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Extended Cesáro operators from generally weighted Bloch spaces to Zygmund space

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ABSTRACT

Let g be a holomorphic function of the unit ball B in the n-dimensional complex space, and denote by T_g the extended Cesáro operator with symbol g. Starting with a brief introduction to well-known results about Cesáro operator, we investigate the boundedness and compactness of T_g from generally weighted Bloch spaces $\mathcal{B}^{\alpha}_{log}$ $(0<\alpha<\infty)$ to Zygmund space $\mathcal{Z}$ in the unit ball, and also present some necessary and sufficient conditions.

1. Introduction

For any $z=(z_1,\dots,z_n)$, $w=(w_1,\dots,w_n)\in C^n$, the inner product is defined by $\langle z,w\rangle=\sum_{k=1}^n z_k\bar{w}_k$. Let B be the unit ball of C^n, the class of all holomorphic functions on B is defined by $H(B)$. For $f\in H(B)$, we write

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1}(z),\dots,\frac{\partial f}{\partial z_n}(z)\right)\quad\text{and}\quad Rf(z)=\langle\nabla f(z),\bar{z}\rangle.$$

Let H^∞ denote the space of all bounded holomorphic functions on the unit ball, equipped with the norm $\|f\|_\infty=\sup_{z\in B}|f(z)|$.

For any $0<\alpha<\infty$, we define the generally weighted Bloch space $\mathcal{B}^{\alpha}_{log}$ as the space of holomorphic functions such that

$$\|f\|_\alpha=\big|f(0)\big|+\sup\left\{\left(1-|z|^2\right)^\alpha\log\frac{4}{1-|z|^2}\big|\nabla f(z)\big|:\ z\in B\right\}<\infty.$$

It is well known that the α-Bloch space $\mathcal{B}^\alpha$ and little α-Bloch space $\mathcal{B}^\alpha_0$ are defined respectively as the space of holomorphic functions such that

$$\|f\|^{\alpha}_{Bloch}=\sup\left\{\left(1-|z|^2\right)^\alpha\big|Rf(z)\big|:\ z\in B\right\}<\infty$$

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and

$$\left\{f \in \mathcal{B}^{\alpha}\colon \lim_{|z|\to 1}\left(1-|z|^2\right)^{\alpha}\left|Rf(z)\right| = 0\right\}$$

with the same norm $\|f\|_{\mathcal{B}^{\alpha}} = |f(0)| + \|f\|^{\alpha}_{Bloch}$.

The Zygmund space $\mathcal{Z}$ in the unit ball consists of those functions whose first order partial derivatives are in the Bloch space $\mathcal{B}^1$. It is well known that (Theorem 7.11 in [24]) $f \in \mathcal{Z}$ if and only if $Rf \in \mathcal{B}^1$, and $\mathcal{Z}$ is a Banach space with the norm

$$\|f\| = \left|f(0)\right| + \|Rf\|^1_{Bloch}.$$

Let $f(z)$ be a holomorphic function on the unit disc D with $Taylor$ expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is defined by

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j} a_k\right) z^j.$$

Despite the simplicity of the definition of $\mathcal{C}[f](z)$, several problems are encountered when characterizing the boundedness and compactness of Cesáro operator between spaces of holomorphic functions. These problems require profound and interesting analytical machinery. Moreover, the study of Cesáro operator has arguably become a major driving force in the development of modern complex analysis. The papers listed in the bibliography are excellent sources for recent developments in the theory of Cesáro operators. It is well known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$ and Bergman space, as well as the Dirichlet space, for the interested readers, we refer to see the papers [1,4,9–18,20–22] and so on.

But the operator $\mathcal{C}$ is not always bounded, in [19], Shi and Ren gave a sufficient and necessary condition for the operator $\mathcal{C}$ to be bounded on mixed norm spaces in the unit disc. It is natural to ask what are the conditions for higher-dimensional case.

A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z}\int_0^z f(t)(\log\frac{1}{1-t})'\,dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesáro operator (also called Volterra-type operator or Riemann–Stieltjes type operator) T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz)Rg(tz)\frac{dt}{t}.$$

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesáro operator is bounded or compact.

