\varepsilon(t)-u^0(t),\partial_t u^\varepsilon(t)-\partial_t u^0(t)\right),$$

with $y^\varepsilon(\tau)=y^0(\tau)=y_\tau\in B_\star$, satisfies the inequality:

$$\left\|\omega(t)\right\|_E\leqslant D\varepsilon^{1-\rho}\mathrm{e}^{R(t-\tau)},\quad \forall t\geqslant\tau,$$

where the positive constants D and R are independent of ε, τ and $y_\tau\in B_\star$. If $d=4$, then $R=R(\rho)$.

In order to study the convergence of the global attractors $\mathcal{A}^\varepsilon$ of the wave equation (3.1) as $\varepsilon\to 0^+$, we actually need a generalization of Lemma 6.3, which applies to the whole family of Eqs. (3.7), with external forces $\hat g=\hat g^\varepsilon\in\mathcal{H}(g^\varepsilon)$. To this end, we state first a general result.

Lemma 6.4. *Let $g(t)$ be translation compact in $L_1^{\mathrm{loc}}(\mathbb{R};H)$, and let $G(t,\tau)=\int_\tau^t g(s)\,ds$ satisfy, for some $\sigma\in\mathbb{R}$, the inequality*:

$$\sup_{t\geqslant\tau,\,\tau\in\mathbb{R}}\left\{\left|G(t,\tau)\right|_{\sigma-1}+\int_t^{t+1}\left|G(s,\tau)\right|_\sigma\,ds\right\}\leqslant\ell.$$

Then, for every $\hat g$ belonging to the hull of g, the function $\hat G(t,\tau)=\int_\tau^t\hat g(s)\,ds$ satisfies the same inequality.

Proof. Let $t\geqslant\tau$ be fixed. Since $\hat g\in\mathcal{H}(g)$, by definition, there is a real sequence $\{\tau_n\}$ such that

$$g(\cdot+\tau_n)\to\hat g(\cdot)\quad\text{in } L_1^{\mathrm{loc}}(\mathbb{R};H),$$

as $n\to\infty$. Thus, setting, for $s\in[t,t+1]$,

$$\psi_n(s)=G(s+\tau_n,\tau+\tau_n),\qquad \hat\psi(s)=\hat G(s,\tau),$$

we have that

$$\sup_{s\in[t,t+1]}\left|\psi_n(s)-\hat\psi(s)\right|\leqslant\int_\tau^{t+1}\left|g(r+\tau_n)-\hat g(r)\right|dr\to 0,$$

as $n\to\infty$, i.e.,

$$\psi_n\to\hat\psi\quad\text{in } L_\infty(t,t+1;H).$$

On the other hand, we know that ψ_n is bounded by ℓ in the space:

$$W_t=L_\infty(t,t+1;H_{\sigma-1})\cap L_1(t,t+1;H_\sigma).$$

Then, by a standard argument of functional analysis (see, e.g., [24]), we conclude that $\hat\psi\in W_t$ and $\|\hat\psi\|_{W_t}\leqslant\ell$. In particular, we learn that

$$\left|\hat G(t,\tau)\right|_{\sigma-1}+\int_t^{t+1}\left|\hat G(s,\tau)\right|_\sigma\,ds\leqslant\ell.$$

Since $t\geqslant\tau$ and $\tau\in\mathbb{R}$ are arbitrary, we are done. □

For any $\varepsilon \in [0,1]$, let

$$\hat{y}^\varepsilon(t) = \big(\hat{u}^\varepsilon(t), \partial_t \hat{u}^\varepsilon(t)\big) = U_{\hat{g}^\varepsilon}(t,\tau) y_\tau$$

be the solution to (3.7) with external force $\hat{g}^\varepsilon = \hat{g}_0 + \varepsilon^{-\rho}\hat{g}_1(\cdot/\varepsilon) \in \mathcal{H}(g^\varepsilon)$ and $y_\tau \in B_\star$. For $\varepsilon > 0$, we consider the deviation

$$\hat{\omega}(t) = \hat{y}^\varepsilon(t) - \hat{y}^0(t).$$

Corollary 6.5. *We have the inequality*:

$$\big\|\hat{\omega}(t)\big\|_E \leqslant D\varepsilon^{1-\rho} \mathrm{e}^{R(t-\tau)}, \quad \forall t \geqslant \tau,$$

for some positive constants D and R independent of ε, τ, $y_\tau \in B_\star$ and $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. If $d = 4$, then $R = R(\rho)$.

