On the simple modules for the restricted Lie superalgebra $sl(n|1)$

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ABSTRACT

The simple modules for the restricted Lie superalgebras are considered. In particular, all the simple modules for $sl(2|1)$ are classified. Also for $n > 2$ and with certain $\chi$, all simple $sl(n|1)$-modules are classified.

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1. Introduction

Since Kac [5] introduced the Lie superalgebras, much progress has been made on the structure theory. Recently, there has been lots of interest on the representation theory of the characteristic zero case. When $\text{char } F = p > 0$, Lie superalgebras were studied in [7,8]. Also the notion of restricted Lie superalgebras was introduced.

In the present work, we introduce for each simple module of a restricted Lie superalgebra a $p$-character, with which we study the simple modules for the special linear superalgebras $sl(n|1)$. The research has been largely motivated by the representation theory of modular Lie algebras (see [1,2,9]).

The paper is organized as follows. Section 2 gives the general set up for the restricted Lie superalgebras. We also study the properties of their simple modules. In Section 3, we classify all the simple modules for $sl(2|1)$ and compute their dimension formulas. In Section 4, we classify the simple modules for $sl(n|1)$ ($n > 2$) with certain $p$-characters, and also determine their dimension formulas.

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2. Preliminaries

2.1. Basic definitions

Let $F$ be an algebraically closed field with $\text{char } F > 5$. Assume $Z_2 = \{0, 1\}$ is the field of two elements. Let $V = V_\theta \oplus V_{\bar{\theta}}$ be a $Z_2$-graded vector space over $F$. We denote by $p(a) = \theta$ the parity of a homogeneous element $a \in V_{\theta}$, $\theta \in Z_2$. We assume throughout this section that the symbol $p(x)$ implies that $x$ is $Z_2$-homogeneous.

A superalgebra is a $Z_2$-graded vector space $A = A_\theta \oplus A_{\bar{\theta}}$ endowed with an algebra structure such that $A_\theta A_{\mu} \subset A_{\theta + \mu}$ for all $\theta, \mu \in Z_2$. A superalgebra $g = g_0 \oplus g_{\bar{\theta}}$ over $F$ is called a Lie superalgebra provided that

(i) $[a, b] = -(-1)^{p(a)p(b)}[b, a],
(ii) [a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]],$

for $a, b \in g_0 \cup g_{\bar{\theta}}$, $c \in g$. 

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Let \( g = g_0 \oplus g_\mathbb{T} \) be a Lie superalgebra. Then the even part \( g_0 \) is a Lie algebra and the odd part \( g_\mathbb{T} \) is a \( g_0 \)-module under the adjoint action. Note that in the case \( \text{char } \mathbb{F} = 2 \), a Lie superalgebra is a \( \mathbb{Z}_2 \)-graded Lie algebra. Thus one usually adopts the convention that \( \text{char } \mathbb{F} = p > 2 \) in the modular case.

Let \( V = V_0 \oplus V_\mathbb{T} \) be a \( \mathbb{Z}_2 \)-graded vector space, \( \dim V_0 = m \), \( \dim V_\mathbb{T} = n \). The algebra \( \text{End}_\mathbb{F}(V) \) consisting of the \( \mathbb{F} \)-linear transformations of \( V \) becomes an associative superalgebra if one defines
\[
\text{End}_\mathbb{F}(V)_\theta := \{ A \in \text{End}_\mathbb{F}(V) \mid A(V_\mu) \subseteq V_{\theta + \mu}, \mu \in \mathbb{Z}_2 \}
\]
for \( \theta \in \mathbb{Z}_2 \). On the vector superspace \( \text{End}_\mathbb{F}(V) = \text{End}_\mathbb{F}(V_0) \oplus \text{End}_\mathbb{F}(V_\mathbb{T}) \) we define a new multiplication \([ , ]\) by
\[
[A, B] := AB - (-1)^{p(A)p(B)}BA \quad \text{for } A, B \in \text{End}_\mathbb{F}(V).
\]
This superalgebra endowed with the new multiplication is a Lie superalgebra, denoted by \( g \). Let \( \gamma \) be a restricted \( g \)-module (see \([11,12]\)). Then \( \gamma \) is strictly triangular on \( V \).

### Lemma 2.2. General properties

#### Lemma 2.1. Let \( g = g_0 \oplus g_\mathbb{T} \) be a restricted \( \mathfrak{m} \)-module and \( \mathfrak{m} = \mathfrak{m}_\mathbb{T} \oplus \mathfrak{m}_\mathbb{T} \) be a simple \( g \)-module. Then there is a unique \( \gamma \in \mathfrak{m}_0^* \) such that \( (x^\theta - x^{[\theta]} - x^\mathbb{T}(x) \cdot 1)\mathfrak{m} = 0 \) for all \( x \in \mathfrak{m}_0 \).

**Proof.** Note that both \( \mathfrak{m}_\mathbb{T} \) and \( \mathfrak{m}_\mathbb{T} \) are \( g_0 \)-modules. Let \( M \subseteq \mathfrak{m}_\mathbb{T} \) be a simple \( g_0 \)-module. Then there exists \( \chi \in \mathfrak{m}_0^* \) such that \( (x^\theta - x^{[\theta]} - x^\mathbb{T}(x) \cdot 1)M = 0 \) for all \( x \in \mathfrak{m}_0 \). Clearly, \( U(g) \mathfrak{m} \) is a \( \mathbb{Z}_2 \)-graded submodule of \( \mathfrak{m} \) if \( 0 \neq m \in M \). Thus \( U(g) \mathfrak{m} = \mathfrak{m} \). Since \( g_0 \) is restricted, we have \( x^\theta - x^{[\theta]} = 0 \). Consequently, \( x^\theta - x^{[\theta]} \in Z(U(g)) \). Then \( (x^\theta - x^{[\theta]} - x^\mathbb{T}(x) \cdot 1)\mathfrak{m} = U(g)(x^\theta - x^{[\theta]} - x^\mathbb{T}(x) \cdot 1)M = 0 \). □

Let \( g \) be a restricted Lie superalgebra. For each \( \chi \in \mathfrak{m}_0^* \), define the \( \chi \)-reduced enveloping algebra of \( g \) by \( u(g, \chi) = U(g)/I_{\chi} \), where \( I_{\chi} \) is the \( \mathbb{Z}_2 \)-graded two-sided ideal of \( U(g) \) generated by elements \( \{x^\theta - x^{[\theta]} - x^\mathbb{T}(x) \cdot 1 \mid x \in \mathfrak{m}_0 \} \). When \( \chi = 0 \), \( u(g, 0) \) is called the restricted universal enveloping algebra of \( g \) (see \([11,12]\) for the Lie algebra situation).

