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An intrinsic Fermat principle on stationary Lorentzian manifolds and applications

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Abstract

In this paper a Fermat principle for Lorentzian manifold endowed with a timelike Killing vector field is formulated. This principle is applied to obtain existence and multiplicity results on the number of light rays joining an event with an integral curve of the Killing vector field. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and statement of the results

In this paper we study lightlike geodesics joining a point with a timelike curve in stationary Lorentzian manifolds. In particular we shall obtain some results on the existence and the multiplicity of such geodesics.

The problem of the number of lightlike geodesics joining a point p with a timelike curve γ on a Lorentzian manifold is motivated by the phenomenon of *gravitational lens*. In General Relativity a space–time is modeled by a 4-dimensional Lorentzian manifold and lightlike geodesics on such a manifold represent the trajectories of light rays. The gravitational lens effect consists in the reception by an observer of two or more images of a light source. It is due to the bending of light rays nearby a heavy mass. As a lens in classical optics, a particular distribution of mass might force the light rays emitted by a source (represented by a timelike curve γ) at different values of its proper time, to converge to the same event on the space–time (represented by a point p).

A natural approach to this problem is based on the extension to General Relativity of the classical Fermat principle in optics:

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the trajectory of a light ray from a source A to a target B is a stationary curve for the travel time among all paths joining the points A and B.

Once such extension has been formulated, several results from critical point theory can be applied to prove existence and multiplicity of light rays.

There is an extensive literature on this subject, where Fermat principles are formulated for different classes of Lorentzian manifolds (see [5] for a brief historical account and for a detailed report of different versions of Fermat principle in General Relativity with applications to gravitational lensing).

The aim of this paper is to study light rays connecting an event with a timelike curve, under intrinsic assumptions on the Lorentzian manifold, in the same spirit of [7], where it is studied the geodesical connectedness of a *stationary* Lorentzian manifold, i.e., a Lorentzian manifold equipped with a timelike Killing vector field.

The paper is organized as follows. In this section we introduce some definitions and we state our results. In Section 2 we develop the variational framework and then we establish the Fermat Principle for a stationary Lorentzian manifold (Theorem 2.5). In Section 3 a number of technical lemmas are collected, as the Palais–Smale condition for the Fermat functionals F_+ and F_- (see (17)). Section 4 is devoted to the proof of the results. In Section 5 we present some application to a certain class of stationary Lorentzian manifold including some relevant space–times as the *Schwarzschild, Reissner–Nordström* and *Kerr* space–times.

Let Λ be a n-dimensional, smooth, connected manifold. Λ is a Lorentzian manifold if it is endowed with a smooth $(0,2)$ tensor field g such that for each $p \in \Lambda$, $g(p): T_p\Lambda \times T_p\Lambda \to \mathbb{R}$ is symmetric, nondegenerate bilinear form of index 1. A tangent vector $v \in T_p\Lambda$ is said *spacelike, lightlike* or *timelike* according to $g(p)[v,v]$ is positive, null, or negative. This tripartition is called the *causal character* of a tangent vector and it is extended to a curve $z: I \to \Lambda$, $I = [\![0,1]\!]$, if its tangent vectors $\dot{z}(s)$, $s \in I$, have the same causal character.

A Lorentzian manifold Λ is said to be *time-oriented* if there exists a continuous timelike vector field on Λ, that is a vector field Y such that $g(p)[Y(p), Y(p)] < 0$ for every $p \in \Lambda$. If Λ is time-oriented, a tangent vector v to Λ at p is said to be *future-pointing* if $g(p)[Y(p), v] < 0$, while it is said *past-pointing* if $g(p)[Y(p), v] > 0$. Analogously a curve is *future-pointing* or *past-pointing* if all its tangent vectors are, respectively, future-pointing or past-pointing.

A vector field Y is a *Killing* field if $L_Y g = 0$, where $L_Y g$ denotes the Lie derivatives of the metric g with respect to Y. Equivalently Y is a Killing vector field if and only if, for all vector fields X and Z on Λ

$$g[\nabla_X Y, Z] = -g[X, \nabla_Z Y], \tag{1}$$

where ∇ is the Levi-Civita connection associated to the metric g. It is well known that Y is a Killing vector field if and only if the stages of its local flows are isometries of (M, g) (see, e.g., [10]).

Definition 1.1. A Lorentzian manifold is said *stationary* if it is endowed with a timelike Killing vector field.

Remark 1.2. Let Λ be a stationary Lorentzian manifold endowed with a timelike Killing vector field Y. Since Y never vanishes, at each point of Λ there exist local coordinates $(x_1, x_2, \ldots, x_{n-1}, t)$ such that

$Y = \frac{\partial}{\partial t}$ on the coordinates neighborhood and the components of the metric, in such a coordinates system, are not depending on the "time coordinate" t (see [8]).

Remark 1.3. Let (Λ, g) be a product Lorentzian manifold such that $\Lambda = \Lambda_0 \times \mathbb{R}$, where Λ_0 is a smooth manifold and g is defined as follows: for any $z = (x, t) \in \Lambda_0 \times \mathbb{R}$ and for any $\zeta = (\xi, \tau) \in T_z\Lambda \equiv T_x\Lambda_0 \times \mathbb{R}$

$$g(z)[\zeta, \zeta] = \langle \xi, \xi \rangle_0 + 2\langle \delta(x), \xi \rangle_0 - \beta(x)\tau^2,$$

where δ and β are respectively a smooth vector field and a positive smooth scalar field on Λ_0, and $\langle \cdot, \cdot \rangle_0$ is a Riemannian metric on Λ_0. (Λ, g) is a stationary manifold, indeed the constant vector field $(x, t) \mapsto (0, 1)$ is a timelike Killing vector field. We will call such a stationary manifold *standard*. When the vector field δ vanishes the standard stationary Lorentzian manifold is called *standard static*. We point out that a stationary Lorentzian manifold has a local structure of standard type (see for instance [7, Appendix C]).

In this paper we assume that Y is complete, that is its flow Ψ is defined on $\Lambda \times \mathbb{R}$. Moreover we assume that the timelike curve γ is an integral curve of Y, i.e., $\gamma : \mathbb{R} \to \Lambda$ and $\dot{\gamma}(s) = Y(\gamma(s))$.

Remark 1.4. Let (Λ, g) be a smooth connected stationary Lorentzian manifold endowed with a timelike Killing vector field Y. Since under a conformal transformation of the metric, a lightlike geodesic is preserved (up to a reparameterization), we can endow Λ with the conformal metric $\langle \cdot, \cdot \rangle$ given by

$$\langle u, v \rangle = -\frac{1}{g(p)[Y(p), Y(p)]} g(p)[u, v],$$

for every $u, v \in T_p\Lambda$. Since the product $g(p)[Y(p), Y(p)]$ is constant along the flow lines of Y, it is easy to see that Y is a timelike Killing vector field also for the metric $\langle \cdot, \cdot \rangle$. Moreover we have

$$\langle Y, Y \rangle = -1. \tag{2}$$

Let us consider the auxiliary metric on Λ defined by

$$\langle u, v \rangle_{(R)} = \langle u, v \rangle + 2\langle u, Y(p) \rangle \langle v, Y(p) \rangle \tag{3}$$

for every $p \in \Lambda$ and $u, v \in T_p\Lambda$. By the wrong way Schwartz inequality (see [10]) it's easy to check that the metric $\langle \cdot, \cdot \rangle_{(R)}$ is Riemannian. Moreover it can be proved that Y is a Killing vector field for the metric $\langle \cdot, \cdot \rangle_{(R)}$.

By the Nash embedding theorem there exists an isometric immersion of the manifold $(\Lambda, \langle \cdot, \cdot \rangle_{(R)})$ in a well defined euclidean space $\mathbb{R}^N$, N depending on the dimension of the manifold Λ. So we shall identify $(\Lambda, \langle \cdot, \cdot \rangle_{(R)})$ with a submanifold of $\mathbb{R}^N$.

