Sub- and superadditive integral means

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Abstract

The integral means are special Cauchy means (see, e.g., [L. Losonczi, On the comparison of Cauchy mean values, J. Inequal. Appl. 7 (2002) 11–24]) depending on one function. The two variable integral means were (independently) defined and studied by Elezović and Pečarić [Differential and integral f-means and applications to digamma function, Math. Inequal. Appl. 3 (2000) 189–196]. The comparison problem of two integral means (under differentiability conditions) was solved by Losonczi [Comparison and subhomogeneity of integral means, Math. Inequal. Appl. 5 (2000) 609–618]. Here we completely characterize the additive, sub- and superadditive integral means of $n \geq 2$ variables.

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1. $n$ variable differential (Lagrange) and integral means

If $f$ is a continuous real function on an (open or proper closed) interval $I$ and $f$ is differentiable on $I^o$ (being the interior of $I$), then for every $x_1, x_2 \in I$, $x_1 < x_2$, there is a point $t \in ]x_1, x_2[$ such that

$$f'(t) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$
This is Lagrange’s mean value theorem.

If \( f' \) is invertible, then \( t \) is unique and

\[
t = (f')^{-1}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right).
\]

This number \( t \) is called the \textit{differential} \( f \)-mean (or Lagrange mean) of \( x_1 \) and \( x_2 \) and denoted by \( D_f(x_1, x_2) \).

Similarly, if \( f : I \to \mathbb{R} \) is continuous and strictly monotonic on \( I \), then for every \( x_1, x_2 \in I, x_1 < x_2 \) there is a point \( s \in ]x_1, x_2[ \) such that

\[
f(s) = \frac{\int_{x_1}^{x_2} f(u) \, du}{x_2 - x_1} \quad \text{thus} \quad s = f^{-1}\left(\frac{\int_{x_1}^{x_2} f(u) \, du}{x_2 - x_1}\right).
\]

This number \( s \) is called the \textit{integral} \( f \)-mean of \( x_1 \) and \( x_2 \) and denoted by \( I_f(x_1, x_2) \).

Clearly (requiring \( D_f, I_f \) to have the mean property or requiring them to be continuous), we have for equal arguments

\[
D_f(x, x) = x, \quad I_f(x, x) = x \quad (x \in I).
\]

As

\[
I_f = D_F \quad \text{with} \quad F(x) = \int_{x_0}^{x} f(t) \, dt,
\]

where \( x \in I \) and \( x_0 \in I^o \) it is enough to study the means \( I_f \) only.

It is possible to define the differential and integral means for several variables. To do so we need to introduce divided differences.

For a function \( f : I \to \mathbb{R}, I \) being a real interval, the divided differences of \( f \) on \textit{distinct} points \( x_i \in I \) are usually defined inductively by

\[
[x_1]_f := f(x_1),
\]

\[
[x_1, \ldots, x_n]_f := \frac{[x_1, \ldots, x_{n-1}]_f - [x_2, \ldots, x_n]_f}{x_1 - x_n} \quad (n = 2, 3, \ldots)
\]

(see, e.g., Aumann and Haupt [1, §3.17], their expression contains an extra factor \( n-1 \) on the right).

This definition must be modified if two or more points of \([x_1, \ldots, x_n]_f \) coincide: if at most \( r \) points \( x_i \) coincide, the definition is then framed on the assumption that \( f \) is \((r-1)\)-times differentiable on \( I \). In the case \( n = 2 \) for example we obtain

\[
[x_1, x_2]_f := \begin{cases} 
\frac{f(x_1) - f(x_2)}{x_1 - x_2} & (x_1 \neq x_2), \\
\frac{f'(x_1)}{x_1 - x_2} & (x_1 = x_2).
\end{cases}
\]

A full definition, as the ratio of two determinants, can be found in Schumaker [10].