Let \( (g, [p]) \) be a restricted Lie superalgebra. Suppose that \( e_1, \ldots, e_m \) and \( f_1, \ldots, f_n \) are ordered bases of \( g_0 \) and \( g_\mathbb{T} \) respectively. Applying similar arguments as that for the modular Lie algebra case, we have that \( u(g, \chi) \) has the following \( \mathbb{F} \)-basis:
\[
\{ f_1^n(1) \cdots f_n^{(m)} e_1^{p(1)} \cdots e_m^{p(m)} | 0 \leq a(i) \leq p - 1; b(j) = 0 \text{ or } 1 \}.
\]

A \( g \)-module \( M \) is called having a \( p \)-character \( \chi \in \mathfrak{m}_0^* \) provided that
\[
x^\theta \cdot m - x^{[\theta]} \cdot m = \chi(x)^p m \quad \text{for all } x \in g_\mathbb{T}, m \in M.
\]

**Lemma 2.1** shows that every simple module has a \( p \)-character. Clearly, \( u(g, \chi) \)-modules may be identified with \( g \)-modules having character \( \chi \).

Let \( g \) be a Lie superalgebra. A subset \( s \subseteq g \) is called a Lie super-set if \( s \subseteq g_0 \cup g_\mathbb{T} \) and \( s \) is closed under the Lie multiplication. Analogous to \([12, \text{Th. } 1.3.1]\), one can prove

**Lemma 2.2.** Let \( V = V_0 \oplus V_\mathbb{T} \) be a \( \mathbb{Z}_2 \)-graded space and \( s \subseteq \text{gl}(V) \) a Lie super-set such that
\begin{itemize}
  \item[(a)] \( s \) is nil.
  \item[(b)] \( g := (s) \) is finite dimensional.
\end{itemize}

Then \( g \) is strictly triangularizable on \( V \).

**Lemma 2.3** (See \([12, \text{Cor. } 1.3.8]\)). Let \( I \) be a finite dimensional \( \mathbb{Z}_2 \)-graded ideal of a Lie superalgebra \( g \) and \( \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \) a simple module of \( g \). If \( x \) acts nilpotently on \( \mathfrak{m} \) for all \( x \in I \), then \( IM = 0 \).
Proof. Let \( W = \{ m \in M | lm = 0 \} \). Since \( I \) is \( \mathbb{Z}_2 \)-graded, \( W \) is a \( \mathbb{Z}_2 \)-graded \( g \)-submodule of \( \mathfrak{M} \). Regard \( \mathfrak{M} \) as an \( l \)-module. Then Lemma 2.2 ensures that \( W \neq 0 \). Therefore, \( W = \mathfrak{M} \). This completes the proof. \( \square \)

Let \( \mathbb{F}_{m \times n} \) denote the set of all \( m \times n \) matrices. Recall that \( gl(m|n) = L_{-1} + L_0 + L_1 \), where

\[
L_1 = \left\{ \begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix} : B \in \mathbb{F}_{m \times n} \right\}, \quad L_{-1} = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in \mathbb{F}_{n \times m} \right\}.
\]

Let \( g = g_{-1} + g_0 + g_1 \subseteq gl(m|n) \) and let \( g^+ = g_0 + g_1 \).

Lemma 2.4. Let \( \mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \) be a simple \( U(g^+) \)-module (\( u(g^+, \chi) \)-module).

\( \mathfrak{M} \neq 0 \).

Proof. Note that \( U(g^+) \subseteq U(g) \). For \( x \in g, x^2 = 0 \), then \( x \in g_1 \) is nilpotent in \( U(g^+) \). Since \( [g_0, g_1] \subseteq g_1 \) and \([g_1, g_1] = 0 \), \( g_1 \) is an ideal of \( g \). Using Lemma 2.3, one gets the desired result. \( \square \)

Let \( g = g_{-1} + g_0 + g_1 \) be a restricted Lie superalgebra, and let \( n^- + h + n^+ \) be the triangular decomposition of \( g_{\mathbb{C}} \). Set \( b^+ = n^+ + h + g_1 \). For each \( \chi \in \mathfrak{g}_{\mathbb{C}}^* \), let \( \lambda \in h^* \) satisfy \( \lambda(h)^{\mathbb{C}} - \lambda(h) = \chi(h) \) for every \( h \in h \). We define the one-dimensional \( b^+ \)-module \( \mathbb{F}_\lambda \) as follows:

\[
\chi (g_{1} + n^+) v_{\lambda} = 0, \quad h v_{\lambda} = \langle \lambda, h \rangle v_{\lambda} \quad \text{for every } h \in h.
\]

Denote \( Z^X(\lambda) = \psi(g, \chi) \otimes u(g^+, \chi) \mathbb{F}_\lambda \). \( Z^X(\lambda) \) is called a Baby Verma module having character \( \chi \).

Definition 2.5. Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded \( g \)-module, and let \( h \) be a maximal torus of \( g_{\mathbb{C}} \) with a basis \( h_1, \ldots, h_n \). If there is a nonzero vector \( v \in V_0 \cup V_1 \), such that each \( n^+ + g_1 \cdot v = 0 \) and \( h \cdot v = \lambda, \lambda_1, \ldots, \lambda_n \). Then \( Z^X(\lambda) \) is called a Baby Verma module having character \( \chi \).

Lemma 2.6. Let \( \chi \in \mathfrak{g}_0^* \) with \( \chi(n^+) = 0 \), and let \( \mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \) be a \( \mathbb{Z}_2 \)-graded \( u(g, \chi) \)-module. Then the following are equivalent:

1. \( \mathfrak{M} \) is nonzero and is generated by each of its maximal vectors.
2. \( \mathfrak{M} \) is simple.