Now let $I = [\![0, 1]\!]$ and let us consider the Sobolev space $H^{1,2}(I, \mathbb{R}^N)$. If p and q are points of Λ, $p \neq q$, we can define the set

$$\Omega^{1,2}_{p,q} \equiv \Omega^{1,2}_{p,q}(\Lambda) = \{ z \in H^{1,2}(I, \mathbb{R}^N) \mid z(I) \subset \Lambda,\ z(0) = p,\ z(1) = q \}.$$

It is well known that $\Omega^{1,2}_{p,q}$ is a smooth Hilbert manifold (see [11]); for every $z \in \Omega^{1,2}_{p,q}$, the tangent space at z to $\Omega^{1,2}_{p,q}$ is given by

$$T_z\Omega^{1,2}_{p,q} = \{ \zeta \in H^{1,2}(I, \mathbb{R}^N) \mid \zeta(s) \in T_{z(s)}\Lambda,\ \zeta(0) = 0,\ \zeta(1) = 0 \}.$$

The *action functional* $f:\Omega^{1,2}_{p,q}\longrightarrow\mathbb{R}$,

$$f(z)=\frac{1}{2}\int_0^1\langle\dot z(s),\dot z(s)\rangle\,\mathrm{d}s$$

is well defined on $\Omega^{1,2}_{p,q}$, indeed we have

$$|\langle v,v\rangle|\leqslant\langle v,v\rangle_{(\mathrm{R})},$$

for every $p\in\Lambda$ and $v\in T_p\Lambda$. Moreover f is smooth and its differential at z is given by

$$f'(z)[\zeta]=\int_0^1\langle\nabla_s\zeta(s),\dot z(s)\rangle\,\mathrm{d}s,$$

where $\nabla_s\zeta$ is the covariant derivative of the field ζ along z, with respect to the Levi-Civita connection associated to the metric $\langle\cdot,\cdot\rangle$. It is well known that a curve $z:I\to\Lambda$ is a critical point of f if and only if z is a geodesic for $(\Lambda,\langle\cdot,\cdot\rangle)$ joining p and q, i.e.,

$$\begin{cases}\nabla_s\dot z=0,\\ z(0)=p,\\ z(1)=q.\end{cases}$$

If the manifold is endowed with a Killing vector field, (1) and the equation $\nabla_s\dot z=0$ imply that the geodesics of $(\Lambda,\langle\cdot,\cdot\rangle)$ satisfy the following conservation law:

$$\langle\dot z,Y(z)\rangle=\text{constant}.\tag{4}$$

Thus we can search the geodesics connecting p and q among the curves in $\Omega^{1,2}_{p,q}$ verifying (4) for almost every $s\in I$. Let us denote with $\mathcal{N}_{p,q}$ the set

$$\mathcal{N}_{p,q}=\bigl\{z\in\Omega^{1,2}_{p,q}\bigm|\exists c_z\in\mathbb{R}\colon\ \langle\dot z,Y(z)\rangle=c_z\text{ a.e. on }I\bigr\}.\tag{5}$$

The following result holds (see [7]):

Proposition 1.5. *Let $(p,q)\in\Lambda\times\Lambda$. The set $\mathcal{N}_{p,q}$ is a closed C^2 submanifold of $\Omega^{1,2}_{p,q}$ and, for every $z\in\mathcal{N}_{p,q}$, the tangent space $T_z\mathcal{N}_{p,q}$ is defined by*

$$T_z\mathcal{N}_{p,q}=\bigl\{\zeta\in T_z\Omega^{1,2}_{p,q}\bigm|\exists c_\zeta\in\mathbb{R}\colon\ \langle\nabla_s\zeta,Y(z)\rangle+\langle\dot z,\nabla_\zeta Y(z)\rangle=c_\zeta\ \textit{a.e. on } I\bigr\}.$$

Now fix $p\in\Lambda$ and consider an integral curve of Y, $\gamma:\mathbb{R}\to\Lambda$. Assume that p is not a point of $\gamma(\mathbb{R})$. Let J^t, $t\in\mathbb{R}$, be the restriction of the action functional $f^t:\Omega^{1,2}_{p,\gamma(t)}\to\mathbb{R}$ to the submanifold $\mathcal{N}_{p,\gamma(t)}$. Moreover let $(J^t)^c$ be the set $\{z\in\mathcal{N}_{p,\gamma(t)}\mid J^t(z)\leqslant c\}$. We introduce the following definition:

Definition 1.6. Let c be a real number, we say that $J^t:\mathcal{N}_{p,\gamma(t)}\to\mathbb{R}$ is *c-precompact* if any sequence $\{z_m\}_{m\in\mathbb{N}}\subset(J^t)^c$ has a subsequence converging in the compact-open topology of Λ.

Notice that if $\{z_m\}_{m\in\mathbb{N}}$ converges to z in the compact-open topology, then $\{z_m\}_{m\in\mathbb{N}}$ converges uniformly to z with respect to the distance on Λ induced by any Riemannian metric of Λ.

Remark 1.7. In [7], the authors prove that if the restriction of the action functional to $\mathcal{N}_{p,q}$ is c-precompact for all $c \in \mathbb{R}$ and for all pairs p, q in Λ, then Λ is *globally hyperbolic*. Nevertheless the global hyperbolicity is in general not sufficient to guarantee geodesical connectedness, not even for stationary Lorentzian manifolds (see [7, Appendix B]).

The notion introduced in Definition 1.6 is essential to obtain our existence and multiplicity results on the light rays joining p and $\gamma(\mathbb{R})$. The existence is stated in the following theorem.

Theorem 1.8. *Let Λ be a connected stationary Lorentzian manifold endowed with a complete Killing vector field Y, p an event on Λ and $\gamma : \mathbb{R} \to \Lambda$ an integral curve of Y such that $p \notin \gamma(\mathbb{R})$. Assume that for each $t \in \mathbb{R}$, $J^t : \mathcal{N}_{p,\gamma(t)} \to \mathbb{R}$ is c-precompact, for all $c \in \mathbb{R}$. Then there exists at least one lightlike geodesic joining p and $\gamma(\mathbb{R})$.*

It is worth to point out that the set $\mathcal{N}_{p,\gamma(t)}$ may be empty (see [7] for an example). However it can be proved that if Λ is connected and the Killing vector field Y is complete then for every pair of points p, $q \in \Lambda$, the set $\mathcal{N}_{p,q}$ is nonempty (see [7, Lemma 5.7]).

The result on the multiplicity of lightlike geodesics joining p and $\gamma(\mathbb{R})$ is contained in the following theorem.

Theorem 1.9. *Under the assumptions of Theorem* 1.8, *assume also that Λ is noncontractible in itself. Then there exist a sequence of future-pointing lightlike geodesics $\{l_m^+\}$ and a sequence of past-pointing lightlike geodesics $\{l_m^-\}$ joining p and $\gamma(\mathbb{R})$.*

Remark 1.10. The results of Theorems 1.8 and 1.9 have been obtained for a standard Lorentzian manifold (see [3] and [12]). There are no results for the general case.

Remark 1.11. Since any reparameterization of a geodesic is an affine transformation, we can state that the lightlike geodesics we find in Theorem 1.9, are geometrically distinct.

Remark 1.12. We recall that the *chronology condition* is said to hold on Λ if Λ contains no closed timelike curves (see [10]). We point out that, differently from [3,12], our results cover the case when γ is a closed curve. So in the present paper Λ may not satisfy the chronology condition.

2. The Fermat principle

Let (Λ, g) be a stationary Lorentzian manifold, let Y be a complete timelike Killing vector field on Λ, let $\gamma : \mathbb{R} \to \Lambda$ be an integral curve of Y and let $p \in \Lambda$, $p \notin \gamma(\mathbb{R})$. In this section we prove a Fermat principle for the lightlike geodesics connecting p and $\gamma(\mathbb{R})$.

We start with a characterization of the submanifold $\mathcal{N}_{p,q}$, proved in [7]. Let $\mathcal{W}$ be the distribution of the vector fields parallel to Y, that is ζ belongs to $\mathcal{W}$ if and only if there exist $z \in \Omega^{1,2}_{p,q}$ and $\mu \in H^{1,2}_0(I, \mathbb{R})$ such that $\zeta(s) = \mu(s)Y(z(s))$. Let $\mathcal{W}_z$ be the subspace of $T_z\Omega^{1,2}_{p,q}$ of the vector fields in $\mathcal{W}$; then

$$\mathcal{N}_{p,q} = \left\{ z \in \Omega^{1,2}_{p,q} \mid f'(z)[\zeta] = 0,\ \forall \zeta \in \mathcal{W}_z \right\}. \tag{6}$$

The following variational principle is based on the above characterization of the manifold $\mathcal{N}_{p,q}$.