Some basic properties of the divided differences are:

1. A divided difference \([x_1, \ldots, x_n]_f \) is independent of the order of its arguments \( x_1, \ldots, x_n \).
2. The second line of the above inductive definition remains valid provided only that
   \( x_1 \neq x_n \).
3. A divided difference is a linear functional of \( f \), i.e., we have
   \[ [x_1, \ldots, x_n]_{af + bg} = a[x_1, \ldots, x_n]_f + b[x_1, \ldots, x_n]_g \]
   for arbitrary constants \( a, b \) and arbitrary (suitably differentiable) functions \( f, g \).
4. (Mean Value Theorem) If \( f \) is \( (n - 1) \)-times differentiable on \( I \) and \( x_i \in I \) \( (i = 1, \ldots, n) \), then there is a \( t \) between the smallest and largest \( x_i \) (strictly between if
   the \( x_i \) are not all the same) such that
   \[ [x_1, \ldots, x_n]_f = \frac{f^{(n-1)}(t)}{(n-1)!}. \]
5. The “Leibniz rule” for divided differences:
   \[ [x_1, \ldots, x_n]_{fg} = \sum_{i=1}^{n} [x_1, \ldots, x_i]_f \cdot [x_i, \ldots, x_n]_g. \]
6. The rule of adding an extra point to a divided difference:
   \[ [x_2, \ldots, x_n]_h = [x_1, \ldots, x_n]_h, \quad h(x) := (x - x_1)g(x). \]
7. Differentiation with respect to a singly-occurring entry results in a repetition of that
   entry:
   \[ \frac{d}{dx_k}[x_1, \ldots, x_n]_f = [x_1, \ldots, x_n, x_k]_f \quad (k = 1, \ldots, n). \]
8. If \( f \) has a continuous \( (n - 1) \)st derivative (\( f \) is analytic), then \( [x_1, \ldots, x_n]_f \) is a con-
   tinuous (analytic) function of \( (x_1, \ldots, x_n) \).
9. If \( f^{(n-1)} \) is continuous, then we have the representation
   \[ [x_1, \ldots, x_n]_f = \int_{S_{n-1}} f^{(n-1)}(t) d\mu, \]
   where
   \[ S_{n-1} := \left\{ \mu = (\mu_1, \ldots, \mu_{n-1}): \mu_k \geq 0, \ k = 1, \ldots, n - 1 \text{ and } \sum_{k=1}^{n-1} \mu_k \leq 1 \right\} \]
   is a simplex in \( \mathbb{R}^{n-1} \) and
   \[ t = x_n + \sum_{k=1}^{n-1} \mu_k (x_k - x_n) = \sum_{k=1}^{n-1} \mu_k x_k + \left( 1 - \sum_{k=1}^{n-1} \mu_k \right) x_n. \]
   This formula is equivalent to the one given by Steffenson [11, p. 17] and it is valid
   even if some (or all) of the points \( x_1, \ldots, x_n \) coalesce.

Supposing that \( f^{(n-1)} \) is invertible, we get from the Mean Value Theorem 4 that
   \[ t = (f^{(n-1)})^{-1}((n-1)! [x_1, \ldots, x_n]_f) \]
is a mean of \( x_1, \ldots, x_n \) which coincides with \( D_f(x_1, x_2) \) if \( n = 2 \). We shall call \( t \) the differential \( f \)-mean (or Lagrange mean) of the numbers \( x_1, \ldots, x_n \) and denote by \( D_f(x_1, \ldots, x_n) \).

Let \( f : I \to \mathbb{R} \) be a continuous strictly monotonic function. The \( f \)-integral mean of \( x = (x_1, \ldots, x_n) \in I^n \) is defined by

\[
I_f(x) := D_F(x),
\]

where

\[
F(x) = \int_{x_0}^x \int_{x_0}^{u_2} \cdots \int_{x_0}^{u_n-2} f(u) \, du_1 \cdots du_{n-2} \quad (x \in I, \ x_0 \in I^0).
\]

The integral means behave very similarly to the quasi-arithmetic means, defined by

\[
M_f(x) := f^{-1}\left(\frac{\sum_{k=1}^n f(x_k)}{n}\right) \quad (x = (x_1, \ldots, x_n) \in I^n, \ n \geq 2).
\]

The two variable integral means were introduced and studied by Elezović and Pečarić [3]. They gave sufficient conditions for their comparison and applied these to obtain some other inequalities. The author [8] found necessary and sufficient conditions for the comparison and subhomogeneity of these integral means. Quasi-arithmetic means are discussed in detail, e.g., by Hardy, Littlewood and Pólya [4].