Suppose \( g \) is a restricted Lie superalgebra. Let \( \text{Aut}(g) \) be the group of restricted automorphisms of \( g \) (an automorphism \( \Phi \) is called restricted if \( \Phi(x)|^p = \Phi(x)|^p \) for each \( x \in \mathfrak{g}_0 \)). Let \( \Phi \in \text{Aut}(g) \). If \( \Phi \in \mathfrak{M}_0 \oplus \mathfrak{M}_1 \), then \( \Phi \) is a \( g \)-module, we denote by \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and a new \( g \)-action given by \( \chi m = \Phi(x)m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple. \( \mathfrak{M} \) is called \( \mathfrak{M}_0 \) \( g \)-module having \( \mathfrak{M} \) as its underlying vector space and the original \( g \)-action given by \( \chi m = \chi m \) for \( x \in g \) and \( m \in \mathfrak{M} \), where the action on the right is the given one. Then \( \mathfrak{M} \) is simple if and only if \( \mathfrak{M}_0 \) is simple.
3. Classification of the simple modules for $sl(2|1)$

In this section we classify the simple modules for the restricted Lie superalgebra $\mathfrak{g} = sl(2|1)$. Let $h_1 = e_{11} + e_{33}$ and $h_2 = e_{22} + e_{33}$. Then $\mathfrak{g}_0 = \langle e_{21}, e_{12}, h_1, h_2 \rangle \cong \mathfrak{gl}_2$. Let $M$ be a simple $u(\mathfrak{g}_0, \chi)$-module with $\chi \in \mathfrak{g}_0^*$. Under the coadjoint action of $\text{Aut}(\mathfrak{g})$, $\chi$ is conjugate to one of the following (see [4, 5.4]):

1. $\chi(h_1) = r, \chi(h_2) = s, \chi(e_{12}) = \chi(e_{21}) = 0$.
2. $\chi(h_1) = \chi(h_2) = r, \chi(e_{12}) = 1$ and $\chi(e_{21}) = 0$.

We refer to the $\chi$ in the case (1) as semisimple, and the $\chi$ in the case (2) as nilpotent.

Let $M$ be a simple $u(\mathfrak{g}_0, \chi)$-module, and let $v \in M$ be a maximal vector of weight $(\eta_1, \eta_2)$. By definition, $h_i v = \eta_i v, i = 1, 2, e_{12} v = e_{13} v = e_{23} v = 0$. Note that $\eta_i \in \mathbb{F}_p$ if and only if $\chi(h_i) = 0, i = 1, 2$. We refer to the weight $(\eta_1, \eta_2)$ as exceptional if $\eta_1 = -1$ or $\eta_2 = 0$.

**Remark.** In case $\eta_1 - \eta_2 \in \mathbb{F}_p$, we also use $\eta_1 - \eta_2$ to represent the smallest nonnegative integer whose residue class modulo $p$ is $\eta_1 - \eta_2$.

3.1. Some formulas

In this subsection we list some formulas to be used in the following. For the brevity, we leave the details of the computation to the interested reader.

Recall the induced module $Z^\chi(M) = u(\mathfrak{g}, \chi) \otimes_{u(\mathfrak{g}_0^*, \chi)} M$. Assume

$$m = 1 \otimes m_1 + e_{31} e_{32} \otimes m_2 \in Z^\chi(M)_{\mathbb{F}}$$

is a maximal vector of weight $(\lambda_1, \lambda_2)$, where $m_i \in M, i = 1, 2$.

From $h_i m = \lambda_i m$ and $h_2 m = \lambda_2 m$, one gets

$$h_1 m_1 = \lambda_1 m_1 \quad h_2 m_1 = \lambda_2 m_1$$
$$h_1 m_2 = (\lambda_1 - 1) m_2 \quad h_2 m_2 = (\lambda_2 - 1) m_2. \quad (\ast)$$

Applying the positive root vectors $e_{12}, e_{13}$ and $e_{23}$ to $m$, we obtain the formulas

$$e_{12} m_1 = 0 \quad e_{12} m_2 = 0. \quad (1)$$
$$\lambda_1 m_2 = 0. \quad (2)$$
$$\lambda_2 m_2 = 0 \quad e_{21} m_2 = 0. \quad (3)$$

Let

$$m' = e_{31} \otimes v_1 + e_{32} \otimes v_2 \in Z^\chi(M)_{\mathbb{F}}$$

be a maximal vector of weight $(\mu_1, \mu_2)$. From $h_i m' = \mu_i m'$, $i = 1, 2$, we get

$$h_1 v_1 = \mu_1 v_1 \quad h_2 v_1 = (\mu_2 - 1) v_1$$
$$h_1 v_2 = (\mu_1 - 1) v_2 \quad h_2 v_2 = \mu_2 v_2. \quad (***)$$

Applying the positive root vectors $e_{12}, e_{13}$ and $e_{23}$ to $m$, we obtain the formulas

$$e_{12} v_1 = 0 \quad e_{12} v_2 = v_1. \quad (4)$$
$$e_{21} v_1 + \mu_2 v_2 = 0. \quad (5)$$

3.2. The simplicity of $Z^\chi(M)$

If $\chi$ is semisimple, then by [9, Prop. 3], $M$ has a unique maximal vector $v$. Assume the weight of $v$ is $(\eta_1, \eta_2)$.

**Proposition 3.1.** Let $\mathfrak{g} = sl(2|1)$ and $\chi \in \mathfrak{g}_0^*$ be semisimple. If $\eta_1 \neq -1$ and $\eta_2 \neq 0$, then $Z^\chi(M)$ is simple.

**Proof.** By Lemma 2.6, it suffices to show that $Z^\chi(M)$ has a unique maximal vector $1 \otimes v$ (up to scalar multiple).

Let $m = 1 \otimes m_1 + e_{31} e_{32} \otimes m_2 \in Z^\chi(M)_{\mathbb{F}}$ be a maximal vector of weight $(\lambda_1, \lambda_2)$. By (1), $m_i (i = 1, 2)$ is a nonzero multiple of $v$ if it is nonzero. If $m_2 \neq 0$, then one would get from (\ast) that $\eta_1 = \lambda_1 - 1$, and hence $\lambda_1 \neq 0$, contrary to (2).

Let $m' = m_{31} \otimes v_1 + e_{32} \otimes v_2 \in Z^\chi(M)_{\mathbb{F}}$ be a maximal vector of weight $(\mu_1, \mu_2)$. Suppose $v_1 \neq 0$. Then by (4) one would get $v_1 = cv$ for some $c \neq 0$. It follows from (***) that $\eta_1 = \mu_1$, so that $\mu_1 \neq -1$, contrary to (5).

Suppose that $v_1 = 0$ but $v_2 \neq 0$. Then by (4) $v_2$ would be maximal. Thus one would get from (***) that $\eta_2 = \mu_2$, and hence $\mu_2 \neq 0$, contrary to (6).