The boundedness and compactness of this operator on weighted Bergman, mixed norm, Hardy, Bloch and Dirichlet spaces in the unit ball have been studied by Xiao [21], Hu [6–8], Zhang [23], Guo and Ren [5], Chang and Stević [2]. In this paper, we will continue this line of research and characterize those g for which T_g is bounded (or compact) from generally weighted Bloch spaces $\mathcal{B}^{\alpha}_{log}$ $(0 < \alpha < \infty)$ to Zygmund space $\mathcal{Z}$. For the proof, we need different method and some complex calculation skills.

2. Some lemmas

In the following, we will use the symbol C to denote a finite positive number which does not depend on variable z and f. In order to prove the main results, we will give some lemmas first.

Lemma 1. *Suppose $f \in \mathcal{B}^{\alpha}_{log}$, for any $z \in B$ we have*

(a) *If $0 < \alpha < 1$, then $|f(z)| \leqslant (1 + \frac{1}{(1-\alpha)\log 4})\|f\|_{\alpha}$;*
(b) *If $\alpha = 1$, then $|f(z)| \leqslant C(\log\log\frac{4}{1-|z|^2})\|f\|_1$;*
(c) *If $\alpha > 1$, then $|f(z)| \leqslant (1 + A(|z|))\|f\|_{\alpha}$, where $A(|z|) = \int_0^{|z|}\frac{du}{(1-u^2)^{\alpha}\log\frac{4}{1-u^2}}$.*

Proof.

$$\left|f(z)\right| = \left|f(0) + \int_0^1 \left\langle\nabla f(tz), \bar{z}\right\rangle dt\right|$$

$$\leqslant \left|f(0)\right| + \int_0^1 \frac{|z|\,dt}{(1-|tz|^2)^\alpha \log \frac{4}{1-|tz|^2}} \cdot \|f\|_\alpha$$
$$\leqslant \|f\|_\alpha + \int_0^{|z|} \frac{du}{(1-u^2)^\alpha \log \frac{4}{1-u^2}} \cdot \|f\|_\alpha.$$

It is obvious for the case $\alpha > 1$.

If $0 < \alpha < 1$, we obtain

$$\left|f(z)\right| \leqslant \|f\|_\alpha + \frac{1}{(1-\alpha)\log 4}\|f\|_\alpha.$$

If $\alpha = 1$, by Schwarz inequality and $\log \frac{4}{1-x} \leqslant 2\log \frac{4}{1-x^2}$ for $x \in [0,1)$, it follows that

$$\left|f(z)\right| \leqslant \|f\|_1 + 2\int_0^{|z|} \frac{du}{(1-u)\log \frac{4}{1-u}} \cdot \|f\|_1$$
$$= \left[2\log\log \frac{4}{1-|z|} + 1 - 2\log\log 4\right]\|f\|_1$$
$$\leqslant \left[2\log\left(2\log \frac{4}{1-|z|^2}\right) + 1 - 2\log\log 4\right]\|f\|_1$$
$$\leqslant \left[2\log\log \frac{4}{1-|z|^2} + 2\log 2 + 1 - 2\log\log 4\right]\|f\|_1.$$

The case $\alpha = 1$ follows by the estimate

$$2\log 2 + 1 - 2\log\log 4 \leqslant C\log\log 4 \leqslant C\log\log \frac{4}{1-|z|^2}. \qquad \square$$

By Lemma 1, Montel theorem and the definition of compact operator, the following lemma follows.

Lemma 2. *Assume that $g \in H(B)$. Then $T_g : \mathcal{B}^\alpha_{log} \to \mathcal{Z}$ is compact if and only if T_g is bounded and for any bounded sequence $(f_k)_{k\in N}$ in $\mathcal{B}^\alpha_{log}$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, $\|T_g f_k\| \to 0$ as $k \to \infty$.*

Lemma 3. *If $(f_k)_{k\in N}$ is a bounded sequence in $\mathcal{B}^\alpha_{log}(0 < \alpha < 1)$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, then we have $\lim_{k\to\infty} \sup_{z\in B} |f_k(z)| = 0$.*

Proof. It is obvious that $\|f\|_{\mathcal{B}^\alpha} < 2\|f\|_\alpha$, so the bounded sequence in $\mathcal{B}^\alpha_{log}(0 < \alpha < 1)$ is also bounded in $\mathcal{B}^\alpha$, and the conclusion follows since the proposition is true for the space $\mathcal{B}^\alpha$, see [23] for the details. $\square$

Lemma 4. *(See [3, Lemma 4].) Let $g \in H(B)$, then*

$$R[T_g f](z) = f(z)Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

The following lemma can be found in [24].