Proof. We observe that (6.1) keeps holding for $\hat{u}^\varepsilon(t)$, as the family $\{U_{\hat{g}^\varepsilon}(t,\tau)\}$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(E \times \mathcal{H}(g^\varepsilon), E)$-continuous (see Section 3). Moreover, if $\varepsilon > 0$,

$$\hat{g}^\varepsilon(t) = \hat{g}_0(t) + \varepsilon^{-\rho}\hat{g}_1(t/\varepsilon),$$

with $\hat{g}_1 \in \mathcal{H}(g_1)$. Hence, due to Lemma 6.4, inequality (6.2) is valid for the corresponding solution $\hat{v}^\varepsilon(t)$ of the linear problem (5.2) with external force $\hat{g}_1(t)$ in place of $g_1(t)$, since the primitive $\hat{G}_1(t,\tau)$ of $\hat{g}_1(t)$ fulfills a condition of the form (5.1). At this point, we simply repeat the proof of Lemma 6.3. □

We are now ready to complete:

Proof of Theorem 6.1. For $\varepsilon > 0$, let y^ε be an arbitrary element of the attractor $\mathcal{A}^\varepsilon$. The representation formula (3.8) of $\mathcal{A}^\varepsilon$ implies the existence of a complete bounded trajectory $\hat{y}^\varepsilon(t)$ of Eq. (3.7), with some external force:

$$\hat{g}^\varepsilon = \hat{g}_0 + \varepsilon^{-\rho}\hat{g}_1(\cdot/\varepsilon) \in \mathcal{H}\big(g^\varepsilon\big), \quad \hat{g}_0 \in \mathcal{H}(g_0), \quad \hat{g}_1 \in \mathcal{H}(g_1)$$

such that

$$\hat{y}^\varepsilon(0) = y^\varepsilon.$$

For some $L \geqslant 0$ to be specified later, consider the vector:

$$y_L := \hat{y}^\varepsilon(-L) \in \mathcal{A}^\varepsilon \subset B_\star.$$

Then, applying Corollary 6.5 with $\tau = -L$, we have that

$$\big\|U_{\hat{g}^\varepsilon}(t,-L)y_L - U_{\hat{g}_0}(t,-L)y_L\big\|_E \leqslant D\varepsilon^{1-\rho}\mathrm{e}^{R(t+L)}, \quad \forall t \geqslant -L. \tag{6.3}$$

On the other hand (see [8]), the set $\mathcal{A}^0$ attracts $U_{\hat{g}_0}(t,\tau)B_\star$, uniformly not only with respect to $\tau \in \mathbb{R}$, but also with respect to $\hat{g}_0 \in \mathcal{H}(g^0)$. Thus, for every $\nu > 0$, there is $T = T(\nu) \geqslant 0$, independent of L and $y_L \in B_\star$ such that

$$\mathrm{dist}_E\big(U_{\hat{g}_0}(t,-L)y_L, \mathcal{A}^0\big) \leqslant \nu, \quad \forall t \geqslant T - L. \tag{6.4}$$

Collecting the two above inequalities, we readily get:

$$\mathrm{dist}_E\big(U_{\hat{g}^\varepsilon}(t,-L)y_L, \mathcal{A}^0\big) \leqslant D\varepsilon^{1-\rho}\mathrm{e}^{R(t+L)} + \nu, \quad \forall t \geqslant T - L.$$

Setting now $L = T$ and choosing $t = 0$, from the simple observation that

$$U_{\hat{g}^\varepsilon}(0,-L)y_L = U_{\hat{g}^\varepsilon}(0,-L)\hat{y}^\varepsilon(-L) = \hat{y}^\varepsilon(0) = y^\varepsilon,$$

we conclude that

$$\mathrm{dist}_E\big(y^\varepsilon, \mathcal{A}^0\big) \leqslant D\varepsilon^{1-\rho}\mathrm{e}^{RT} + \nu.$$

Therefore, as $y^\varepsilon \in \mathcal{A}^\varepsilon$ is arbitrary,

$$\limsup_{\varepsilon\to 0^+}\big\{\mathrm{dist}_E\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big)\big\} \leqslant \nu.$$

But $\nu > 0$ is also arbitrary, and this provides the desired conclusion. □

Remark 6.6. With a similar proof, we can extend the result formulated in Remark 5.2 for the case $2 \leqslant d < 3$ as follows. If the function $G_1(t,\tau)$ satisfies (5.1) with

$$\vartheta = 3(1 - 2/d) < 1,$$

then the global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ in the space $E^{\vartheta-1}$. Namely,

$$\lim_{\varepsilon\to 0^+} \big\{\mathrm{dist}_{E^{\vartheta-1}}\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big)\big\} = 0.$$

The details are left to the reader.