Proposition 2.1. *Let J be the restriction of f to $\mathcal{N}_{p,q}$, then a curve $z \in \Omega^{1,2}_{p,q}$ is a geodesic on Λ if and only if $z \in \mathcal{N}_{p,q}$ and z is a critical point of J.*

Proof. If z is a geodesic, then $\langle \dot z, Y(z)\rangle$ is a constant, hence $z \in \mathcal{N}_{p,q}$. Moreover z is a critical point for f and for J, too. Now, assume that $z \in \mathcal{N}_{p,q}$ is a critical point for J. If we prove that

$$T_z\Omega^{1,2}_{p,q} = T_z\mathcal{N}_{p,q} \oplus \mathcal{W}_z,$$

by (6) we have that z is also a critical point for f, hence it is a geodesic. Let $\tilde\zeta \in T_z\Omega^{1,2}_{p,q}$, we have to prove that there exist $\mu \in H^{1,2}_0(I,\mathbb{R})$ and $\zeta \in T_z\mathcal{N}_{p,q}$ such that

$$\tilde\zeta = \mu Y(z) + \zeta.$$

The field $\zeta = \tilde\zeta - \mu Y(z)$ belongs to $T_z\mathcal{N}_{p,q}$, if and only if the equation

$$\langle \nabla_s\tilde\zeta, Y(z)\rangle - \mu' - \mu\langle \nabla_s Y(z), Y(z)\rangle + \langle \dot z, \nabla_{\tilde\zeta} Y(z)\rangle - \mu\langle \dot z, \nabla_{Y(z)} Y(z)\rangle = C, \tag{7}$$

is satisfied for some constant C. Since Y is a Killing vector field, we have

$$-\mu\langle \dot z, \nabla_{Y(z)} Y(z)\rangle = \mu\langle \nabla_s Y(z), Y(z)\rangle$$

and

$$\langle \dot z, \nabla_{\tilde\zeta} Y(z)\rangle = -\langle \tilde\zeta, \nabla_s Y(z)\rangle.$$

Therefore (7) becomes

$$\langle \nabla_s\tilde\zeta, Y(z)\rangle - \mu' - \langle \tilde\zeta, \nabla_s Y(z)\rangle = C.$$

Then μ is given by

$$\mu(s) = \int_0^s \big(\langle \nabla_s\tilde\zeta, Y(z)\rangle - \langle \tilde\zeta, \nabla_s Y(z)\rangle - C\big)\,\mathrm{d}r. \tag{8}$$

Clearly $\mu \in H^{1,2}(I,\mathbb{R})$, $\mu(0) = 0$ and, setting

$$C = \int_0^1 \big(\langle \nabla_s\tilde\zeta, Y(z)\rangle - \langle \tilde\zeta, \nabla_s Y(z)\rangle\big)\,\mathrm{d}s, \tag{9}$$

we have $\mu(1) = 0$, too. □

Let us denote by $\Psi : \Lambda \times \mathbb{R} \to \Lambda$ the flow generated by the vector field Y. Let $q = \gamma(0)$ and $t \in \mathbb{R}$. Moreover, consider the point $\gamma(t)$ and the map $\mathcal{F}^t : \Omega^{1,2}_{p,q} \to \Omega^{1,2}_{p,\gamma(t)}$ which maps z into the curve z^t defined by

$$z^t(s) = \Psi\big(z(s), ts\big). \tag{10}$$

Proposition 2.2. *The map $\mathcal{F}^t$ is a diffeomorphism and its inverse map is given by $\mathcal{F}^{-t}$. Moreover let $\mathcal{J}^t$ be the restriction of $\mathcal{F}^t$ to $\mathcal{N}_{p,q}$, then $\mathcal{J}^t$ is a diffeomorphism from $\mathcal{N}_{p,q}$ to $\mathcal{N}_{p,\gamma(t)}$.*

Proof. Let us denote by j the identity map of the interval I. Then the map $\mathcal{F}^t$ is given by

$$z \mapsto (z, tj) \mapsto \Psi \circ (z, tj) = z^t,$$

hence $\mathcal{F}^t$ is a smooth map from $\Omega^{1,2}_{p,q}$ to $\Omega^{1,2}_{p,\gamma(t)}$ (see [11, p. 323, Theorem (4)]). Clearly $\mathcal{F}^t$ has inverse map given by $\mathcal{F}^{-t}$, hence it is a diffeomorphism.

Let us denote by $\mathrm{d}_x\Psi(x_0, u_0)$ the differential of Ψ with respect to the variable $x \in \Lambda$, evaluated at the point (x_0, u_0), and by $\mathrm{d}_u\Psi(x_0, u_0)$ the differential of Ψ with respect to the variable $u \in \mathbb{R}$, evaluated at the point (x_0, u_0). Since Ψ is the flow of Y, it results

$$\mathrm{d}_u\Psi(x_0, u_0)[1] = Y\big(\Psi(x_0, u_0)\big) \tag{11}$$

and

$$\mathrm{d}_x\Psi\big(z(s), ts\big)\big[Y\big(z(s)\big)\big] = Y\big(\Psi\big(z(s), ts\big)\big). \tag{12}$$

Differentiating Eq. (10), since $\mathrm{d}_x\Psi$ is an isometry, Eqs. (11) and (12) give

$$\big\langle \dot z^t, Y\big(z^t\big)\big\rangle = \big\langle \mathrm{d}_x\Psi[\dot z], Y\big(z^t\big)\big\rangle + \big\langle \mathrm{d}_u\Psi[t], Y\big(z^t\big)\big\rangle = \langle \dot z, Y(z)\rangle + t\langle Y(z), Y(z)\rangle = \langle \dot z, Y(z)\rangle - t. \tag{13}$$

By (13) we deduce that $z \in \mathcal{N}_{p,q}$ if and only if $\mathcal{F}^t(z) \in \mathcal{N}_{p,\gamma(t)}$. Therefore $\mathcal{F}^t(\mathcal{N}_{p,q}) = \mathcal{N}_{p,\gamma(t)}$. So $\mathcal{J}^t = \mathcal{F}^t{}_{|\mathcal{N}_{p,q}}$ is actually a diffeomorphism from $\mathcal{N}_{p,q}$ to $\mathcal{N}_{p,\gamma(t)}$. □

By using (2), (11), (12) and the conservation of scalar product by $\mathrm{d}_x\Psi$, the action functional on $\Omega^{1,2}_{p,\gamma(t)}$ evaluated at $z^t = \mathcal{F}^t(z)$ can be written in the following form:

$$\begin{aligned} f^t\big(z^t\big) &= \frac{1}{2}\int_0^1 \big\langle \mathrm{d}_x\Psi(z, ts)[\dot z] + \mathrm{d}_u\Psi(z, ts)[t], \mathrm{d}_x\Psi(z, ts)[\dot z] + \mathrm{d}_u\Psi(z, ts)[t]\big\rangle\,\mathrm{d}s \\ &= \frac{1}{2}\left(\int_0^1 \langle \dot z, \dot z\rangle\,\mathrm{d}s + 2\int_0^1 t\langle \dot z, Y(z)\rangle\,\mathrm{d}s - t^2\right). \end{aligned} \tag{14}$$

Let $H^t : \Omega^{1,2}_{p,q} \to \mathbb{R}$ be the smooth functional defined as

$$H^t(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle\,\mathrm{d}s + t\int_0^1 \langle \dot z, Y(z)\rangle\,\mathrm{d}s - \frac{1}{2}t^2.$$

Clearly, by (14), it results $f^t \circ \mathcal{F}^t = H^t$. Moreover the chain rule applied to the map $f^t \circ \mathcal{F}^t$ implies that

$$\big(f^t\big)'\big(z^t\big)\big[\zeta^t\big] = \big(H^t\big)'(z)[\zeta], \tag{15}$$

for every $z \in \Omega^{1,2}_{p,q}$, $z^t = \mathcal{F}^t(z)$, $\zeta \in T_z\Omega^{1,2}_{p,q}$ and $\zeta^t = \mathrm{d}\mathcal{F}^t(z)[\zeta]$. Now consider the restriction G^t of H^t to $\mathcal{N}_{p,q}$; G^t is given by

$$G^t(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle\,\mathrm{d}s + t\langle \dot z, Y(z)\rangle - \frac{1}{2}t^2, \tag{16}$$

for all $z \in \mathcal{N}_{p,q}$. The following proposition on the critical points of the functional G^t is a consequence of (15).