Lemma 5. *Let $f \in \mathcal{B}^1$, then*

$$\left|f(z)\right| \leqslant C\log \frac{4}{1-|z|^2}\|f\|_{\mathcal{B}^1}$$

for any $z \in B$. Furthermore, if $f \in \mathcal{B}^1_0$, then

$$\lim_{|z|\to 1}\left|f(z)\right|\left\{\log \frac{4}{1-|z|^2}\right\}^{-1} = 0.$$

Lemma 6. *Let $f \in \mathcal{Z}$, then*

$$\left|Rf(z)\right| \leqslant C \log \frac{4}{1-|z|^2} \|f\|$$

for any $z \in B$.

Proof. Since $f \in \mathcal{Z}$ implies $Rf \in \mathcal{B}^1$, by Lemma 5, and note that $Rf(0) = 0$ we have

$$\left|Rf(z)\right| \leqslant C \log \frac{4}{1-|z|^2} \|Rf\|_{\mathcal{B}^1} = C \log \frac{4}{1-|z|^2} \|Rf\|^1_{Bloch} \leqslant C \log \frac{4}{1-|z|^2} \|f\|. \qquad \square$$

3. The boundedness and compactness for the case $0 < \alpha < 1$

Theorem 1. *Let $g \in H(B)$, $0 < \alpha < 1$. Then the following statements are equivalent*:

(a) $T_g : \mathcal{B}^\alpha_{log} \to \mathcal{Z}$ *is bounded*;
(b) $T_g : \mathcal{B}^\alpha_{log} \to \mathcal{Z}$ *is compact*;
(c) $g \in \mathcal{Z}$.

Proof. (b) $\Rightarrow$ (a) is obvious.

Note that $RRT_g f = Rf \cdot Rg + f \cdot RRg$, (a) $\Rightarrow$ (c) follows by taking the test function $f = 1$, which is in $\mathcal{B}^\alpha_{log}$.

Next we show (c) $\Rightarrow$ (b). Assume $(f_k)_{k\in N}$ is a sequence in $\mathcal{B}^\alpha_{log}$ such that $\sup_{k\in N} \|f_k\|_\alpha \leqslant M$ and that $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. For any given $\epsilon > 0$, it is clear that there exists a $\delta \in (0,1)$ such that

$$\left(1-|z|^2\right)^{1-\alpha} < \epsilon,$$

whenever $\delta < |z| < 1$. Now let $\Omega = \{z \in B\colon\ |z| \leqslant \delta\}$ and note that $T_g f_k(0) = 0$, we can get

$$\begin{aligned}
\|T_g f_k\| &= \sup_{z\in B}\left(1-|z|^2\right)\left|RR(T_g f_k)(z)\right| \\
&= \sup_{z\in B}\left(1-|z|^2\right)\left|Rf_k(z)\cdot Rg(z) + f_k(z)\cdot R(Rg)(z)\right| \\
&\leqslant \sup_{z\in B}\left(1-|z|^2\right)\left(\left|Rf_k(z)\cdot Rg(z)\right| + \left|f_k(z)\cdot R(Rg)(z)\right|\right) \\
&\leqslant \sup_{z\in \Omega}\left(1-|z|^2\right)\left|Rg(z)\right|\cdot\left|Rf_k(z)\right| + \sup_{z\in B-\Omega}\left(1-|z|^2\right)\left|Rg(z)\right|\cdot\left|Rf_k(z)\right| + \|g\|\cdot\sup_{z\in B}\left|f_k(z)\right| \\
&\leqslant C\sup_{z\in \Omega}\left(1-|z|^2\right)\left|Rf_k(z)\right|\cdot\log\frac{4}{1-|z|^2}\|g\| + C\sup_{z\in B-\Omega}\left(1-|z|^2\right)^{1-\alpha}\|f_k\|_\alpha\cdot\|g\| + \|g\|\cdot\sup_{z\in B}\left|f_k(z)\right|.
\end{aligned}$$

The last inequality follows by Lemma 6. With the uniform convergence of f_k and the Cauchy estimates, then owing to Lemmas 3 and 2, the conclusion follows by letting $k \to \infty$. $\square$