Thus, we conclude that there is no maximal vector in $Z^\chi(M)_{\mathbb{F}}$, and $Z^\chi(M)$ has a unique maximal vector $1 \otimes v$. \qed
Let \( N = N_0 \oplus N_1 \) be a \( sl(2|1) \)-module. Recall that a nonzero element \( v \in N_0 \cup N_1 \) is called a maximal vector if \( e_i v = 0 \) whenever \( 1 \leq i < j \leq 3 \), and \( h_i v = c_i v, i = 1, 2 \). Recall that \( g_0 = \langle e_{12}, e_{21}, h_1, h_2 \rangle \cong gl_2 \). A nonzero element \( v \in N_0 \cup N_1 \) is referred to as a \( gl_2 \)-maximal vector or being \( gl_2 \)-maximal, if \( e_{12} v = 0 \) and if \( h_i v = c_i v, i = 1, 2 \). Thus a maximal vector is \( gl_2 \)-maximal, but the converse is not true.

Suppose \( \chi \) is nilpotent. Let \( v \in M \) be a \( gl_2 \)-maximal vector of weight \((\eta_1, \eta_2)\). Then we get \((e_{11} - e_{22}) v = (\eta_1 - \eta_2) v\). Since \( \chi(e_{21}) \neq 0 \), we have
\[
M = \langle v, e_{21} v, \ldots, e_{2}^{p-1} v \rangle.
\]

If \( \eta_1 - \eta_2 \in \mathbb{F}_p \) and \( \eta_1 - \eta_2 \neq p - 1 \), then \( M \) has two \( gl_2 \)-maximal vectors \( v \) and \( e_{21}^{\eta_1 - \eta_2 + 1} v \) (see [4, 5.4]). A short calculation then shows that the weight of the second \( gl_2 \)-maximal vector is \((\eta_2 - 1, \eta_1 + 1)\).

**Remark.** Clearly the assumption \( \eta_1 \neq -1 \) and \( \eta_2 \neq 0 \) also holds for the maximal vector \( e_{21}^{\eta_1 - \eta_2 + 1} v \) once it is assumed for \( v \).

**Proposition 3.2.** Let \( g = sl(2|1) \) and \( \chi \in g_0^\ast \) be nilpotent. If \( \eta_1 \neq -1 \) and \( \eta_2 \neq 0 \), then \( Z^\chi(M) \) is simple.

**Proof.** Let \( m = 1 \otimes m_1 + e_{31}e_{32} \otimes m_2 \in Z^\chi(M)_0 \) be a maximal vector of weight \((\lambda_1, \lambda_2)\). If \( m_2 \neq 0 \), then by (1) \( m_2 \) would be maximal. By the remark above we may assume \( m_2 = v \). Then we would get from (\(*\)\) that \( \eta_1 = \lambda_1 - 1 \), and hence \( \lambda_1 \neq 0 \), contrary to (2).

Suppose \( v' = e_{31} \otimes v_1 + e_{32} \otimes v_2 \in Z^\chi(M)_1 \) is a maximal vector. In a similar fashion, one obtains \( v_1 = v_2 = 0 \), i.e., there is no maximal vector in \( Z^\chi(M)_1 \). Thus, the maximal vector in \( Z^\chi(M) \) must be in the form \( 1 \otimes v \in Z^\chi(M)_0 \), where \( v \) is maximal in \( M \). It follows that \( Z^\chi(M) \) is simple. \( \square \)

In summary, we have

**Theorem 3.3.** Assume \( \eta_1 \neq -1 \) and \( \eta_2 \neq 0 \). Then \( Z^\chi(M) \) is a simple \( u(g, \chi) \)-module. If \( \chi \) is semisimple, then \( Z^\chi(M) \)'s with different weights are nonisomorphic. If \( \chi \) is nilpotent, then two \( Z^\chi(M) \)'s with weight \((\lambda_1, \lambda_2) \) and \((\eta_1, \eta_2) \) respectively are isomorphic if and only if \((\lambda_1, \lambda_2) = (\eta_1 - 1, \eta_1 + 1)\).

**Proof.** If \( \chi \) is semisimple, then by the proof of Proposition 3.1 each \( Z^\chi(M) \) contains a unique maximal vector \( 1 \otimes v \). Therefore, two \( Z^\chi(M) \)'s with different weights are nonisomorphic. If \( \chi \) is nilpotent, then from the proof of Proposition 3.2 \( Z^\chi(M) \) contains the maximal vector \( 1 \otimes v \), and also \( 1 \otimes e_{21}^{\eta_1 - \eta_2 + 1} v \) in case \( \eta_1 - \eta_2 \in \mathbb{F}_p \) and \( \eta_1 - \eta_2 \neq p - 1 \). Which establishes the last claim of the theorem. \( \square \)

3.3. The exceptional weights

3.3.1. The maximal vectors in \( Z^\chi(M) \)

**Lemma 3.4.** Let \( \chi \in g_0 \) be semisimple.

(a) Suppose \( \eta_1 = -1 \) and \( \eta_2 \neq -1, 0 \). Then \( Z^\chi(M)_0 \) has a unique maximal vector \( 1 \otimes v \), and \( Z^\chi(M)_1 \) has a unique maximal vector\( v' = e_{31} \otimes v - e_{32} \otimes (\eta_2 + 1)^{-1} e_{21} v \).

(b) If \( \eta_1 = -1 \) and \( \eta_2 = -1 \), then \( Z^\chi(M)_0 \) has two maximal vectors \( 1 \otimes v \) and \( e_{31} e_{32} \otimes v \), and \( Z^\chi(M)_1 \) has no maximal vectors.

(c) If \( \eta_2 = 0 \), then \( Z^\chi(M)_0 \) has a unique maximal vector \( 1 \otimes v \), and \( Z^\chi(M)_1 \) has a unique maximal vector \( e_{32} \otimes v \).

**Proof.** We shall only prove the statement (a). By applying similar arguments, one can get (b) and (c).

Let \( m = 1 \otimes m_1 + e_{31} e_{32} \otimes m_2 \in Z^\chi(M)_0 \) be a maximal vector. If \( m_2 \neq 0 \), then \( m_2 \) would be maximal by (1). So we would get from (\(*\)\) that \( \eta_2 = \lambda_2 - 1 \), and hence \( \lambda_2 \neq 0 \), contrary to (3). Thus \( 1 \otimes v \) is the unique maximal vector in \( Z^\chi(M)_0 \).