Remark 6.7. It is worth noting that the translation compactness of $g_0(t)$ and $g_1(t)$ in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$ in the proof of Theorem 6.1 is used only to apply Proposition 3.7. Thus, in view of Remark 3.8, the hypotheses on $g_0(t)$ and $g_1(t)$ can be weakened.

In the next example, we show that the conclusion of Theorem 6.1 is false if $\rho = 1$.

Example 6.8. Let e be the normalized eigenvector corresponding to a given eigenvalue λ of A, and consider the equation:

$$\partial_t^2 u + Au + \gamma\partial_t u = -\varepsilon^{-1}\sin(t/\varepsilon)e,$$

which is a particular case of (3.1), with $f \equiv g_0 \equiv 0$, $g_1(t) = -\sin(t)$ and $\rho = 1$. Here,

$$G_1(t,\tau) = \big(\cos(t) - \cos(\tau)\big)e,$$

which clearly satisfies (5.1) (with $\vartheta = 1$). This equation admits a unique complete bounded trajectory, given by:

$$\xi(t) = \big(\bar u(t), \partial_t \bar u(t)\big) = \big(\Lambda(t)e, \Lambda'(t)e\big),$$

where

$$\Lambda(t) = \frac{(\varepsilon - \lambda\varepsilon^3)\sin(t/\varepsilon) + \gamma\varepsilon^2\cos(t/\varepsilon)}{1 + (\gamma^2 - 2\lambda)\varepsilon^2 + \lambda^2\varepsilon^4},$$

and

$$\Lambda'(t) = \frac{(1 - \lambda\varepsilon^2)\cos(t/\varepsilon) - \gamma\varepsilon\sin(t/\varepsilon)}{1 + (\gamma^2 - 2\lambda)\varepsilon^2 + \lambda^2\varepsilon^4}.$$

Since $g_1(t)$ is 2π-periodic,

$$\mathcal{H}\big(g^\varepsilon\big) = \bigcup_{\tau\in[0,2\pi\varepsilon)} g^\varepsilon(\cdot + \tau).$$

Hence, from the representation (3.8) of the attractor $\mathcal{A}^\varepsilon$, we conclude that

$$\mathcal{A}^\varepsilon = \bigcup_{\tau\in[0,2\pi\varepsilon)} \xi(-\tau).$$

Observe that, for ε small,

$$\bar u(t) \sim \varepsilon\sin(t/\varepsilon)e, \qquad \partial_t \bar u(t) \sim \cos(t/\varepsilon)e.$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \big\|\mathcal{A}^\varepsilon\big\|_E = 1.$$

On the other hand, the linear homogeneous equation,

$$\partial_t^2 u + Au + \gamma\partial_t u = 0,$$

is well known to generate an exponentially stable linear semigroup. This implies that $\mathcal{A}^0 = \{0\}$. In particular, the convergence $\mathrm{dist}_E(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0$ cannot occur.

Remark 6.9. Incidentally, the above example also shows that the constraint $\rho \leqslant 1$ is essential. Indeed, if $\rho > 1$, it is clear that the uniform boundedness of $\mathcal{A}^\varepsilon$ is not to be expected anymore.

7. Hölder continuity at $\varepsilon = 0$ of $\mathcal{A}^\varepsilon$

In this final section, we provide an explicit estimate of the form $M\varepsilon^\eta$ for the Hausdorff semidistance $\operatorname{dist}_E(\mathcal{A}^\varepsilon, \mathcal{A}^0)$, assuming that the global attractor $\mathcal{A}^0$ is exponential. We begin with a definition.

Definition 7.1. The global attractor $\mathcal{A}$ of a dynamical process $\{U(t,\tau)\}$ in the space X is said to be *exponential with rate* $\varkappa > 0$, if there is an increasing positive function $\mathcal{Q}$ such that, for any bounded set B in X,

$$\operatorname{dist}_X\big(U(t,\tau)B, \mathcal{A}\big) \leqslant \mathcal{Q}\big(\|B\|_X\big)\mathrm{e}^{-\varkappa(t-\tau)}, \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R}.$$

Then, we have the following result.