The aim of this paper is to characterize the (generalized) sub- and superadditive integral means of \( n \geq 2 \) variables on suitable intervals \( I \), that is to give necessary and sufficient conditions for the inequality

\[
I_f(x + y) \leq I_g(x) + I_h(y) \quad (x, y \in I^n)
\]

and its reverse.

2. Sub- and superadditive integral means

Let \( DM(I) \) be the set of all functions \( f : I \to \mathbb{R} \) which have nonvanishing derivative on \( I \). As usual \( C_n(I) \) denotes the set of all functions \( f : I \to \mathbb{R} \) which have continuous \( n \)th derivative on \( I \).

We remark that \( f \in DM(I) \) is a sufficient condition for \( I_f(x) \) to exists for every \( x \in I^n \) and for the formula

\[
I_f(x) = f^{-1}\left((n-1)! \int_{S_{n-1}} f(t) \, d\mu\right)
\]

(1)

to be valid. As \( \int_{S_{n-1}} d\mu = 1/(n-1)! \), we can rewrite this as

\[
\int_{S_{n-1}} \left(f(t) - f(I_f(x))\right) \, d\mu = 0.
\]

This formula could also serve as the definition of \( I_f(x) \).
Let $\mathcal{D}M_2(I) := \mathcal{D}M(I) \cap C_2(I)$ and for any function $f \in \mathcal{D}M_2(I)$ define $f^*$ by

$$f^*(x) := \frac{f''(x)}{f'(x)} \quad (x \in I).$$

**Theorem 1.** Suppose that $I$ is one of the intervals $[0, \infty], [-\infty, 0[, \mathbb{R}$ and $n \geq 2$ is a fixed integer, $f, g, h \in \mathcal{D}M_2(I)$ and let

$$K(x, y) := f \left[ g^{-1}(x) + h^{-1}(y) \right] \quad (x, y \in I).$$

The inequality

$$I_f(x + y) \leq I_g(x) + I_h(y) \quad (x, y \in I^n)$$

holds if and only if one of the conditions

$$\begin{cases} 
(i) & f^*(x + y) \leq g^*(x) \quad (x, y \in I) \\
(ii) & f^*(x + y) \leq h^*(y) \quad (x, y \in I) \\
(iii) & f^*(x + y)(g^*(x) + h^*(y)) \leq g^*(x)h^*(y) \quad (x, y \in I)
\end{cases}$$

either $f' > 0$ on $I$ and $K$ is concave on $I \times I$, or $f' < 0$ on $I$ and $K$ is convex on $I \times I$, is satisfied.

**Proof.** First we prove that (3) is necessary for (2) to hold.

Let $x, y \in I$ be arbitrary fixed values and

$$G(u, v) := I_g(u, v, \ldots, v) + I_h(v, \ldots, v) - I_f(u + v, x + y, \ldots, x + y) \quad (u, v \in I).$$

By (2), $G(u, v) \geq 0$ and $G(x, y) = 0$ thus $G$ has a minimum at $(x, y)$. Therefore the inequalities

$$\partial_1^2 G(x, y) \geq 0, \quad \partial_2^2 G(x, y) \geq 0, \quad \partial_1^2 G(x, y) \partial_2^2 G(x, y) - \partial_1 \partial_2 G(x, y)^2 \geq 0$$

are necessary for (2) to hold. By (1), we have

$$I_f(u, x, \ldots, x) = f^{-1}\left((n-1)! \int_{s_{n-1}} f(s) \, d\mu\right) \quad \text{with } s = \mu_1 u + (1 - \mu_1)x.$$

Differentiating behind the integral sign, we get
\[
\begin{align*}
\partial_1 I_f(u, x, \ldots, x) &= \frac{(n-1)! \int_{S_{n-1}} \mu_1 f'(s) d\mu}{f'(I_f)}, \\
\partial_1^2 I_f(u, x, \ldots, x) &= \frac{(n-1)! \int_{S_{n-1}} \mu_1^2 f''(s) d\mu}{f'(I_f)} \\
&= \frac{[(n-1)! \int_{S_{n-1}} \mu_1 f'(s) d\mu]^2 f''(I_f)}{f'(I_f)^3},
\end{align*}
\]

where \( I_f = I_f(u, x, \ldots, x) \). Hence, using \( I_f(x, x, \ldots, x) = x \) and the equations

\[
\int_{S_{n-1}} \mu_1 d\mu = \frac{1}{n!}, \quad \int_{S_{n-1}} \mu_1^2 d\mu = \frac{2}{(n+1)!}
\]