Let
\[
m' = e_{31} \otimes v_1 + e_{32} \otimes v_2 \in Z^\chi(M)_1
\]
be a maximal vector. Suppose \( v_1 \neq 0 \). By (4), \( v_1 \) is maximal. Then by (\(*\*)\) we get \( \eta_2 = \mu_2 - 1 \), which implies that \( \mu_2 \neq 0 \). Hence we get from (6) that \( v_2 = -(\eta_2 + 1)^{-1} e_{21} v_1 \). If \( v_2 = 0 \) and \( v_2 \neq 0 \), then \( v_2 \) would be maximal by (4), and (\(*\*)\) would imply that \( \mu_2 = \eta_2 \neq 0 \), contrary to (6). Therefore, \( Z^\chi(M)_1 \) has a unique maximal vector
\[
m'' = e_{31} \otimes v - (\eta_2 + 1)^{-1} e_{32} \otimes e_{21} v. \]
Lemma 3.5. Let $\chi \in g_0^*$ be nilpotent.

(1) If $\eta_1 = -1$ and $\eta_2 \neq 0$, $-1$, then $Z^\chi(M)_1$ has a unique maximal vector

$$e_{31} \otimes v - (\eta_2 + 1)^{-1} e_{32} \otimes e_{21} v.$$

(2) If $\eta_1 = -1$ and $\eta_2 = -1$, then $Z^\chi(M)_1$ has a unique maximal vector $e_{32} \otimes e_{21} v$.

(3) If $\eta_1 = -1$ and $\eta_2 = 0$, then $Z^\chi(M)_1$ has two unique maximal vectors:

$$e_{31} \otimes v - e_{32} \otimes e_{21} v, \quad e_{32} \otimes v.$$

(4) If $\eta_2 = 0$ and $\eta_1 \neq -1, -2$, then $Z^\chi(M)_1$ has two maximal vectors:

$$e_{31} \otimes e_{21}^{\eta_1+1} v - (\eta_1 + 2)^{-1} e_{32} \otimes e_{21}^{\eta_1+2} v, \quad e_{32} \otimes v.$$

(5) If $\eta_2 = 0$ and $\eta_1 = -2$, then $Z^\chi(M)_1$ contains a unique maximal vector $e_{32} \otimes v$.

3.3.2. The simple quotients of $Z^\chi(M)$

Lemma 3.6. Let $g = g_0 + g_1$ be a Lie superalgebra and $M = M_0 \otimes M_1$ be a $g$-module. Suppose both $M_0$ and $M_1$ are simple $g_0$-modules. If there are $x, y \in g_1$ such that $xM_0 \neq 0$ and $yM_1 \neq 0$, then $M$ is simple.

The proof is obvious.

Theorem 3.7. If $\chi$ is semisimple, then $Z^\chi(M)$ has a unique maximal submodule $Z = Z_0 \oplus Z_1$. Set $L^\chi(M) = Z^\chi(M)/Z$.

(a) If $\eta_1 = -1$ and $\eta_2 \neq 0, -1$, then $Z$ is generated by $m' = e_{31} \otimes v - (\eta_2 + 1)^{-1} e_{32} \otimes e_{21} v$, and

$$\dim L^\chi(M) = \begin{cases} 2(\eta_1 - \eta_2 + 1) + 1, & \text{if } \eta_2 \in \mathbb{F}_p \\ 2p, & \text{if } \eta_2 \not\in \mathbb{F}_p. \end{cases}$$

(b) If $\eta_1 = -1$ and $\eta_2 = 0$, then $Z$ is generated by $e_{32} \otimes v$, and $\dim L^\chi(M) = 2p$.

(c) If $\eta_1 = -1$ and $\eta_2 = -1$, then $Z = \mathbb{F}e_{31} e_{32} \otimes v$, and $\dim L^\chi(M) = 3$.

(d) If $\eta_1 \neq -1$ and $\eta_2 = 0$, then $Z$ is generated by $e_{32} \otimes v$, and

$$\dim L^\chi(M) = \begin{cases} 2\eta_1 + 1, & \text{if } \eta_1 \in \mathbb{F}_p \text{ and } 0 \leq \eta_1 < p - 1 \\ 2p, & \text{if } \eta_1 \not\in \mathbb{F}_p. \end{cases}$$

Proof. We shall only give the proof of assertion (a). (b)–(d) can be proved by applying similar arguments. By Lemma 3.4, $Z^\chi(M)_1$ has a unique maximal vector

$$m' = e_{31} \otimes v - (\eta_2 + 1)^{-1} e_{32} \otimes e_{21} v.$$

Let $Z = Z_0 + Z_1$ be the submodule of $Z^\chi(M)$ generated by $m'$. Then $Z_1$ is spanned by all the $e_{21}^k m'$’s. A short calculation shows that

$$e_{21}^k m' = (1 + k (\eta_2 + 1)^{-1}) e_{31} \otimes e_{21}^k v - (\eta_2 + 1)^{-1} e_{32} \otimes e_{21}^{k+1} v, \quad 0 \leq k \leq p - 1.$$

Note that $e_{21}^{\eta_1 - \eta_2} m' = 0$ if $\eta_2 \not\in \mathbb{F}_p$. Then we obtain

$$Z_1 = \begin{cases} (m', e_{21}^1 m', \ldots, e_{21}^{\eta_1 - \eta_2 - 1} m'), & \text{if } \eta_2 \in \mathbb{F}_p \\ (m', e_{21}^1 m', \ldots, e_{21}^{p-1} m'), & \text{if } \eta_2 \not\in \mathbb{F}_p. \end{cases}$$

Since $[e_{21}, e_{31} e_{32}] = 0$, we have

$$Z_0 = \begin{cases} (e_{31} e_{32} \otimes v, \ldots, e_{31} e_{32} \otimes e_{21}^{\eta_1 - \eta_2} v), & \text{if } \eta_2 \in \mathbb{F}_p \\ (e_{31} e_{32} \otimes v, \ldots, e_{31} e_{32} \otimes e_{21}^{p-1} v), & \text{if } \eta_2 \not\in \mathbb{F}_p. \end{cases}$$