Theorem 7.2. *Let the assumptions of Theorem* 6.1 *hold. Besides, assume that the global attractor* $\mathcal{A}^0$ *is exponential with rate* $\varkappa > 0$. *Then,*

$$\operatorname{dist}_E\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant M\varepsilon^\eta, \tag{7.1}$$

where $M = M(g_0, g_1, \rho) \geqslant 0$ *and* $\eta = \varkappa(1-\rho)/(\varkappa + R)$, *with* R *as in Corollary* 6.5.

Proof. We preliminary observe (see [8]) that, if $\mathcal{A}^0$ is exponential with rate $\varkappa > 0$, then

$$\operatorname{dist}_E\big(U_{\hat{g}^0}(t,\tau)B_\star, \mathcal{A}^0\big) \leqslant D_\star \mathrm{e}^{-\varkappa(t-\tau)}, \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R},$$

for some $D_\star = \mathcal{Q}(\|B_\star\|_E) > 0$, uniformly as $\hat{g}^0 \in \mathcal{H}(g^0)$. Thus, we can repeat the proof of Theorem 6.1, replacing now (6.4) by the explicit estimate

$$\operatorname{dist}_E\big(U_{\hat{g}^0}(t,-L)y_L, \mathcal{A}^0\big) \leqslant D_\star \mathrm{e}^{-\varkappa(t+L)}, \quad \forall t \geqslant -L.$$

For $t = 0$, this inequality, along with (6.3), immediately give:

$$\operatorname{dist}_E\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant D\varepsilon^{1-\rho}\mathrm{e}^{RL} + D_\star \mathrm{e}^{-\varkappa L}.$$

Since the above estimate holds for every $L \geqslant 0$, setting:

$$\eta = \varkappa(1-\rho)/(\varkappa + R),$$

and letting L be such that

$$\varepsilon^{1-\rho}\mathrm{e}^{RL} = \mathrm{e}^{-\varkappa L},$$

we eventually obtain:

$$\operatorname{dist}_E\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant M\varepsilon^\eta,$$

with $M = D + D_\star$. $\square$

Remark 7.3. Again (cf. Remark 6.6), if $2 \leqslant d < 3$ and $G_1(t,\tau)$ satisfies assumption (5.1) with $\vartheta = 3(1 - 2/d) < 1$, then we have the estimate:

$$\operatorname{dist}_{E^{\vartheta-1}}\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant M\varepsilon^\eta.$$

We conclude the section by presenting two examples of averaged wave equation whose global attractors $\mathcal{A}^0$ are exponential. The function $g_1(t)$ appearing in the sequel is supposed to comply with (5.1).

7.1. Autonomous wave equations with regular attractors

Assume that the function g_0 is independent of time, that is, $g_0(t) \equiv g_0 \in H$. Then, the wave equation (3.1) for $\varepsilon = 0$ is autonomous, and the map,

$$S(t) := U_{g_0}(t, 0), \quad t \geqslant 0,$$

is a strongly continuous semigroup in E, due to the translation invariance property (cf. [8])

$$U_{g_0}(t,\tau) = U_{g_0}(t-\tau,0), \quad t \geqslant \tau.$$

In that case, the global attractor $\mathcal{A}^0$ is strictly invariant for the semigroup, i.e.,

$$S(t)\mathcal{A}^0 = \mathcal{A}^0, \quad \forall t \geqslant 0,$$

and there exists the global Lyapunov function,

$$\mathcal{L}(y^0) = |u^0|_1^2 + |\partial_t u^0|^2 + 2\mathcal{F}(u^0) - 2\langle g_0, u^0\rangle,$$

satisfying, for every solution $y^0(t) = (u^0(t), \partial_t u^0(t))$ to (3.1), the identity

$$\mathcal{L}(y^0(t)) - \mathcal{L}(y^0(\tau)) = -\gamma \int_\tau^t |\partial_t u^0(s)|^2\, ds.$$