Proposition 2.3. *Let $z \in \mathcal{N}_{p,q}$. Then z is a critical point of G^t if and only if $z^t = \mathcal{F}^t(z)$ is a critical point of J^t.*

Proof. The map G^t is equal to $J^t \circ \mathcal{J}^t$ and, since $\mathcal{J}^t$ is a diffeomorphism from $\mathcal{N}_{p,q}$ to $\mathcal{N}_{p,\gamma(t)}$, the chain rule yields the thesis. □

The equation $G^t(z) = 0$ defines the following functionals F_+ and F_- on the manifold $\mathcal{N}_{p,q}$:

$$F_+(z) = \langle \dot z, Y(z)\rangle + \sqrt{\langle \dot z, Y(z)\rangle^2 + \int_0^1 \langle \dot z, \dot z\rangle \, \mathrm{d}s}$$

$$= -\langle \dot z, Y(z)\rangle_{(\mathrm{R})} + \sqrt{\int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})} \, \mathrm{d}s - \langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2}, \tag{17}$$

$$F_-(z) = \langle \dot z, Y(z)\rangle - \sqrt{\langle \dot z, Y(z)\rangle^2 + \int_0^1 \langle \dot z, \dot z\rangle \, \mathrm{d}s}$$

$$= -\langle \dot z, Y(z)\rangle_{(\mathrm{R})} - \sqrt{\int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})} \, \mathrm{d}s - \langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2}. \tag{18}$$

Proposition 2.4. *The functional F_+ is well defined on $\mathcal{N}_{p,q}$, it is smooth and for every $z \in \mathcal{N}_{p,q}$*

$$F_+'(z)[\zeta] = \langle \nabla_s \zeta, Y(z)\rangle + \langle \dot z, \nabla_\zeta Y(z)\rangle + \frac{\langle \dot z, Y(z)\rangle[\langle \nabla_s \zeta, Y(z)\rangle + \langle \dot z, \nabla_\zeta Y(z)\rangle] + \int_0^1 \langle \dot z, \nabla_s \zeta\rangle \, \mathrm{d}s}{\sqrt{\langle \dot z, Y(z)\rangle^2 + \int_0^1 \langle \dot z, \dot z\rangle \, \mathrm{d}s}}, \tag{19}$$

for all $\zeta \in T_z\mathcal{N}_{p,q}$.

Proof. The non obvious part of the proposition is to prove that

$$\langle \dot z, Y(z)\rangle^2 + \int_0^1 \langle \dot z, \dot z\rangle \, \mathrm{d}s = \int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})} \, \mathrm{d}s - \langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2 > 0,$$

for every $z \in \mathcal{N}_{p,q}$. From the Schwartz inequality we deduce

$$\int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})} \, \mathrm{d}s - \langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2 = \int_0^1 \left(\langle \dot z, \dot z\rangle_{(\mathrm{R})} - \langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2\right) \mathrm{d}s \geqslant 0, \tag{20}$$

for all $z \in \mathcal{N}_{p,q}$. Thus we have only to prove that

$$\langle \dot z, Y(z)\rangle_{(\mathrm{R})}^2 = \langle \dot z, \dot z\rangle_{(\mathrm{R})} \text{ a.e. on } I, \tag{21}$$

never holds. By contrary assume that (21) holds, then a trivial computation shows that, $\dot{z} = \langle \dot{z}, Y(z)\rangle_{(\mathrm{R})} Y(z)$. Let $a = \langle \dot{z}, Y(z)\rangle_{(\mathrm{R})}$, $a \in \mathbb{R}$, then we can write $\dot{z} = aY(z)$, almost everywhere on I. Since $Y(z)$ is a continuous vector field along z, we have that $\dot{z}$ is continuous, too. Therefore $z(I)$ is contained in the support of the flow line α of Y passing through p. But $z(1) = q \in \gamma(\mathbb{R})$ and γ is a flow line of Y. This means that $\alpha(\mathbb{R})$ and $\gamma(\mathbb{R})$ intersect, hence by the theorem about the uniqueness for the integral curves of a smooth field through a fixed point, they coincide. This is in contradiction with the assumption $p \notin \gamma(\mathbb{R})$. □

An analogous proposition holds for the functional F_-.

Theorem 2.5 (Fermat principle). *The curve $l : I \to \Lambda$ is a lightlike geodesic joining p and $\gamma(\mathbb{R})$ if and only if there exists a couple $(z, t) \in \mathcal{N}_{p,q} \times \mathbb{R}$ such that $t = F_+(z)$ (respectively, $t = F_-(z)$), z is a critical point of F_+ (respectively, F_-) and $l = \mathcal{F}^t(z)$.*

Proof. Define the functional $\mathcal{G} : \mathcal{N}_{p,q} \times \mathbb{R} \to \mathbb{R}$, by setting

$$\mathcal{G}(z, t) = G^t(z).$$

Let l be a lightlike geodesic joining p and $\gamma(\mathbb{R})$ and let $t \in \mathbb{R}$ be such that $l(1) = \gamma(t)$. Let $z = \mathcal{F}^{-t}(l)$, where $\mathcal{F}^{-t}$ is the inverse of $\mathcal{F}^t$. Since $l \in \mathcal{N}_{p,\gamma(t)}$, z is a curve in $\mathcal{N}_{p,q}$ and $l = \mathcal{F}^t(z)$; moreover $J^t(l) = 0$ implies $G^t(z) = 0$, that is $t = F_+(z)$ or $t = F_-(z)$. Let $t = F_+(z)$, then it results

$$\mathcal{G}\big(z, F_+(z)\big) = 0. \tag{22}$$

Differentiating Eq. (22) we get, for all $z \in \mathcal{N}_{p,q}$,

$$\mathcal{G}_z\big(z, F_+(z)\big) + \mathcal{G}_t\big(z, F_+(z)\big)F'_+(z) = 0, \tag{23}$$

where $\mathcal{G}_z(z, F_+(z))$ and $\mathcal{G}_t(z, F_+(z)$ denote, respectively, the differential of $\mathcal{G}$ with respect to the variable $z \in \mathcal{N}_{p,q}$ and to the variable $t \in \mathbb{R}$, evaluated at the point $(z, F_+(z))$. Since $(J^t)'(l) = 0$ and $\mathcal{G}_z(z, F_+(z)) = (G^{F_+(z)})'(z)$, Proposition 2.3 implies

$$\mathcal{G}_z\big(z, F_+(z)\big) = 0.$$

Thus from (23), we get $\mathcal{G}_t(z, F_+(z))F'_+(z) = 0$. Now if

$$0 = \mathcal{G}_t\big(z, F_+(z)\big),$$

then, since

$$\mathcal{G}_t\big(z, F_+(z)\big) = \langle \dot{z}, Y\rangle - t = \langle \dot{z}, Y\rangle - F_+(z) = -\sqrt{\langle \dot{z}, Y(z)\rangle^2 + \int_0^1 \langle \dot{z}, \dot{z}\rangle \,\mathrm{d}s},$$

it would be

$$\langle \dot{z}, Y\rangle^2 + \int_0^1 \langle \dot{z}, \dot{z}\rangle \,\mathrm{d}s = 0, \tag{24}$$

but we have shown in the proof of Proposition 2.4 that (24) never holds. Conversely if $(z,t)\in\mathcal{N}_{p,q}\times\mathbb{R}$ is such that $t=F_+(z)$ and z is a critical point of F_+, then $\mathcal{G}(z,F_+(z))=0$ and consequently $J^{F_+(z)}(z^{F_+(z)})=0$. Moreover by (23) it follows that $\mathcal{G}_z(z,F_+(z))=0$, hence $(G^{F_+(z)})'(z)=0$ and by Proposition 2.3, $(J^{F_+(z)})'(z^{F_+(z)})=0$. Therefore $l=z^{F_+(z)}=\mathcal{J}^{F_+(z)}(z)$ is a lightlike geodesic joining p and $\gamma(\mathbb{R})$. □

Remark 2.6. If $z\in\mathcal{N}_{p,q}$ and $t=F_+(z)$, then substituting the value of t in (13) shows that $\langle\dot z^t, Y(z^t)\rangle$ is a negative constant. Thus we have that the critical points of F_+ are mapped by $\mathcal{J}$ into future-pointing lightlike geodesics. Analogously the critical points of F_- correspond to past-pointing lightlike geodesics, joining p and $\gamma(\mathbb{R})$.