4. The boundedness and compactness for the case $\alpha = 1$

Theorem 2. *Suppose $g \in H(B)$, $\alpha = 1$. Then $T_g : \mathcal{B}^1_{log} \to \mathcal{Z}$ is bounded if and only if*

$$\sup_{z\in B}\left(1-|z|^2\right)\log\log\frac{4}{1-|z|^2}\left|RRg(z)\right| < +\infty.$$

Proof. Note the condition $\sup_{z\in B}(1-|z|^2)\log\log\frac{4}{1-|z|^2}|RRg(z)| < +\infty$ implies $g \in \mathcal{Z}$. The sufficiency follows by the estimate

$$\begin{aligned}
\|T_g f\| &\leqslant \sup_{z\in B}\left(1-|z|^2\right)\left(\left|Rf(z)\cdot Rg(z)\right| + \left|f(z)\cdot R(Rg)(z)\right|\right) \\
&\leqslant \sup_{z\in B}\left(1-|z|^2\right)\left|Rg(z)\right|\cdot\frac{\|f\|_1}{(1-|z|^2)\log\frac{4}{1-|z|^2}} + C\sup_{z\in B}\left(1-|z|^2\right)\log\log\frac{4}{1-|z|^2}\left|R(Rg)(z)\right|\cdot\|f\|_1 \\
&\leqslant C\|f\|_1\cdot\|g\| + \sup_{z\in B}\left(1-|z|^2\right)\log\log\frac{4}{1-|z|^2}\left|RRg(z)\right|\cdot\|f\|_1.
\end{aligned}$$

To prove the converse, suppose T_g is bounded, by taking the test function $f = 1$, we get $g = T_g 1 \in \mathcal{Z}$.

Setting $f_w(z) = \log\log\frac{4}{1-\langle z,w\rangle}$ for any $w \in B$, an easy calculation shows that $f_w \in \mathcal{B}^1_{log}$, and $M = \sup_{w\in B}\|f_w\|_1 < \infty$. In fact, for any $1 \leqslant j \leqslant n$, we have

$$\frac{\partial f_w}{\partial z_j}(z) = \frac{w_j}{(1-\langle z,w\rangle)\log\frac{4}{1-\langle z,w\rangle}}.$$

From which it follows that

$$\left(1-|z|^2\right)\log\frac{4}{1-|z|^2}\left|\frac{\partial f_w}{\partial z_j}(z)\right| \leqslant 2\left(1-|z|\right)\log\frac{4}{1-|z|}\frac{1}{|1-\langle z,w\rangle|\cdot\log\frac{4}{|1-\langle z,w\rangle|}}.$$

For any $z \in Q$ with $Q = \{z \in B\colon\ |1-\langle z,w\rangle| \geqslant 1\}$, since $t\log\frac{4}{t}$ is increasing on $(0,1]$, we obtain

$$\sup_{z\in Q}\left(1-|z|^2\right)\log\frac{4}{1-|z|^2}\left|\frac{\partial f_w}{\partial z_j}(z)\right| \leqslant \frac{2\log 4}{\log 2} = 4. \tag{1}$$

And for $z \in B - Q$, also owing to $t\log\frac{4}{t}$ is increasing on $(0,1]$, we have

$$\left|1-\langle z,w\rangle\right|\log\frac{4}{|1-\langle z,w\rangle|} \geqslant \left(1-|z|\right)\log\frac{4}{1-|z|}$$

and then

$$\sup_{z\in B-Q}\left(1-|z|^2\right)\log\frac{4}{1-|z|^2}\left|\frac{\partial f_w}{\partial z_j}(z)\right| \leqslant 2. \tag{2}$$

Combining (1) and (2), we obtain

$$\sup_{z\in B}\left(1-|z|^2\right)\log\frac{4}{1-|z|^2}\left|\frac{\partial f_w}{\partial z_j}(z)\right| \leqslant 4.$$