If we further assume that the stationary equation

$$Aw + f(w) = g_0$$

has a finite number of solutions $\{w_1, \ldots, w_N\}$ in H^1, and each w_i is hyperbolic (see, e.g., [2,19]), then $\mathcal{A}^0$ is the union of the unstable manifolds $\mathcal{M}^u(w_i)$ issuing from w_i. Namely,

$$\mathcal{A}^0 = \bigcup_{i=1}^N \mathcal{M}^u(w_i).$$

Besides, the global attractor $\mathcal{A}^0$ is exponential with some rate $\varkappa > 0$. Finally, adopting the trajectory approximation method given in [32], it is possible to show that, for ε small, the global attractor $\mathcal{A}^\varepsilon$ is also exponential, with some rate $\varkappa_0 \leqslant \varkappa$. Using this fact, it is not hard to prove that an inequality of the form (7.1) holds as well for the *symmetric Hausdorff distance*, that is,

$$\operatorname{dist}_E^{\mathrm{sym}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \operatorname{dist}_E(\mathcal{A}^\varepsilon, \mathcal{A}^0) + \operatorname{dist}_E(\mathcal{A}^0, \mathcal{A}^\varepsilon) \leqslant M_0 \varepsilon^{\eta_0},$$

for some $M_0 \geqslant M$ and $\eta_0 \leqslant \eta$.

7.2. Sine-Gordon type equations with a flat nonlinearity

Let $d = 2$, i.e.,

$$|f'(u)| \leqslant K,$$

for some $K \geqslant 0$. Moreover, let

$$K < \lambda_1 \quad \text{and} \quad \gamma^2 > 2\big(\lambda_1 - \sqrt{\lambda_1^2 - K^2}\big).$$

Under the above assumptions, exploiting the techniques of [12], one can prove that, for every function $g \in L_1^b(\mathbb{R}; H)$, the wave equation (3.7) with external force $g(t)$ has a unique global solution $\xi_g \in C_b(\mathbb{R}; E)$. In other words, the kernel $\mathcal{K}_g$ consists of the unique element ξ_g. Besides, this solution is exponentially stable; namely, there is a constant $\varkappa > 0$, independent of $g \in L_1^b(\mathbb{R}; H)$, such that, for every $y_\tau \in E$,

$$\|U_g(t,\tau)y_\tau - \xi_g(t)\|_E \leqslant \mathcal{Q}(\|y_\tau\|_E)\mathrm{e}^{-\varkappa(t-\tau)}, \quad \forall t \geqslant \tau,$$

for some increasing positive function $\mathcal{Q}$, depending on $\|g\|_{L_1^b}$. Taking the external force $g(t) = g^\varepsilon(t)$ translation compact in $L_1^{\mathrm{loc}}(\mathbb{R}; H)$, we conclude that, for every $\varepsilon \in [0,1]$, the global attractors $\mathcal{A}^\varepsilon$ of the processes $\{U_\varepsilon(t,\tau)\}$ have the form:

$$\mathcal{A}^\varepsilon = \{\xi_{\hat{g}^\varepsilon}(0) \mid \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)\},$$

and they are exponential with rate $\varkappa$. Indeed, since

$$g_t^\varepsilon := g^\varepsilon(\cdot - t) \in \mathcal{H}(g^\varepsilon), \quad \forall t \in \mathbb{R},$$

it is apparent that

$$\mathrm{dist}_E\big(U_\varepsilon(t,\tau)y_\tau, \mathcal{A}^\varepsilon\big) \leqslant \big\| U_\varepsilon(t,\tau)y_\tau - \xi_{g_t^\varepsilon}(0)\big\|_E = \big\| U_\varepsilon(t,\tau)y_\tau - \xi_{g^\varepsilon}(t)\big\|_E.$$

Along the lines of [12], it can also be shown that the deviation estimate of Corollary 6.5 holds with $R = 0$. Thus, for every $y_\tau \in B_\star$,

$$\big\| U_{\hat{g}^\varepsilon}(t,\tau)y_\tau - U_{\hat{g}^0}(t,\tau)y_\tau\big\|_E \leqslant D\varepsilon^{1-\rho}, \quad \forall t \geqslant \tau.$$

Recasting the proof of Theorem 7.2 accordingly, we find that

$$\mathrm{dist}_E\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant M\varepsilon^{1-\rho}.$$

Note that the constant $\eta = 1 - \rho$ is independent of the exponential rate $\varkappa$. Finally, since the global attractors $\mathcal{A}^\varepsilon$ are exponential with rate $\varkappa$, it can be shown that the above inequality holds for the symmetric Hausdorff distance as well, that is,

$$\mathrm{dist}_E^{\mathrm{sym}}\big(\mathcal{A}^\varepsilon, \mathcal{A}^0\big) \leqslant M\varepsilon^{1-\rho},$$

as in the previous example.

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