(see [6, p. 224] for the details), we obtain

\[
\partial_1^2 I_f(x, \ldots, x) = \frac{n-1}{n^2(n+1)} f^*(x).
\]

Now, the necessity of (3) follows from

\[
\begin{align*}
\partial_1^2 G(x, y) &= \partial_1^2 I_f(x, \ldots, x) - \partial_1^2 I_f(x + y, \ldots, x + y) \\
&= \frac{n-1}{n^2(n+1)} (g^*(x) - f^*(x + y)), \\
\partial_2^2 G(x, y) &= \partial_2^2 I_f(y, \ldots, y) - \partial_2^2 I_f(x + y, \ldots, x + y) \\
&= \frac{n-1}{n^2(n+1)} (h^*(y) - f^*(x + y)), \\
\partial_1 \partial_2 G(x, y) &= -\partial_1^2 I_f(x + y, \ldots, x + y) = \frac{n-1}{n^2(n+1)} f^*(x + y)
\end{align*}
\]

by (7).

Next we prove that (3) is equivalent to (4)–(6).

Suppose that (3) holds and assume for the sake of definiteness that \( f'(x) > 0 \ (x \in I) \).

Then we have

\[
\begin{align*}
\partial_1 K(x, y) &= \frac{f'(g^{-1}(x) + h^{-1}(y))}{g'(g^{-1}(x))}, \\
\partial_2 K(x, y) &= \frac{f'(g^{-1}(x) + h^{-1}(y))}{h'(h^{-1}(y))}, \\
\partial_1^2 K(x, y) &= \frac{f''(g^{-1}(x) + h^{-1}(y))}{g'(g^{-1}(x))^2} - \frac{f'(g^{-1}(x) + h^{-1}(y)) f''(g^{-1}(x))}{g'(g^{-1}(x))^3} \\
&= \frac{f''(g^{-1}(x) + h^{-1}(y))}{g'(g^{-1}(x))^2} \left[ f^*(g^{-1}(x) + h^{-1}(y)) - g^*(g^{-1}(x)) \right], \\
\partial_2^2 K(x, y) &= \frac{f''(g^{-1}(x) + h^{-1}(y))}{h'(h^{-1}(y))^2} \left[ f^*(g^{-1}(x) + h^{-1}(y)) - h^*(h^{-1}(y)) \right], \\
\partial_1 \partial_2 K(x, y) &= \frac{f''(g^{-1}(x) + h^{-1}(y))}{g'(g^{-1}(x)) h'(h^{-1}(y))} = \frac{f'(g^{-1}(x) + h^{-1}(y))}{g'(g^{-1}(x)) h'(h^{-1}(y))} \\
&\times f^*(g^{-1}(x) + h^{-1}(y)).
\end{align*}
\]
Now the inequalities $\partial_1^2 K(x, y) \leq 0$, $\partial_2^2 K(x, y) \leq 0$ are equivalent to (3)(i), (ii) respectively, while

$$\partial_1^2 K(x, y) \partial_2^2 K(x, y) - (\partial_1 \partial_2 K(x, y))^2 \geq 0$$

can easily be shown to be equivalent to (3)(iii). It is well known that the last three inequalities are equivalent to the concavity of $K$ (see, e.g., [2]) proving the equivalence of (3) and (4) (the case when $f' < 0$ is similar).

The inequality (5) can easily be rewritten as the convexity/concavity of $K$ by (4), proving the equivalence of these two conditions (see also Beck [2]).

The equivalence of (5) and (6) can be found in Losonczi [5] but can also be proved using the convexity/concavity criterion

$$\partial_1 K(u, v)(s - u) + \partial_2 K(u, v)(t - v) \leq K(s, t) + K(u, v)$$

(see [9]).

Finally we prove that (6) is sufficient for (2).