With the obvious estimate $|\nabla f(z)| \leqslant \sum_{j=1}^n|\frac{\partial f_w}{\partial z_j}(z)|$, it follows that

$$\begin{aligned}\infty &> \|T_g\|\cdot\|f_w\|_1 \geqslant \|T_g f_w\| \\ &= \sup_{z\in B}\left(1-|z|^2\right)\left|Rf_w(z)\cdot Rg(z) + f_w(z)RRg(z)\right| \\ &\geqslant \left(1-|w|^2\right)\left|Rf_w(w)\cdot Rg(w) + f_w(w)RRg(w)\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(1-|w|^2\right)\log\log\frac{4}{1-|w|^2}\left|RRg(w)\right| &\leqslant \left(1-|w|^2\right)\left|Rf_w(w)\cdot Rg(w)\right| + \|T_g\|\cdot\|f_w\|_1 \\ &\leqslant \left(1-|w|^2\right)\frac{|w|^2}{(1-|w|^2)\log\frac{4}{1-|w|^2}}\left|Rg(w)\right| + C \\ &\leqslant C\|g\| + C.\end{aligned}$$

Since w is arbitrary, the proof of this theorem is completed. □

Theorem 3. *Suppose $g \in H(B)$, $\alpha = 1$. Then $T_g : \mathcal{B}^1_{log} \to \mathcal{Z}$ is compact if and only if*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\log\log\frac{4}{1-|z|^2}\left|RRg(z)\right| = 0. \tag{3}$$

Proof. We prove the sufficiency first. Assume condition (3) holds, we get $Rg \in \mathcal{B}^1_0$. By Lemma 5 and (3), for any given $\epsilon > 0$, there exists a δ $(0 < \delta < 1)$, such that both

$$\left(1-|z|^2\right)\log\log\frac{4}{1-|z|^2}\left|RRg(z)\right| < \epsilon \quad\text{and}\quad \left|Rg(z)\right|\left\{\log\frac{4}{1-|z|^2}\right\}^{-1} < \epsilon$$

whenever $\delta < |z| < 1$. Let $K = \{z \in B\colon\ |z| \leqslant \delta\}$, and for any sequence $(f_k)_{k\in N}$ with $\sup_{k\in N}\|f_k\|_1 \leqslant C$ and $f_k \to 0$ uniformly on compact subsets of B. Notice that $T_g f(0) = 0$, then

$$\begin{aligned}\|T_g f_k\| &= \sup_{z\in B}\big(1-|z|^2\big)\big|Rf_k(z)\cdot Rg(z)+f_k(z)\cdot R(Rg)(z)\big| \\ &\leqslant \sup_{z\in B}\big(1-|z|^2\big)\big(\big|Rf_k(z)\cdot Rg(z)\big|+\big|f_k(z)\cdot R(Rg)(z)\big|\big) \\ &\leqslant \sup_{z\in K}\big(1-|z|^2\big)\big|Rg(z)\big|\cdot\big|Rf_k(z)\big|+\sup_{z\in B-K}\big(1-|z|^2\big)\big|Rg(z)\big|\cdot\big|Rf_k(z)\big|+\sup_{z\in K}\big(1-|z|^2\big)\big|RRg(z)\big|\cdot\big|f_k(z)\big| \\ &\quad + C\sup_{z\in B-K}\big(1-|z|^2\big)\log\log\frac{4}{1-|z|^2}\big|RRg(z)\big|\cdot\|f_k\|_1 \\ &\leqslant C\sup_{z\in K}\big(1-|z|^2\big)\big|Rf_k(z)\big|\cdot\log\frac{4}{1-|z|^2}\|g\|+\sup_{z\in B-K}\frac{|Rg(z)|}{\log\frac{4}{1-|z|^2}}\|f_k\|_1+\sup_{z\in K}\big|f_k(z)\big|\cdot\|g\|+C\epsilon.\end{aligned}$$

With the uniform convergence of f_k and Cauchy estimates we get $\|T_g f_k\|\to 0$ as $k\to\infty$. Owing to Lemma 2, T_g is compact.