Remark 2.7. Whenever the Lorentzian manifold is not stationary, different version of Fermat principles have been formulated (see [1,4–6]).

3. The Palais–Smale condition for F_+

In this section we shall prove some technical lemmas which are needed to prove the results of this paper. We shall direct our attention only on F_+. Indeed the same arguments hold for F_-.

We first recall a basic lemma contained in [7]. We report its proof for the convenience of the reader.

Lemma 3.1. *Let $t\in\mathbb{R}$ and consider the functional $J^t:\mathcal{N}_{p,\gamma(t)}\to\mathbb{R}$. If J^t is c-precompact for all $c\in\mathbb{R}$, then for any $c\in\mathbb{R}$ there exists a positive constant $D(c)$ such that*

$$\sup_{z\in(J^t)^c}\left|\langle\dot z, Y(z)\rangle\right|\leqslant D(c).$$

Proof. Let $\{z_m\}_{m\in\mathbb{N}}$ be a sequence contained in $(J^t)^c$ such that

$$\lim_{m\to\infty}\left|\langle\dot z_m, Y(z_m)\rangle\right|=\sup_{z\in(J^t)^c}\left|\langle\dot z, Y(z)\rangle\right|.$$

We have to prove that the sequence $\{|\langle\dot z_m, Y(z_m)\rangle|\}_{m\in\mathbb{N}}$ is bounded. By the c-precompactness, passing to a subsequence, we can assume that z_m converges uniformly to a curve $z:I\to\Lambda$. Therefore $\{z_m\}_{m\in\mathbb{N}}$ is definitively contained in a compact neighborhood V of $z([\![0,1]\!])$. The local structure of a stationary manifold (see Remark 1.3) allows us to choose a finite number of local charts of the manifold Λ

$$\left(U_k, x_k^1,\dots,x_k^{n-1},t_k\right)_{1\leqslant k\leqslant r},$$

such that

- $\{U_k\}_{1\leqslant k\leqslant r}$ is a covering of V and, for every $k\in\{1,\dots,r\}$,
  $$U_k=\Lambda_{0,k}\times\,]\!]-\varepsilon_k,\varepsilon_k[\![,$$
  where $\Lambda_{0,k}$ is a submanifold of U_k and ε_k a positive real number;
- for every $k\in\{1,\dots,r\}$, $Y|_{U_k}=\frac{\partial}{\partial t_k}$ and, setting $\mathbf{x}_k=(x_k^1,\dots,x_k^{n-1})$, the Lorentzian metric $\langle\cdot,\cdot\rangle$ on U_k is given by
  $$\langle(v,\tau),(v,\tau)\rangle_{\mathbf{x}_k,t}=g_{0,k}(\mathbf{x}_k)[v,v]+g_{0,k}(\mathbf{x}_k)[\delta_k(\mathbf{x}_k),v]\tau-\tau^2,\qquad(25)$$

where $g_{0,k}$ denotes the Riemannian metric induced by $\langle\cdot,\cdot\rangle$ on $\Lambda_{0,k}$ and δ_k is a smooth vector field on $\Lambda_{0,k}$;

- $\max_k \sup_{U_k} \sqrt{g_{0,k}(\mathbf{x}_k)[\delta_k(\mathbf{x}_k),\delta_k(\mathbf{x}_k)]} = D_0 < +\infty$;
- there exists a finite sequence $0 = a_0 < a_1 < \cdots < a_r = 1$ such that definitively $z_m([\![a_{k-1},a_k]\!]) \subset U_k$, for every $k \in \{1,\dots,r\}$.

We set

$$\triangle_k = \sup_{p_1,p_2\in U_k} |t_k(p_1) - t_k(p_2)|,$$

and

$$\triangle = \max_k \triangle_k.$$

Notice that, by the compactness of V, we can assume $\triangle_k < +\infty$ for all k, hence also $\triangle < +\infty$. Let us denote $g_{0,k}$ with $\langle\cdot,\cdot\rangle_{0,k}$. For m large enough and $s \in [\![a_{k-1},a_k]\!]$ we have $z_m(s) = (\mathbf{x}_{k,m}(s), t_{k,m}(s))$ and $Y(z_m) = (0,1)$. Then for $s \in [\![a_{k-1},a_k]\!]$ we have

$$\langle \dot z_m, Y(z_m)\rangle = \langle (\dot{\mathbf{x}}_{k,m}, \dot t_{k,m}), (0,1)\rangle = \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k} - \dot t_{k,m}. \tag{26}$$

Integrating (26) over $[a_{k-1},a_k]$ gives

$$\langle \dot z_m, Y(z_m)\rangle = \frac{1}{a_k - a_{k-1}}\left(\int_{a_{k-1}}^{a_k} \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s - t_{k,m}(a_k) + t_{k,m}(a_{k-1})\right). \tag{27}$$

Since $\langle \dot z_m, Y(z_m)\rangle$ is a constant, for every $k = 1,2,\dots,r$, we have

$$\left|\langle \dot z_m, Y(z_m)\rangle\right| \leqslant \frac{1}{a_k - a_{k-1}}\left(D_0 \int_{a_{k-1}}^{a_k} \sqrt{\langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}}\,\mathrm{d}s + \triangle\right).$$

Thus the lemma is proved if we show that the sequence of real numbers

$$\left\{\int_{a_{k-1}}^{a_k} \sqrt{\langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}}\,\mathrm{d}s\right\}_{m\in\mathbb{N}} \tag{28}$$

is bounded for at least one value of k. From (25) and (26) we obtain

$$\begin{aligned}\int_{a_{k-1}}^{a_k} \langle \dot z_m, \dot z_m\rangle\,\mathrm{d}s &= \int_{a_{k-1}}^{a_k} \left(\langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k} + 2\langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\dot t_{k,m} - \dot t_{k,m}^2\right)\mathrm{d}s \\ &= \int_{a_{k-1}}^{a_k} \left(\langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k} + \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}^2 - \langle \dot z_m, Y(z_m)\rangle^2\right)\mathrm{d}s.\end{aligned} \tag{29}$$

Substituting (27) in (29), we get

$$\int_{a_{k-1}}^{a_k} \langle \dot{z}_m, \dot{z}_m\rangle\,\mathrm{d}s = \int_{a_{k-1}}^{a_k} \langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s + \int_{a_{k-1}}^{a_k} \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}^2\,\mathrm{d}s$$
$$- \frac{1}{a_k - a_{k-1}}\left(\int_{a_{k-1}}^{a_k} \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s\right)^2$$
$$+ \frac{2(t_{k,m}(a_k) - t_{k,m}(a_{k-1}))}{a_k - a_{k-1}} \int_{a_{k-1}}^{a_k} \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s - \frac{(t_{k,m}(a_k) - t_{k,m}(a_{k-1}))^2}{a_k - a_{k-1}}.$$

By the Hölder's inequality we have

$$\int_{a_{k-1}}^{a_k} \langle \dot{z}_m, \dot{z}_m\rangle\,\mathrm{d}s \geqslant \int_{a_{k-1}}^{a_k} \langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s + \frac{2(t_{k,m}(a_k) - t_{k,m}(a_{k-1}))}{a_k - a_{k-1}} \int_{a_{k-1}}^{a_k} \langle \delta_k(\mathbf{x}_{k,m}), \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s$$
$$- \frac{(t_{k,m}(a_k) - t_{k,m}(a_{k-1}))^2}{a_k - a_{k-1}}. \tag{30}$$

Summing (30) over k we obtain

$$2c \geqslant \int_0^1 \langle \dot{z}_m, \dot{z}_m\rangle\,\mathrm{d}s \geqslant \sum_{k=1}^r \int_{a_{k-1}}^{a_k} \langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}\,\mathrm{d}s - 2\triangle D_0 \sum_{k=1}^r \frac{1}{a_k - a_{k-1}} \int_{a_{k-1}}^{a_k} \sqrt{\langle \dot{\mathbf{x}}_{k,m}, \dot{\mathbf{x}}_{k,m}\rangle_{0,k}}\,\mathrm{d}s$$
$$- \triangle^2 \sum_{k=1}^r \frac{1}{a_k - a_{k-1}}. \tag{31}$$

By (31) it follows that the sequences (28) are bounded for all k, which proves the lemma. □

Now we pass to prove the following lemma that we will use in the proof of the Palais–Smale condition for F_+.