Let $x, y \in \mathbb{I}^n$ and substitute

$$s = x_n + \sum_{k=1}^{n-1} \mu_k (x_k - x_n), \quad u = I_g(x),$$

$$t = y_n + \sum_{k=1}^{n-1} \mu_k (y_k - y_n), \quad v = I_h(y)$$

into (6) and integrate over $S_{n-1}$. We obtain that

$$\frac{\int_{S_{n-1}} (f(s + t) - f(I_g(x) + I_f(y))) \, d\mu}{f'(I_g(x) + I_f(y))} \leq \frac{\int_{S_{n-1}} (g(s) - g(I_g(x))) \, d\mu}{g'(I_g(x))} + \frac{\int_{S_{n-1}} (h(t) - h(I_h(y))) \, d\mu}{h'(I_h(y))}.$$  

Here the right-hand side is zero, therefore, if e.g. $f' > 0$ we conclude that

$$\int_{S_{n-1}} (f(s + t) - f(I_g(x) + I_h(y))) \, d\mu \leq 0,$$

or integrating the second term and applying $f^{-1}$ to both sides,

$$f^{-1}\left((n - 1)! \int_{S_{n-1}} f \left( x_n + y_n + \sum_{k=1}^{n-1} \mu_k (x_k + y_k - (x_n + y_n)) \right) \, d\mu \right) \leq I_g(x) + I_h(y)$$

which is exactly the required inequality (2). ∎
Remark 1. We remark that Theorem 1 remains valid if we reverse the inequality signs in (2), (3)(i), (3)(ii), (5), (6) and interchange the words concave and convex in (4), or, if we replace the inequality signs in (2), (3)(i)--(iii), (5), (6) with equality sign and replace the words concave/convex in (4) by the word linear.

Remark 2. (6) is a special case of a more general sufficient condition for the general comparison of Cauchy mean values (claimed in Losonczi [7], Theorem 6 without proof). Due to an oversight in the intended proof, Theorem 6 is valid only in the case when \( k(x, y) = x + y \).

3. Sub- and superadditive integral means on various intervals

In this section we study the inequality (2) in the case when \( g = h = f \) and \( I = \mathbb{R}_+, \mathbb{R}_-, \mathbb{R} \).

Theorem 2. Suppose that \( n \geq 2 \) is a fixed integer, \( f \in D_M^2(\mathbb{R}_+) \). The inequality

\[
I_f(x + y) \leq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_+^n)
\]

holds if and only if there exists an \( x^* \) with \( 0 \leq x^* \leq \infty \) such that

\[
f^*(x) \begin{cases} > 0 & \text{if } x \in ]0, x^* [ \cap \mathbb{R}_+, \\ = 0 & \text{if } x \in [x^*, \infty [ \cap \mathbb{R}_+ \end{cases}
\]

(where we agree that \( ]a, a[ = \emptyset, [a, a[ = \emptyset \) for \( -\infty \leq a \leq \infty \)) and

\[
\frac{1}{f^*(x)} + \frac{1}{f^*(y)} \leq \frac{1}{f^*(x + y)} \quad (x, y, x + y \in ]0, x^* [ \cap \mathbb{R}_+).
\]

Proof. By Theorem 1, (8) is equivalent to (3). In our case (3)(i) and (ii) both mean that \( f^* \) is decreasing.

We claim that \( f^*(x) \geq 0 \) for \( x \in \mathbb{R}_+ \). Otherwise there were an \( x_0 \in \mathbb{R}_+ \) such that \( f^*(x_0) < 0 \). Then, \( f^*(x) < 0 \) for \( x \geq x_0 \) as \( f^* \) is decreasing. For \( x, y \geq x_0 \) we get from (3)(iii) (dividing it by \( f^*(x)f^*(y)f^*(x + y) < 0 \)) and from the negativity of \( f^* \) that

\[
\frac{1}{f^*(x)} > \frac{1}{f^*(x)} + \frac{1}{f^*(y)} \geq \frac{1}{f^*(x + y)} \quad (x, y \geq x_0),
\]

which implies that \( f^*(x) < f^*(x + y) \), i.e., \( f^* \) is strictly increasing on the interval \([x_0, \infty[\) which is a contradiction, proving our claim.

Case 1: If \( f^*(x) = 0 \) (\( x \in \mathbb{R}_+ \)), then (10) with \( x^* = 0 \) is clearly sufficient for (8) as (3)(i)--(iii) are obviously satisfied.