Thus we get

$$L^\chi(M)_0 = \begin{cases} (1 \otimes v, \ldots, 1 \otimes e_{21}^{\eta_1 - \eta_2} v), & \text{if } \eta_2 \in \mathbb{F}_p \\ (1 \otimes v, \ldots, 1 \otimes e_{21}^{p-1} v), & \text{if } \eta_2 \not\in \mathbb{F}_p. \end{cases}$$

and

$$L^\chi(M)_1 = \begin{cases} (e_{32} \otimes v, \ldots, e_{32} \otimes e_{21}^{\eta_1 - \eta_2} v, e_{31} \otimes e_{21}^{\eta_1 - \eta_2} v), & \text{if } \eta_2 \in \mathbb{F}_p \\ (e_{31} \otimes v, \ldots, e_{31} \otimes e_{21}^{p-2} v, e_{32} \otimes v), & \text{if } \eta_2 \not\in \mathbb{F}_p. \end{cases}$$
In case \( \eta_2 \not\in \mathbb{F}_p \), each simple \( u(\mathfrak{g}_0, \chi) \)-module is \( p \)-dimensional by \([4, 5.4]\). Then both \( L^x(M)_0 \) and \( L^x(M)_1 \) are simple. If \( \eta_2 \in \mathbb{F}_p \), a straightforward calculation then shows that \( L^x(M)_0 \) has a unique \( gl_2 \)-maximal vector \( 1 \otimes \overline{v} \) of weight \( \eta_1 - \eta_2 \), and \( L^x(M)_1 \) has a unique \( gl_2 \)-maximal vector \( \overline{e}_{32} \otimes \overline{v} \) of weight \( \eta_1 - \eta_2 + 1 \). Therefore, both \( L^x(M)_0 \) and \( L^x(M)_1 \) are simple \( u(\mathfrak{g}_0, \chi) \)-modules.

Note that \( e_{32} \overline{1} \otimes \overline{v} = e_{32} \otimes \overline{v} \neq 0 \). In the case \( \eta_2 \notin \mathbb{F}_p \), we have \( e_{13}e_{31} \otimes e_{21}^{n_1-n_2}v = \eta_2 \overline{1} \otimes e_{21}^{n_1-n_2}v \neq 0 \). In the case \( \eta_2 \notin \mathbb{F}_p \), we have \( e_{23}e_{32} \otimes \overline{v} = \eta_2 \overline{1} \otimes \overline{v} \neq 0 \). Then \( \text{Lemma 3.6} \) shows that \( L^x(M) \) is simple. Hence, \( Z \) is a maximal submodule. Since each proper submodule of \( Z^x(M) \) contains a maximal vector, hence \( m', Z \) is the unique maximal submodule. \( \square \)

**Proposition 3.8.** Let \( \chi \in \mathfrak{g}_0^* \) be semisimple. Then \( L^x(M) \) has a unique maximal vector \( \overline{1} \otimes \overline{v} \).

**Proof.** We shall only prove the assertion for the case \( \eta_1 = -1 \) and \( \eta_2 \neq 0, -1 \). The assertion for the rest cases can be proved similarly.

By \([9, \text{Prop. 3}]\), \( L^x(M)_i \), \( i = 0, 1 \), has a unique \( gl_2 \)-maximal vector. By the proof of \( \text{Theorem 3.7} \), \( \overline{1} \otimes \overline{v} \) is the only \( gl_2 \)-maximal vector in \( L^x(M)_0 \), and is also maximal. \( L^x(M)_1 \) contains a unique \( gl_2 \)-maximal vector \( \overline{e}_{32} \otimes \overline{v} \). But it is not maximal, since \( e_{23}e_{32} \otimes \overline{v} = \eta_2 \overline{1} \otimes \overline{v} \) and \( \eta_2 \neq 0 \). \( \square \)

**Theorem 3.9.** Assume \( \chi \) is nilpotent and \( \chi(h_1) = \chi(h_2) = 0 \). Let \( M \) be a simple \( u(\mathfrak{g}_0, \chi) \)-module with exceptional weight. Then \( Z^x(M) \) has a unique maximal submodule \( Z = Z_0 \oplus Z_1 \). Set \( L^x(M) = Z^x(M)/Z \). Then \( \text{dim } L^x(M) = 2p \). In particular, we have:

1. If \( \eta_1 = 1 \) and \( \eta_2 \neq 0, -1 \), then \( Z \) is generated by
   \[ e_{31} \otimes v \ominus (\eta_2 + 1)^{-1}e_{32} \otimes e_{21}v. \]
2. If \( \eta_1 = -1 \) and \( \eta_2 = -1 \), then \( Z \) is generated by \( e_{32} \otimes e_{21}v. \)
3. If \( \eta_1 = -1 \) and \( \eta_2 = 0 \), then \( Z \) is generated by \( e_{31} \otimes v \ominus e_{32} \otimes e_{21}v \), and also generated by \( e_{32} \otimes v \).
4. If \( \eta_1 \neq -1, -2 \) and \( \eta_2 = 0 \), then \( Z \) is generated by \( e_{32} \otimes v \), and also generated by
   \[ e_{31} \otimes e_{21}^{n_1+2}v - (\eta_1 + 2)^{-1}e_{32} \otimes e_{21}^{n_1+2}v. \]
5. If \( \eta_1 = -2 \) and \( \eta_2 = 0 \), then \( Z \) is generated by \( e_{32} \otimes v \).

**Proof.** (1) By \( \text{Lemma 3.5} \), \( Z^x(M)_1 \) has a unique maximal vector
   \[ m' = e_{31} \otimes v \ominus (\eta_2 + 1)^{-1}e_{32} \otimes e_{21}v. \]
   Note that
   \[ e^{k}_{21}m' = (1 + k(\eta_2 + 1)^{-1})e_{31} \otimes e_{21}v - (\eta_2 + 1)^{-1}e_{32} \otimes e_{21}^{k+1}v, \quad 0 \leq k < \eta_1 - \eta_2 \]
   \[ e_{21}^{\eta_2}m' = -(\eta_2 + 1)^{-1}e_{32} \otimes e_{21}^{\eta_2-1}v \]
   \[ \ldots \]
   \[ e_{21}^{\eta_1}m' = -(\eta_2 + 1)^{-1}(e_{32} \otimes v - \eta_2 e_{31} \otimes e_{21}^{-1}v). \]
   It is also easy to see that
   \[ Z_1 = \langle m', e_{21}m', \ldots, e_{21}^{\eta_2-1}m' \rangle \]
   and
   \[ Z_0 = \langle e_{31}e_{32} \otimes v, \ldots, e_{31}e_{32} \otimes e_{21}^{\eta_2-1}v \rangle. \]
   This gives us
   \[ L^x(M)_0 = \langle \overline{1} \otimes v, \ldots, 1 \otimes e_{21}^{\eta_2-1}v \rangle \]
   and
   \[ L^x(M)_1 = \langle e_{31} \otimes v, \ldots, e_{31} \otimes e_{21}^{\eta_2-1}v \rangle. \]
   Since \( e_{31}e_{32} \otimes v = -1 \otimes v \neq 0 \), \( \text{Lemma 3.6} \) shows that \( L^x(M) \) is simple. Suppose \( \mathfrak{m} \) is a proper submodule of \( Z^x(M) \). Then \( \mathfrak{m} \) contains \( m' \), hence we get \( \mathfrak{m} = Z \). So \( Z \) is the unique maximal submodule of \( Z^x(M) \).