Lemma 3.2. *Assume that for every $t \in \mathbb{R}$ the functional J^t is c-precompact for all $c \in \mathbb{R}$. Let $\{z_m\}_{m\in\mathbb{N}} \subset \mathcal{N}_{p,q}$ and $C > 0$ such that*

$$|F_+(z_m)| \leqslant C. \tag{32}$$

Then

$$\sup_m |\langle \dot{z}_m, Y(z_m)\rangle| < +\infty.$$

Proof. By contradiction, if $\sup_m |\langle \dot{z}_m, Y(z_m)\rangle| = +\infty$ then (17) and (32) implies the existence of a subsequence, which will be denoted again by $\{z_m\}_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} \langle \dot{z}_m, Y(z_m)\rangle = -\infty = \lim_{m\to\infty} -\langle \dot{z}_m, Y(z_m)\rangle_{(\mathrm{R})}. \tag{33}$$

Moreover (17) and (32) yields

$$\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s - \langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})} \leqslant \langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})} + C^2 + 2C\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})}. \tag{34}$$

From (3), (16) and (34) we get

$$\begin{aligned}\frac{1}{2}\int_0^1 \langle \dot z^t_m, \dot z^t_m\rangle\,\mathrm{d}s &= \frac{1}{2}\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s - \langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})} - t\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} - \frac{1}{2}t^2\\ &\leqslant \frac{1}{2}C^2 + C\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} - t\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} - \frac{1}{2}t^2.\end{aligned}$$

Thus fix $t > C$. From (33) it follows

$$\lim_{m\to\infty} J^t\big(z^t_m\big) = -\infty,$$

and there exists $C_1 \in \mathbb{R}$ such that for every $m \in \mathbb{N}$, $J^t(z^t_m) \leqslant C_1$. Hence, by Lemma 3.1,

$$\sup_m \big|\big\langle \dot z^t_m, Y\big(z^t_m\big)\big\rangle\big| < +\infty. \tag{35}$$

On the other hand (13) and (33) imply that $\sup_m |\langle \dot z^t_m, Y(z^t_m)\rangle| = +\infty$, in contradiction with (35). □

Now we can pass to the proof of the Palais–Smale condition for the functional F_+. We recall that a smooth functional f defined on a Hilbert manifold (M, g) satisfies the Palais–Smale condition if every sequence $\{z_m\}_{m\in\mathbb{N}}$, such that $\{f(z_m)\}_{m\in\mathbb{N}}$ is bounded and $\lim_{m\to\infty} \|f'(z_m)\| = 0$ (here $\|f'(z_m)\|$ denotes the norm of the operator $f'(z_m)$ in the Hilbert space $T_{z_m}M$), contains a converging subsequence.

Theorem 3.3. *Assume that for any $t \in \mathbb{R}$ the functional J^t is c-precompact for all $c \in \mathbb{R}$. Then F_+ satisfies the Palais–Smale condition.*

Proof. Let $\{z_m\}_{m\in\mathbb{N}}$ be a sequence of curves contained in $\mathcal{N}_{p,q}$ and $C > 0$ such that

$$|F_+(z_m)| \leqslant C, \tag{36}$$

$$\|F'_+(z_m)\| \to 0. \tag{37}$$

We have

$$|F_+(z_m)| = \left| -\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} + \sqrt{\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s - \langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})}}\right| \leqslant C$$

thus by Lemma 3.2 there exists a constant $C_1 > 0$ such that

$$\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s \leqslant C_1. \tag{38}$$

By (38) we deduce that $\{z_m\}_{m\in\mathbb{N}}$ is bounded in $H^{1,2}(I,\mathbb{R}^N)$ hence, passing to a subsequence, there exists $z\in H^{1,2}(I,\mathbb{R}^N)$ such that $z_m\to z$ weakly in $H^{1,2}(I,\mathbb{R}^N)$. By the definition of $\langle\cdot,\cdot\rangle_{(\mathrm{R})}$ and (38) it follows that

$$\sup_m \int_0^1 \langle \dot z_m, \dot z_m\rangle \,\mathrm{d}s < +\infty. \tag{39}$$

Then, by the c-precompactness, we can extract another subsequence converging uniformly to a curve in Λ. Then $z\in \Omega^{1,2}_{p,q}(\Lambda)$. Let us denote by $A(z)$ the functional

$$A(z)=\int_0^1 \langle \dot z,\dot z\rangle_{(\mathrm{R})}\,\mathrm{d}s-\int_0^1 \langle \dot z, Y(z)\rangle^2_{(\mathrm{R})}\,\mathrm{d}s.$$

Consider now the functional $\widetilde F_+:\Omega^{1,2}_{p,q}(\Lambda)\to\mathbb{R}$

$$\widetilde F_+(z)=-\int_0^1 \langle \dot z, Y(z)\rangle_{(\mathrm{R})}\,\mathrm{d}s+\sqrt{A(z)}.$$

Arguing as in the proof of Proposition 2.4 (now the vector field $\frac{\dot z}{\langle \dot z, Y(z)\rangle_{(\mathrm{R})}}$ is continuous over the subset of the points in I where $\langle \dot z, Y(z)\rangle_{(\mathrm{R})}$ does not vanish) $\widetilde F_+$ is smooth on $\Omega^{1,2}_{p,q}(\Lambda)$ and its restriction to $\mathcal{N}_{p,q}$ is equal to F_+. For every $z\in\Omega^{1,2}_{p,q}(\Lambda)$ and for any $\tilde\zeta\in T_z\Omega^{1,2}_{p,q}(\Lambda)$, the differential of $\widetilde F_+$ in z at $\tilde\zeta$ is given by

$$\begin{aligned}\widetilde F_+'(z)[\tilde\zeta]=&-\int_0^1 \big\langle \nabla^{(\mathrm{R})}_s\tilde\zeta, Y(z)\big\rangle_{(\mathrm{R})}\,\mathrm{d}s-\int_0^1 \big\langle \dot z, \nabla^{(\mathrm{R})}_{\tilde\zeta}Y(z)\big\rangle_{(\mathrm{R})}\,\mathrm{d}s\\ &+\frac{\int_0^1\langle \nabla^{(\mathrm{R})}_s\tilde\zeta,\dot z\rangle_{(\mathrm{R})}\,\mathrm{d}s-\int_0^1\langle \dot z, Y(z)\rangle_{(\mathrm{R})}[\langle \nabla^{(\mathrm{R})}_s\tilde\zeta, Y(z)\rangle_{(\mathrm{R})}+\langle \dot z,\nabla^{(\mathrm{R})}_{\tilde\zeta}Y(z)\rangle_{(\mathrm{R})}]\,\mathrm{d}s}{\sqrt{A(z)}},\end{aligned} \tag{40}$$

where $\nabla^{(\mathrm{R})}$ denotes that Levi-Civita connection with respect to the Riemannian metric $\langle\cdot,\cdot\rangle_{(\mathrm{R})}$. Since z_m converges to z weakly in $H^{1,2}(I,\mathbb{R}^N)$, there exist two sequences $\{\tilde\zeta_m\}_{m\in\mathbb{N}}$ and $\{\nu_m\}_{m\in\mathbb{N}}$ in $H^{1,2}(I,\mathbb{R}^N)$, such that $\tilde\zeta_m\in T_{z_m}\Omega^{1,2}_{p,q}(\Lambda)$, $z_m-z=\tilde\zeta_m+\nu_m$, $\tilde\zeta_m\to 0$ weakly in $H^{1,2}(I,\mathbb{R}^N)$, $\nu_m\to 0$ strongly in $H^{1,2}(I,\mathbb{R}^N)$ (see [9, Proposition 2.9.6]). Moreover, as in the proof of Proposition 2.1, we can define two sequences $\{\zeta_m\}_{m\in\mathbb{N}}$ and $\{\mu_m\}_{m\in\mathbb{N}}$ such that for every $m\in\mathbb{N}$, $\zeta_m\in T_{z_m}\mathcal{N}_{p,q}$, $\mu_m\in H^{1,2}_0(I,\mathbb{R}^N)$ and $\tilde\zeta_m=\zeta_m+\mu_m Y$. Since $\{\tilde\zeta_m\}_{m\in\mathbb{N}}$ is bounded in $T\Omega^{1,2}_{p,q}$, also $\{\zeta_m\}_{m\in\mathbb{N}}$ is bounded in $T\mathcal{N}_{p,q}$. Indeed from the equality $\zeta_m=\tilde\zeta_m-\mu_m Y$, it is sufficient to show that the sequence $\{\mu_m Y\}_{m\in\mathbb{N}}$ is bounded in $T\Omega^{1,2}_{p,q}$. The field Y satisfies

$$\langle Y(z), Y(z)\rangle_{(\mathrm{R})}=1, \tag{41}$$

thus

$$\int_0^1 \langle \mu_m Y(z_m), \mu_m Y(z_m)\rangle_{(\mathrm{R})}\,\mathrm{d}s=\int_0^1 \mu_m^2\,\mathrm{d}s. \tag{42}$$