Case 2: If there exists an \( x_1 \in \mathbb{R}_+ \) such that \( f^*(x_1) > 0 \), then let

\[
x^* = \sup \{ x \in \mathbb{R}_+: f^*(x) > 0 \}.
\]

We have \( 0 < x^* \leq \infty \).
Subcase 2.1: If $x^* = \infty$, then $f^*(x) > 0$, $x \in \mathbb{R}_+$. In this subcase (3)(iii) can be written in the form (10), and (10) implies that $f^*$ is (strictly) decreasing, i.e., (3)(i) and (ii) hold. Thus we proved that (10) is necessary and sufficient for (8) (or (3)).

Subcase 2.2: If $0 < x^* < \infty$, then $f^*$ is positive on $]0, x^*[\cap \mathbb{R}_+$ and zero on $]x^*, \infty]$, i.e., (9) hold.

For $x, y, x + y \in ]0, x^*[\cap \mathbb{R}_+$ we can rewrite (3)(iii) in the form (10), proving that (10) is necessary for (3) or (8). We show that it is sufficient too. First, (10) implies that $f^*$ is (strictly) decreasing on $]0, x^*[\cap \mathbb{R}_+$ and since $f^*$ is zero on $]x^*, \infty[\cap \mathbb{R}_+$ and continuous, it is decreasing on $\mathbb{R}_+$, proving that (3)(i) and (ii) hold. To complete the proof we show that (3)(iii) holds too. We can decompose $\mathbb{R}_+ \times \mathbb{R}_+$ as

\begin{align*}
\mathbb{R}_+ \times \mathbb{R}_+ &= \bigcup_{i=1}^5 D_i, \\
D_1 &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+: x, y, x + y \in ]0, x^*[\right arrow \}, \\
D_2 &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+: x, y \in ]0, x^*[\cup [x^*, 2x^*]\right arrow \}, \\
D_3 &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+: x, y \in [x^*, \infty]\right arrow \}, \\
D_4 &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+: x \in [x^*, \infty]\cap [0, x^*[\right arrow \}, \\
D_5 &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+: y \in [x^*, \infty]\cap [0, x^*[\right arrow \}.
\end{align*}

(3)(iii) holds on $D_1$ by (10). It holds on $D_2$ as its left hand side is zero and the right one is positive. Finally, (3)(iii) holds on $D_3, D_4, D_5$ as the left hand side is zero and so is the right-hand side ($f^*(x + y) = 0$ and at least one of the factors $f^*(x), f^*(y)$ is zero). 

**Theorem 3.** Suppose that $n \geq 2$ is a fixed integer, $f \in \mathcal{D}_{M_2}(\mathbb{R}_+)$. The inequality

\begin{equation}
I_f(x + y) \geq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_+^n) \tag{11}
\end{equation}

holds if and only if there exists an $x^*$ with $0 \leq x^* \leq \infty$, such that

\begin{equation}
f^*(x) \begin{cases} < 0 & \text{if } x \in ]0, x^*[\cap \mathbb{R}_+ \\
= 0 & \text{if } x \in [x^*, \infty]\cap \mathbb{R}_+ \end{cases} \tag{12}
\end{equation}

and

\begin{equation}
\frac{1}{f^*(x)} + \frac{1}{f^*(y)} \geq \frac{1}{f^*(x + y)} \quad (x, y, x + y \in ]0, x^*[\cap \mathbb{R}_+). \tag{13}
\end{equation}

The proof is similar to that of Theorem 1, thus we omit it.

The subadditivity of $I_f$ on $\mathbb{R}_-$

\begin{equation}
I_f(x + y) \leq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_-^n) \tag{14}
\end{equation}

can be reduced to the superadditivity of $I_f$ on $\mathbb{R}_+$ where $\tilde{f}(u) = f(-u) \: (u \in \mathbb{R}_+)$. Let $x = -u$, then
\[ I_f(x) = f^{-1}\left( (n-1)! \int_{S_{n-1}} f\left( x_n + \sum_{k=1}^{n-1} \mu_k (x_k - x_n) \right) d\mu \right) \]

\[ = f^{-1}\left( (n-1)! \int_{S_{n-1}} f\left( -u_n + \sum_{k=1}^{n-1} \mu_k (-u_k + u_n) \right) d\mu \right) \]

\[ = -f^{-1}\left( (n-1)! \int_{S_{n-1}} f\left( u_n + \sum_{k=1}^{n-1} \mu_k (u_k - u_n) \right) d\mu \right) = -I_f(u) \]

hence (14) goes over into

\[ I_f(u + v) \geq I_f(u) + I_f(v) \quad (u, v \in \mathbb{R}_n^+) \] (15)