(2)-(5) can be proved similarly. Details are left to the interested reader. \( \square \)

**Proposition 3.10.** Suppose \( \chi \) is nilpotent and \( \chi(h_1) = \chi(h_2) = 0 \). Then \( L^x(M) \) has a unique maximal vector \( \overline{1} \otimes \overline{v} \) if \( \eta_1 - \eta_2 = p - 1 \), while \( L^x(M) \) has two maximal vectors \( \overline{1} \otimes \overline{v}, 1 \otimes e_{21}^{\eta_2-1}v, \) if \( \eta_1 - \eta_2 \neq p - 1. \)
Proof. We shall only give the proof of the assertion for the case \( \eta_1 = -1 \) and \( \eta_2 \neq 0, -1 \). The assertion for the rest cases is left to the reader.

By [4, 5.4], each simple \( u(g_\ell, \chi) \)-module contains two \( g_\ell \)-maximal vectors: \( v, e_{21}^{\eta_1-\eta_2+1} v \). In the present case, \( L^2(M)_0 \) has maximal vectors \( 1 \otimes v, 1 \otimes e_{21}^{\eta_1-\eta_2+1} v \). \( L^2(M)_1 \) has \( g_\ell \)-maximal vectors \( e_{21}^{\eta_1+\eta_2+2} v \) and \( e_{21}^{\eta_1+\eta_2+1} v \), but neither of which is maximal, since

\[
e_{21} \cdot e_{21}^{\eta_1-\eta_2+1} v = 1 \otimes e_{21}^{\eta_1-\eta_2+1} v \neq 0
\]

and

\[
e_{21} \cdot e_{21}^{\eta_1+\eta_2+2} v = 1 \otimes e_{21}^{\eta_1+\eta_2+2} v = 1 \otimes v \neq 0. \quad \square
\]

Let \( M \) be a simple \( u(g_0, \chi) \)-module generated by the \( g_\ell \)-maximal vector of weight \( (\eta_1, \eta_2) \), and let \( L^2(M)(\eta_1, \eta_2) \) denote the simple quotient of \( Z^x(M) \). Then we have

**Corollary 3.11.** If \( \chi \) is nilpotent and \( \chi(h_1) = \chi(h_2) = 0 \), then

\[
L^2(M)(\eta_1, \eta_2) \cong L^2(M)(\eta_2 - 1, \eta_1 + 1).
\]

Notice that if \( \chi \) is nilpotent, and if \( \chi(h_1) = \chi(h_2) = r \neq 0 \), then \( (\eta_1, \eta_2) \) is not exceptional.

Let \( g = g_\ell \oplus g_0 \oplus g_1 \subseteq gl(m|n) \) be a restricted Lie superalgebra. Recall \( g^+ = g_0 + g_1 \). Then we have

**Lemma 3.12.** Let \( \mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \) be a simple \( u(g', \chi) \)-module. Then there is a simple \( u(g^+, \chi) \)-submodule contained in either \( \mathfrak{M}_0 \) or \( \mathfrak{M}_1 \).

**Proof.** Let \( v \in \mathfrak{M}_0 \cup \mathfrak{M}_1 \) be a maximal vector. Without loss of generality we assume \( v \in \mathfrak{M}_0 \). Then \( u(g_0, \chi)v \subseteq \mathfrak{M}_0 \) is a \( u(g^+, \chi) \)-submodule. So it contains a simple \( u(g^+, \chi) \)-submodule. \( \square \)

To summarize, we have

**Theorem 3.13.** Let \( g = sl(2|1) \) and \( \chi \in g_0^* \) with \( \chi(n^+) = 0 \). Let \( M \) be a simple \( u(g_0, \chi) \)-module having a maximal vector of weight \( (\eta_1, \eta_2) \).

1. If \( (\eta_1, \eta_2) \) is not exceptional, then we have that every simple \( u(g, \chi) \)-module is isomorphic to \( Z^x(M) \). In this case we let \( L^2(M) = Z^x(M) \).

2. If \( \chi \) is semisimple and \( (\eta_1, \eta_2) \) is exceptional, then each simple \( u(g, \chi) \)-module is isomorphic to some \( L^2(M) \) given in **Theorem 3.7**.

3. For each semisimple \( \chi \), there are \( p^2 \) distinct (up to isomorphism) simple \( u(g, \chi) \)-modules. They are represented by \( \{L^2(M)|M \in \mathfrak{S}\} \), where \( \mathfrak{S} \) is a complete set of distinct simple \( u(g_0, \chi) \)-modules.

4. In case \( \chi \) is nilpotent and \( \chi(h_1) = \chi(h_2) = 0 \), then each simple module with exceptional weight is isomorphic to some \( L^2(M) \) given in **Proposition 3.8**. In addition,

\[
\dim L^2(M) = 2p.
\]

5. If \( \chi \) is nilpotent, there are totally \( \frac{p^2 + p}{2} \) distinct simple \( u(g_0, \chi) \)-modules. They are represented by \( \{L^2(M)|M \in \mathfrak{S}\} \), where \( \mathfrak{S} \) is a complete set of distinct simple \( u(g_0, \chi) \)-modules.

**Proof.** Let \( \mathfrak{M} \) be a simple \( u(g, \chi) \)-module, and let \( M \subseteq \mathfrak{M}_0 \) be a simple \( u(g^+, \chi) \)-submodule. By assumption, there is a \( g_\ell \)-maximal vector \( v \in M \) of weight \( (\eta_1, \eta_2) \). The inclusion map \( M \rightarrow \mathfrak{M} \) induces a \( \mathbb{Z}_2 \)-graded \( u(g, \chi) \)-module epimorphism \( \psi : Z^x(M) \rightarrow \mathfrak{M} \).

In case \( (\eta_1, \eta_2) \) is not exceptional, since \( Z^x(M) \) is simple, \( \psi \) is an isomorphism. This proves (1).

In case \( (\eta_1, \eta_2) \) is exceptional, \( Z^x(M) \) has a unique maximal submodule by **Theorems 3.7** and **3.9**. Then \( \mathfrak{M} \) is isomorphic to the unique simple quotient \( L^x(M) \). This proves (2) and (4).

(3) is an immediate consequence of (2) and **Proposition 3.8**.