Furthermore

$$\int_0^1 \big\langle \nabla^{(R)}_{\dot z_m} \mu_m Y(z_m), \nabla^{(R)}_{\dot z_m} \mu_m Y(z_m)\big\rangle_{(R)} \, ds = \int_0^1 \big(\mu'_m\big)^2 \, ds + 2\int_0^1 \mu_m \mu'_m \big\langle Y(z_m), \nabla^{(R)}_{\dot z_m} Y(z_m)\big\rangle_{(R)} \, ds$$
$$+ \int_0^1 \mu_m^2 \big\langle \nabla^{(R)}_{\dot z_m} Y(z_m), \nabla^{(R)}_{\dot z_m} Y(z_m)\big\rangle_{(R)} \, ds.$$

By (41) we obtain, $\langle \nabla^{(R)}_{\dot z_m} Y(z_m), Y(z_m)\rangle_{(R)} = 0$, for almost every $s \in I$, thus we get

$$\int_0^1 \big\langle \nabla^{(R)}_{\dot z_m} \mu_m Y(z_m), \nabla^{(R)}_{\dot z_m} \mu_m Y(z_m)\big\rangle_{(R)} \, ds = \int_0^1 \big(\mu'_m\big)^2 \, ds + \int_0^1 \mu_m^2 \big\langle \nabla^{(R)}_{\dot z_m} Y(z_m), \nabla^{(R)}_{\dot z_m} Y(z_m)\big\rangle_{(R)} \, ds. \tag{43}$$

Let us denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively, the L^2 and the L^∞ norm. If the sequence $\{\|\mu'_m\|_2\}_{m\in\mathbb{N}}$ is bounded then also $\{\|\mu_m\|_2\}_{m\in\mathbb{N}}$ and $\{\|\mu_m\|_\infty\}_{m\in\mathbb{N}}$ are bounded. By (38), the curves z_m have images contained in a compact set of Λ and, since Y is smooth,

$$\left\{ \int_0^1 \mu_m^2 \big\langle \nabla^{(R)}_{\dot z_m} Y(z_m), \nabla^{(R)}_{\dot z_m} Y(z_m)\big\rangle_{(R)} \, ds \right\}_{m\in\mathbb{N}},$$

is bounded. Therefore, from (43), $\{\mu_m Y(z_m)\}_{m\in\mathbb{N}}$ is bounded in $T\Omega^{1,2}_{p,q}$ if $\{\|\mu'_m\|_2\}_{m\in\mathbb{N}}$ is bounded. Since $\langle \dot z, Y(z)\rangle = -\langle \dot z, Y(z)\rangle_{(R)}$, the manifold $\mathcal{N}_{p,q}$ is equivalently defined as

$$\mathcal{N}_{p,q} = \big\{ z \in \Omega^{1,2}_{p,q} \;\big|\; \exists c_z \in \mathbb{R}\colon \langle \dot z, Y(z)\rangle_{(R)} = c_z \text{ a.e. on } I \big\}.$$

By such a definition, for all $z \in \mathcal{N}_{p,q}$, we have that

$$T_z\mathcal{N}_{p,q} = \big\{ \zeta \in T_z\Omega^{1,2}_{p,q} \;\big|\; \exists c_\zeta \in \mathbb{R}\colon \big\langle \nabla^{(R)}_s \zeta, Y(z)\big\rangle_{(R)} + \big\langle \dot z, \nabla^{(R)}_s Y(z)\big\rangle_{(R)} = c_\zeta \text{ a.e.} \big\}.$$

As in the proof of Proposition 2.1, we can write μ_m by means of (8) and define the constants C_m as in (9), where now $\nabla^{(R)}_{\dot z_m}$ and $\langle\cdot,\cdot\rangle_{(R)}$ take, respectively, the roles of ∇_s and $\langle\cdot,\cdot\rangle$. Then it follows:

$$|C_m| \leqslant \big\| \nabla^{(R)}_{\dot z_m} \tilde\zeta_m \big\|_2 + \|\tilde\zeta_m\|_2 \big\| \nabla^{(R)}_{\dot z_m} Y(z_m)\big\|_2 \tag{44}$$

and

$$\int_0^1 \big(\mu'_m\big)^2 \, ds \leqslant 3\big\| \nabla^{(R)}_{\dot z_m} \tilde\zeta_m \big\|_2^2 + 3\|\tilde\zeta_m\|_\infty^2 \big\| \nabla^{(R)}_{\dot z_m} Y(z_m)\big\|_2^2 + 3C_m^2. \tag{45}$$

By (38), (44) and (45), it follows that $\{\|\mu'_m\|_2\}_{m\in\mathbb{N}}$ is bounded and consequently $\{\zeta_m\}_{m\in\mathbb{N}}$ is bounded in $T\mathcal{N}_{p,q}$. Therefore, (37) implies that

$$F'_+(z_m)[\zeta_m] \to 0. \tag{46}$$

Since Y is a Killing field we have $\langle \dot z, \nabla^{(R)}_Y Y(z)\rangle_{(R)} = -\langle \nabla^{(R)}_s Y(z), Y(z)\rangle_{(R)}$ and $\langle \nabla^{(R)}_s Y(z), \dot z\rangle_{(R)} = 0$. Moreover (41) implies $\langle \nabla^{(R)}_s Y(z), Y(z)\rangle_{(R)} = 0$. So recalling that $\langle \dot z_m, Y(z_m)\rangle_{(R)}$ is a constant s-a.e., from

(40), we easily obtain

$$\widetilde{F}'_+(z_m)[\mu_m Y] = 0, \tag{47}$$

for every $m \in \mathbb{N}$. Since it results

$$\widetilde{F}'_+(z_m)[\tilde{\zeta}_m] = \widetilde{F}'_+(z_m)[\zeta_m + \mu_m Y],$$

from (46) and (47), we get

$$\widetilde{F}'_+(z_m)[\tilde{\zeta}_m] \to 0,$$

that is

$$\begin{aligned}\widetilde{F}'_+(z_m)[\tilde{\zeta}_m] = &-\int_0^1 \bigl\langle \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m, Y(z_m)\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s - \int_0^1 \bigl\langle \dot{z}_m, \nabla^{(\mathrm{R})}_{\tilde{\zeta}_m} Y(z_m)\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s + \frac{\int_0^1 \langle \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m, \dot{z}_m\rangle_{(\mathrm{R})}\,\mathrm{d}s}{\sqrt{A(z_m)}} \\ &+ \frac{-\langle \dot{z}_m, Y(z_m)\rangle_{(\mathrm{R})}\int_0^1 \langle \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m, Y(z_m)\rangle_{(\mathrm{R})} + \langle \dot{z}_m, \nabla^{(\mathrm{R})}_{\tilde{\zeta}_m} Y(z_m)\rangle_{(\mathrm{R})}\,\mathrm{d}s}{\sqrt{A(z_m)}} \to 0.\end{aligned} \tag{48}$$

Since $\tilde{\zeta}_m$ converges weakly and uniformly to 0, z_m uniformly to z and (38) holds, we deduce that

$$-\int_0^1 \bigl\langle \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m, Y(z_m)\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s - \int_0^1 \bigl\langle \dot{z}_m, \nabla^{(\mathrm{R})}_{\tilde{\zeta}_m} Y(z_m)\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s \to 0. \tag{49}$$