Since

\[ \tilde{f}^*(u) := \tilde{f}''(u) = -f''(-u) = -f^*(-u) \quad (u \in \mathbb{R}_+) \]

we can deduce from Theorem 2 the next

**Theorem 4.** Suppose that \( n \geq 2 \) is a fixed integer, \( f \in DM_2(\mathbb{R}_-) \). The inequality

\[ I_f(x + y) \leq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_n) \] (16)

holds if and only if there exists an \( x^* \) with \(-\infty \leq x^* \leq 0\) such that

\[ f^*(x) \begin{cases} > 0 & \text{if } x \in ]x^*, 0[ \cap \mathbb{R}_- , \\ = 0 & \text{if } x \in ]-\infty, x^*] \cap \mathbb{R}_- , \end{cases} \] (17)

and

\[ \frac{1}{f^*(x)} + \frac{1}{f^*(y)} \leq \frac{1}{f^*(x + y)} \quad (x, y, x + y \in ]x^*, 0[ \cap \mathbb{R}_- ). \] (18)

The superadditivity of \( I_f \) on \( \mathbb{R}_- \) can be characterized similarly.

Finally we deal with the sub- and superadditivity of \( I_f \) on \( \mathbb{R} \).

**Theorem 5.** Suppose that \( n \geq 2 \) is a fixed integer, \( f \in DM_2(\mathbb{R}) \). Any of the two inequalities

\[ I_f(x + y) \leq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_n) \] (19)

\[ I_f(x + y) \geq I_f(x) + I_f(y) \quad (x, y \in \mathbb{R}_n) \] (20)

hold if and only if \( I_f \) is the arithmetic mean,

\[ I_f(x) = \frac{x_1 + \cdots + x_n}{n} \quad (x \in \mathbb{R}_n). \]

**Proof.** By Theorem 1, (19) holds if and only if

\[ \begin{cases} f^*(x + y) \leq f^*(x) & (x, y \in \mathbb{R}) \text{ and} \\ f^*(x + y) \leq f^*(y) & (x, y \in \mathbb{R}) \text{ and} \\ f^*(x + y)(f^*(x) + f^*(y)) \leq f^*(x)f^*(y) & (x, y \in \mathbb{R}). \end{cases} \]
(i) and (ii) mean that \( f^* \) is both increasing and decreasing on \( \mathbb{R} \), therefore it is a constant function, \( f^*(x) = c \ (x \in \mathbb{R}) \). This function satisfies (iii) if and only if \( 2c^2 \leq c^2 \) hence \( c = 0 = f^*(x) \) and \( f(x) = ax + b \ (x \in \mathbb{R}) \) where \( a \neq 0, b \in \mathbb{R} \) are arbitrary constants. We have

\[
\int_{S_{n-1}} f(t) d\mu = \int_{S_{n-1}} \left( a \left( x_n + \sum_{k=1}^{n-1} \mu_k(x_k - x_n) \right) + b \right) d\mu \\
= (ax_n + b) \int_{S_{n-1}} d\mu + \sum_{k=1}^{n-1} (x_k - x_n) \int_{S_{n-1}} \mu_k d\mu \\
= ax_n + b + \frac{n-1}{(n-1)!} + \sum_{k=1}^{n-1} \frac{x_k - x_n}{n!},
\]

therefore

\[
I_f(x) = \frac{(n-1)!}{a} \left[ \frac{ax_n + b}{(n-1)!} + \sum_{k=1}^{n-1} \frac{x_k - x_n}{n!} \right] - \frac{b}{a} = x_n + \sum_{k=1}^{n-1} \frac{x_k - x_n}{n} = \frac{\sum_{k=1}^{n} x_k}{n}
\]
as we claimed. For (20) the proof is similar. \( \square \)

**Remark 3.** On the basis of Theorem 1, Theorems 2–5 remain valid if we replace in each of them \( I_f \) by \( M_f \).

**References**