(5) If \( \chi \) is nilpotent, then we obtain \( L^2(M)(\eta_1, \eta_2) \cong L^2(M)(\eta_2 - 1, \eta_1 + 1) \) from **Theorem 3.3** and **Corollary 3.11**. Note that \( (\eta_1, \eta_2) = (\eta_2 - 1, \eta_1 + 1) \) if and only if \( \eta_2 - \eta_1 = 1 \), so we have totally \( \frac{p^2 + p}{2} \) distinct simple \( u(g, \chi) \)-modules. \( \square \)

Let \( g = sl(2|1) \) and \( \chi \in g_0^* \), and let \( \mathfrak{M} = \mathfrak{M}_0 + \mathfrak{M}_1 \) be a simple \( u(g, \chi) \)-module. Recall that there exists \( \Phi \in Aut(g) \) such that \( \chi^\Phi \) is either semisimple or nilpotent. Then \( \mathfrak{M}^\Phi \) is a simple \( u(g, \chi^\Phi) \)-module. By the above, \( \mathfrak{M}^\Phi \) is isomorphic to certain \( L^x^\Phi(M) \). Two simple \( u(g, \chi) \)-modules \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \) are isomorphic if and only if \( \mathfrak{M}_1^\Phi \) and \( \mathfrak{M}_2^\Phi \) are isomorphic or, equivalently, their corresponding \( L^x^\Phi(M) \)'s are isomorphic. Therefore, there are totally \( \frac{p^2}{2} \) distinct simple \( u(g, \chi) \)-modules if \( \chi^\Phi \) is semisimple, while there are totally \( \frac{p^2 + p}{2} \) distinct simple \( u(g, \chi) \)-modules if \( \chi^\Phi \) is nilpotent.
4. Simple modules for $sl(n|1)$, $n > 2$

In this section we determine the simple modules for the restricted Lie superalgebra $\mathfrak{g} = sl(n|1)$, $n > 2$. Recall $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, and $\mathfrak{g}^+ = \mathfrak{g}_0 + \mathfrak{g}_1$. We denote $h_i = e_n + e_{n+1,i+1}, i = 1, \ldots, n$. Then $\mathfrak{h} = (h_1, h_2, \ldots, h_n)$ is a maximal torus of $\mathfrak{g}_0$. Let $\mathfrak{n}^+$ denote $\sum_{l \leq n} \mathfrak{g}_l \subseteq \mathfrak{g}_0$. Let $\chi \in \mathfrak{g}_0^*$ and $M$ be a simple $u(\mathfrak{g}_0, \chi)$-module. Set $Z^\chi(M) = u(\mathfrak{g}, \chi) \otimes_{u(\mathfrak{g}^+, \chi)} M$.

**Theorem 4.1.** If $\chi(h_1) \chi(h_2) \cdots \chi(h_n) \neq 0$ and $\chi(n^+) = 0$, then $Z^\chi(M)$ is simple.

**Proof.** For $l = \{1, \ldots, i\} \subseteq \{1, \ldots, n\}$, we denote the product $e_{n+1,i} \cdots e_{n+1,0} \in u(\mathfrak{g}, \chi)$ by $e_l$. Assume

$$m = \sum_{l \text{ even}} e_l \otimes m_l \in Z^\chi(M)\hat{\otimes}$$

is a maximal vector of weight $(\lambda_1, \ldots, \lambda_n)$, where each $m_l \in M$. Let $|ij| = \text{Card}[s \in \{1, 2, \ldots, n\} : s < j]$. Then we get in $u(\mathfrak{g}, \chi)$ the following identities:

$$[e_{ij}, e_l] = \begin{cases} (-1)^{|ij|+1} e_{ij} e_l, & \text{if } i < j, \text{ and } i < l < j; \\ 0, & \text{otherwise}. \end{cases}$$

For each $l = 1, \ldots, n$, we have

$$\lambda_l m = h_l \sum e_l \otimes m_l = \sum e_l \otimes h_l m_l + \sum [h_l, e_l] \otimes m_l = \sum e_l \otimes h_l m_l + \sum (|l| - \delta_{00}) e_l \otimes m_l.$$  

This gives us $h_l m_l = (\lambda_l - |l| \delta_{00}) m_l$, $l = 1, \ldots, n$.

Assume $i \leq j$ for every $j \in \mathfrak{l}$. Then it is easy to get

$$[e_{i,n+1}, e_l] = e_{i,n+1} [e_{i,n+1}] - (-1)^{|i|} e_{i} e_{i,n+1} = \begin{cases} \sum_{j=1}^k (-1)^{|i|} e_{i,j} e_{i,j}, & \text{if } i \not\in l \\ h_i e_{i,j} + \sum_{j=2}^k (-1)^{|i|} e_{i,j} e_{i,j}, & \text{if } i \in l. \end{cases}$$

Applying $e_{1,n+1}$ to $m$, we have

$$0 = e_{1,n+1} m = \sum_{l1} h_l e_{l1} \otimes m_l + \sum_{l \not\in l1} (-1)^{|i|} e_{i,j} \otimes m_{i,j} \otimes 1_1$$  

$$= \sum_{l \not\in l1} e_{l1} \otimes \lambda_l m_l + \sum_{j=1}^k (-1)^{|i|} e_{i,j} \otimes m_{i,j} \otimes 1_1.$$  

Fixing $l$ with $1 \leq l$, and taking all the terms involving $e_{l1} \otimes m_l$, we obtain

$$e_{l1} \otimes \left( \lambda_l + \sum_{j=1}^k (-1)^{|i|+j-1} \right) m_l = 0.$$  

Note that $\lambda_l \not\in \mathfrak{p}_0$, so we must have $m_l = 0$ for each $l$ with $1 \leq l$. Thus by induction, we get $m_l = 0$ whenever $l$ is nonempty, hence $m = 1 \otimes m_0 \in 1 \otimes M$. Applying a similar argument, one obtains that $Z^\chi(M)\hat{\otimes}$ contains no maximal vectors. Then $Z^\chi(M)$ is simple by Lemma 2.6. □

Applying a similar argument as that used in the proof of Theorem 3.13, we have

**Corollary 4.2.** Let $\mathfrak{g} = sl(n|1)$ and $\chi \in \mathfrak{g}_0^*$. If $\chi(h_1) \cdots \chi(h_n) \neq 0$ and $\chi(n^+) = 0$, then there are totally $p^n$ distinct simple $u(\mathfrak{g}, \chi)$-modules. They are represented by $[Z^\chi(M)|M \in \mathfrak{S}]$, where $\mathfrak{S}$ is a complete set of simple $u(\mathfrak{g}_0, \chi)$-modules.
References