Now we turn to prove the necessity. For any given sequence $(z_k)_{k\in N}$ in B with $|z_k|\to 1$ as $k\to\infty$, we set

$$h_k(z)=\left(\log\log\frac{4}{1-|z_k|^2}\right)^{-1}\left(\log\log\frac{4}{1-\langle z,z_k\rangle}\right)^2.$$

An easy calculation shows that $\sup_k\|h_k\|_1<\infty$, and $h_k\to 0$ uniformly on compact subsets of B, as in Theorem 2, we have

$$\begin{aligned}\|T_g h_k\| &= \sup_{z\in B}\big(1-|z|^2\big)\big|Rh_k(z)\cdot Rg(z)+h_k(z)RRg(z)\big| \\ &\geqslant \big(1-|z_k|^2\big)\big|Rh_k(z_k)\cdot Rg(z_k)+h_k(z_k)RRg(z_k)\big|.\end{aligned}$$

Therefore,

$$\begin{aligned}\big(1-|z_k|^2\big)\log\log\frac{4}{1-|z_k|^2}\big|RRg(z_k)\big| &\leqslant \big(1-|z_k|^2\big)\big|Rh_k(z_k)\cdot Rg(z_k)\big|+\|T_g h_k\| \\ &\leqslant \frac{2|z_k|^2}{\log\frac{4}{1-|z_k|^2}}\big|Rg(z_k)\big|+\|T_g h_k\|. \end{aligned}\tag{4}$$

By Lemma 2, we just need to show

$$\lim_{k\to\infty}\big|Rg(z_k)\big|\left\{\log\frac{4}{1-|z_k|^2}\right\}^{-1}=0. \tag{5}$$

For this purpose, set

$$f_k(z)=\frac{1-|z_k|^2}{(1-\langle z,z_k\rangle)\log\frac{4}{1-\langle z,z_k\rangle}}-\frac{(1-|z_k|^2)^2}{(1-\langle z,z_k\rangle)^2\log\frac{4}{1-\langle z,z_k\rangle}},$$

then $f_k(z_k)=0$, f_k satisfy the conditions in Lemma 2, and a little calculation shows that $Rf_k(z_k)=\frac{|z_k|^2}{1-|z_k|^2}\{\log\frac{4}{1-|z_k|^2}\}^{-1}$. Therefore

$$\begin{aligned}\|T_g f_k\| &= \sup_{z\in B}\big(1-|z|^2\big)\big|Rf_k(z)\cdot Rg(z)+f_k(z)RRg(z)\big| \\ &\geqslant \big(1-|z_k|^2\big)\big|Rf_k(z_k)\cdot Rg(z_k)+f_k(z_k)RRg(z_k)\big| \\ &= |z_k|^2\big|Rg(z_k)\big|\left\{\log\frac{4}{1-|z_k|^2}\right\}^{-1}.\end{aligned}$$

Letting $k\to\infty$, (5) follows. And by (4), we obtain

$$\lim_{k\to\infty}\big(1-|z_k|^2\big)\log\log\frac{4}{1-|z_k|^2}\big|RRg(z_k)\big|=0.$$

Since the sequence z_k is arbitrary, we complete the proof of the theorem. □

5. The boundedness and compactness for the case $\alpha > 1$

Theorem 4. *Suppose $g \in H(B)$, $\alpha > 1$. Then $T_g : \mathcal{B}^{\alpha}_{log} \to \mathcal{Z}$ is bounded if and only if following conditions hold:*

(a) $\sup_{z\in B}(1-|z|^2)^{1-\alpha}\{\log\frac{4}{1-|z|^2}\}^{-1}|Rg(z)| < \infty$;
(b) $\sup_{z\in B}(1-|z|^2)A(|z|)|RRg(z)| < \infty$;
(c) $g \in \mathcal{Z}$.

Proof. Since we have

$$\|T_g f\| \leqslant \sup_{z\in B}\big(1-|z|^2\big)\big(\big|Rf(z)\cdot Rg(z)\big| + \big|f(z)\cdot R(Rg)(z)\big|\big)$$
$$\leqslant \sup_{z\in B}\big(1-|z|^2\big)\bigg[\frac{|Rg(z)|}{(1-|z|^2)^{\alpha}\log\frac{4}{1-|z|^2}}\|f\|_{\alpha} + \big|RRg(z)\big|\big(1+A\big(|z|\big)\big)\cdot\|f\|_{\alpha}\bigg]$$

then the conditions (a), (b) and (c) imply T_g is bounded.

Now we turn to the necessity. Condition (c) follows by taking the test function $f = 1$, which is in $\mathcal{B}^{\alpha}_{log}$.

Next we show if T_g is bounded, condition (a) must hold. In fact, for $w \in B$ with $|w| \leqslant \frac{1}{2}$, note that $g \in \mathcal{Z}$, by Lemma 6, we have

$$\sup_{|w|\leqslant\frac{1}{2}}\big(1-|w|^2\big)^{1-\alpha}\bigg\{\log\frac{4}{1-|w|^2}\bigg\}^{-1}\big|Rg(w)\big| \leqslant C\sup_{|w|\leqslant\frac{1}{2}}\big(1-|w|^2\big)^{1-\alpha}\|g\| \leqslant C.$$

For $w \in B$ with $|w| > \frac{1}{2}$, set

$$h_w(z) = \frac{1-|w|^2}{(1-\langle z,w\rangle)^{\alpha}\log\frac{4}{1-\langle z,w\rangle}} - \frac{(1-|w|^2)^2}{(1-\langle z,w\rangle)^{\alpha+1}\log\frac{4}{1-\langle z,w\rangle}}.$$

An easy calculation shows that $h_w \in \mathcal{B}^{\alpha}_{log}$, $\sup_{1>|w|>\frac{1}{2}}\|h_w\|_{\alpha} < \infty$, $h_w(w) = 0$ and $Rh_w(w) = \frac{|w|^2}{(1-|w|^2)^{\alpha}\log\frac{4}{1-|w|^2}}$. Therefore,

$$\|T_g h_w\| = \sup_{z\in B}\big(1-|z|^2\big)\big|RRT_g h_w(z)\big|$$
$$\geqslant \big(1-|w|^2\big)\big|Rh_w(w)Rg(w)\big|$$
$$\geqslant \frac{1}{4}\big(1-|w|^2\big)^{1-\alpha}\bigg\{\log\frac{4}{1-|w|^2}\bigg\}^{-1}\big|Rg(w)\big|.$$

Since w is arbitrary, we obtain

$$\sup_{1>|w|>\frac{1}{2}}\big(1-|w|^2\big)^{1-\alpha}\bigg\{\log\frac{4}{1-|w|^2}\bigg\}^{-1}\big|Rg(w)\big| \leqslant C. \tag{6}$$

Combining (6) and (7), condition (a) follows.

Finally we show the condition (b) holds. We also discuss it in two cases.

For $|w| \leqslant \frac{1}{2}$, we have

$$\sup_{|w|\leqslant\frac{1}{2}}\big(1-|w|^2\big)A\big(|w|\big)\big|RRg(w)\big| \leqslant C\|g\|. \tag{7}$$

For $1 > |w| > \frac{1}{2}$, set

$$f_w(z) = \int_0^{\langle z,w\rangle}\frac{dt}{(1-t)^{\alpha}\log\frac{4}{1-t}}.$$

Then $\frac{\partial f}{\partial z_j}(z) = \frac{w_j}{(1-\langle z,w\rangle)^{\alpha}\log\frac{4}{1-\langle z,w\rangle}}$, and $Rf_w(z) = \frac{\langle z,w\rangle}{(1-\langle z,w\rangle)^{\alpha}\log\frac{4}{1-\langle z,w\rangle}}$.

A little calculation shows that $f_w \in \mathcal{B}^{\alpha}_{log}$ and $\sup_{w\in B}\|f_w\|_{\alpha} < \infty$. Since

$$\|T_g f_w\| = \sup_{z\in B}\big(1-|z|^2\big)\big|RRT_g f_w(z)\big| \geqslant \big(1-|w|^2\big)\big|Rf_w(w)Rg(w) + f_w(w)RRg(w)\big|,$$

it follows that

$$\left(1-|w|^2\right)\left|RRg(w)\right| \int_0^{|w|^2} \frac{dt}{(1-t)^\alpha \log \frac{4}{1-t}} \leqslant \|T_g f_w\| + \left(1-|w|^2\right)\left|Rf_w(w)Rg(w)\right| < \infty.$$

Note that there is a positive constant $0 < C < 1$ such that

$$\int_0^{\frac{1}{2}} \frac{2t\,dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}} > C \int_0^{\frac{1}{2}} \frac{dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}}$$

and then

$$\begin{aligned}
\int_0^{|w|^2} \frac{dt}{(1-t)^\alpha \log \frac{4}{1-t}} &= \int_0^{|w|} \frac{2t\,dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}} \\
&= \int_0^{\frac{1}{2}} \frac{2t\,dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}} + \int_{\frac{1}{2}}^{|w|} \frac{2t\,dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}} \\
&\geqslant C \int_0^{|w|} \frac{dt}{(1-t^2)^\alpha \log \frac{4}{1-t^2}}.
\end{aligned}$$

By condition (a) and the argument above we obtain

$$\sup_{1>|w|>\frac{1}{2}} \left(1-|w|^2\right)\left|RRg(w)\right| A\left(|w|\right) \leqslant C. \tag{8}$$

The conclusion follows by combining (8) and (9). □

Theorem 5. *Suppose $g \in H(B)$, $\alpha > 1$. Then $T_g : \mathcal{B}^\alpha_{log} \to \mathcal{Z}$ is compact if and only if following conditions hold*:

(a) $\lim_{|z|\to 1}(1-|z|^2)^{1-\alpha}\{\log \frac{4}{1-|z|^2}\}^{-1}|Rg(z)| = 0$;
(b) $\lim_{|z|\to 1}(1-|z|^2)A(|z|)|RRg(z)| = 0$;
(c) $g \in \mathcal{Z}$.

Proof. From (a) and (b), we have for any given $\epsilon > 0$, there exists a $\delta (0 < \delta < 1)$, such that both

$$\left(1-|z|^2\right)^{1-\alpha} \left\{\log \frac{4}{1-|z|^2}\right\}^{-1} \left|Rg(z)\right| < \epsilon \quad \text{and} \quad \left(1-|z|^2\right) A\left(|z|\right)\left|RRg(z)\right| < \epsilon$$

whenever $\delta < |z| < 1$. Let $K = \{z \in B\colon\ |z| \leqslant \delta\}$, and for any sequence $(f_k)_{k\in N}$ with $\sup_{k\in N} \|f_k\|_\alpha \leqslant C$ and $f_k \to 0$ uniformly on compact subsets of B,

$$\begin{aligned}
\|T_g f_k\| &\leqslant \sup_{z\in B}\left(1-|z|^2\right)\left(\left|Rf_k(z)\cdot Rg(z)\right| + \left|f_k(z)\cdot R(Rg)(z)\right|\right) \\
&\leqslant \sup_{z\in K} |Rf_k|\left(1-|z|^2\right) \log \frac{4}{1-|z|^2} \|g\| + \sup_{z\in B-K} \left(1-|z|^2\right) \frac{|Rg(z)|}{(1-|z|^2)^\alpha \log \frac{4}{1-|z|^2}} \|f\|_\alpha \\
&\quad + \sup_{z\in K}\left|f_k(z)\right| \cdot \|g\| + \sup_{z\in B-K} 2\left(1-|z|^2\right)\left|RRg(z)\right| A\left(|z|\right) \cdot \|f\|_\alpha.
\end{aligned}$$

The compactness of T_g follows by letting $k \to \infty$.

Now we turn to the necessity. For any given sequence $(z_k)_{k\in N}$ in B with $|z_k| \to 1$ as $k \to \infty$, set

$$h_k(z) = \frac{1-|z_k|^2}{(1-\langle z, z_k\rangle)^\alpha \log \frac{4}{1-\langle z, z_k\rangle}} - \frac{(1-|z_k|^2)^2}{(1-\langle z, z_k\rangle)^{\alpha+1} \log \frac{4}{1-\langle z, z_k\rangle}}.$$

It is easy to check (h_k) satisfy the conditions in Lemma 2. A similar argument of Theorem 4, we can get

$$\left(1-|z_k|^2\right)^{1-\alpha}\left\{\log\frac{4}{1-|z_k|^2}\right\}^{-1}\left|Rg(z_k)\right| \leqslant 4\|T_g h_k\|,$$

letting $k \to \infty$, condition (a) follows. And then set

$$f_k(z) = \left(\int_0^{|z_k|^2}\frac{dt}{(1-t)^\alpha\log\frac{4}{1-t}}\right)^{-1}\left(\int_0^{\langle z,z_k\rangle}\frac{dt}{(1-t)^\alpha\log\frac{4}{1-t}}\right)^2,$$

which also satisfy conditions in Lemma 2, and a minor modification of Theorem 4 can show that the condition (b) holds, we omit the details here. For the condition (c), we just take the test function $f = 1$, so $g = T_g 1 \in \mathcal{Z}$. This completes the proof of the theorem. □

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