Recalling that the sequences $\{\langle \dot{z}_m, Y(z_m)\rangle_{(\mathrm{R})}\}$ and $\{A(z_m)\}$ are bounded and multiplying both hand sides of (48) by $\sqrt{A(z_m)}$, from (49), we obtain

$$\int_0^1 \bigl\langle \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m, \dot{z}_m\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s \to 0.$$

Thus, since $z_m - z = \tilde{\zeta}_m - \nu_m$, we have

$$\begin{aligned}\int_0^1 \langle \dot{z}_m - \dot{z}, \dot{z}_m - \dot{z}\rangle_{(\mathbb{R}^N)}\,\mathrm{d}s &= \int_0^1 \langle \dot{z}_m - \dot{z}, \dot{\tilde{\zeta}}_m + \dot{\nu}_m\rangle_{(\mathbb{R}^N)}\,\mathrm{d}s \\ &= \int_0^1 \bigl\langle \dot{z}_m - \dot{z}, \nabla^{(\mathrm{R})}_{\dot{z}_m}\tilde{\zeta}_m\bigr\rangle_{(\mathrm{R})}\,\mathrm{d}s + \int_0^1 \langle \dot{z}_m - \dot{z}, \dot{\nu}_m\rangle_{(\mathbb{R}^N)}\,\mathrm{d}s \to 0,\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^N}$ denotes the euclidean product in $\mathbb{R}^N$, $\dot{\tilde{\zeta}}_m$ and $\dot{\nu}_m$ the derivatives of the vector fields $\tilde{\zeta}_m$ and ν_m in $\mathbb{R}^N$. Therefore, $z_m \to z$ strongly in $H^{1,2}(I, \mathbb{R}^N)$. Hence there exists a subsequence of $\{\dot{z}_m\}_{m\in\mathbb{N}}$ which converges almost everywhere to $\dot{z}$. Consequently $\langle \dot{z}, Y(z)\rangle_{(\mathrm{R})}$ is a constant almost everywhere on I and $z \in \mathcal{N}_{p,q}$. □

4. Proof of Theorems 1.8 and 1.9

In this section we will prove the Theorems 1.8 and 1.9. In the next lemma we show that F_+ is bounded from below.

Lemma 4.1. *Under the assumptions of Theorem* 1.8, F_+ *is bounded from below.*

Proof. Let $\{z_m\}$ be a minimizing sequence for F_+ and assume by contradiction that

$$\lim_{m\to\infty} F_+(z_m) = -\infty. \tag{50}$$

Then, for any m large enough,

$$\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s \leqslant 2\langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})}. \tag{51}$$

Moreover from (50) we deduce that

$$\lim_{m\to\infty} -\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} = -\infty. \tag{52}$$

Now, let $t \in \mathbb{R}$, $t > 0$. By (3), (16) and (51) we get

$$\frac{1}{2}\int_0^1 \langle \dot z^t_m, \dot z^t_m\rangle\,\mathrm{d}s = \frac{1}{2}\int_0^1 \langle \dot z_m, \dot z_m\rangle_{(\mathrm{R})}\,\mathrm{d}s - \langle \dot z_m, Y(z_m)\rangle^2_{(\mathrm{R})} - t\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} - \frac{1}{2}t^2$$
$$\leqslant -t\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})} - \frac{1}{2}t^2.$$

Therefore it results

$$\lim_{m\to\infty} J^t\big(z^t_m\big) = -\infty,$$

and there exists a constant $c \in \mathbb{R}$ such that for any $m \in \mathbb{N}$ $J^t(z^t_m) \leqslant c$. By Lemma 3.1, it is

$$\sup_m \big|\big\langle \dot z^t_m, Y\big(z^t_m\big)\big\rangle\big| < +\infty.$$

Then (13) implies that

$$\sup_m |\langle \dot z_m, Y(z_m)\rangle_{(\mathrm{R})}| < +\infty,$$

in contradiction with (52). □

Proof of Theorem 1.8. By Lemma 4.1, the functional F_+ is bounded from below. Moreover, by Theorem 3.3, it satisfies the Palais–Smale condition. Finally the sublevels of F_+ are complete metric spaces. Indeed if $\{z_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $(F_+)^c$, then it converges to a curve $z \in H^{1,2}(I, \mathbb{R}^N)$. Since F_+ is bounded from below, the assumptions of Lemma 3.2 are verified, hence it is

$$\sup_m |\langle \dot z_m, Y(z_m)\rangle| < +\infty.$$

Arguing as in the proof of Theorem 3.3, we get (39). By the c-precompactness, we have that the sequence $\{z_m\}_{m\in\mathbb{N}}$ has a subsequence converging uniformly to a curve in Λ. Such a curve must coincide with z, hence $z \in \mathcal{N}_{p,q}$. Thus the sublevels of F_+ are complete metric spaces. By a well known theorem in Critical Point Theory (see, e.g., [9]), these properties of F_+ imply that it attains its infimum at a point z on $\mathcal{N}_{p,q}$. By Theorem 2.5 such a minimum point provides a lightlike geodesic, $z^{F_+(z)} = \mathcal{F}^{F_+(z)}(z)$, joining p and $\gamma(F_+(z))$. □

The proof of Theorem 1.9 is based on the Ljusternik–Schnirelmann category. We recall that if X is a topological space and A is a subspace of X, the Ljusternik–Schnirelmann category of A in X, denoted by $\operatorname{cat}_X A$, is the minimum number of closed, contractible subsets of X, covering A. If A is not covered by a finite number of closed, contractible subsets of X, we set $\operatorname{cat}_X A = +\infty$. Moreover we will denote by $\operatorname{cat} X$ the category of X in X.

Proof of Theorem 1.9. Since Λ is not contractible in itself, a well known result by Fadell and Husseini (see [2]) says that $\operatorname{cat} \Omega^{1,2}_{p,q} = +\infty$. By the completeness of Y, it can be proved that also $\operatorname{cat} \mathcal{N}_{p,q} = +\infty$ (see [7, p. 186]). Moreover $\mathcal{N}_{p,q}$ is an Hilbert manifold and F_+ is bounded from below, satisfies the Palais–Smale condition and has sublevels which are complete metric spaces. Therefore, by a standard argument in Critical Point Theory F_+ has infinitely many critical points and it diverges on the set of its critical points. Thus there exists a sequence $\{z_m\}_{m\in\mathbb{N}}$ of critical points of F_+ which, by Theorem 2.5, provides a sequence $\{l_m^+\}_{m\in\mathbb{N}}$ of lightlike geodesic of Λ, such that, for every $m \in \mathbb{N}$, $l_m^+ = \mathcal{J}^{F_+(z_m)}(z_m)$ and l_m^+ joins p and $\gamma(F_+(z_m))$. Moreover, by Remark 2.6, we can conclude that the sequence $\{l_m^+\}$ consists of future-pointing lightlike geodesics. □

Remark 4.2. The result on the existence and multiplicity of light rays connecting p and $\gamma(\mathbb{R})$ and pointing into the past can be obtained using the functional F_- instead of F_+.

5. Application to standard stationary Lorentzian manifolds

In Remark 1.3 we recalled the definition of standard stationary Lorentzian manifolds. Clearly $\frac{\partial}{\partial t}$ is a timelike Killing vector field for such manifolds. Its integral curves are the vertical lines $r \in \mathbb{R} \mapsto (x_0, r) \in \Lambda$, for all $x_0 \in \Lambda_0$.

In this subsection the coefficient β will be assumed constant and equal to 1 (see Remark 1.4). Moreover we require the metric g to satisfy the following assumptions:

- the Riemannian manifold Λ_0 is complete;
- the vector field δ is bounded, that is there exists a positive constant $D_0 \in \mathbb{R}$ such that

$$\sup_{x\in M_0} \sqrt{\langle \delta(x), \delta(x)\rangle_0} \leqslant D_0.$$

Let $p = (\bar{x}, \bar{t}) \in \Lambda = M_0 \times \mathbb{R}$ and consider the vertical line γ through the point $q = (x_0, 0)$, $\bar{x} \neq x_0$. Fix $r \in \mathbb{R}$ and set $\triangle = r - \bar{t}$. In this setting the manifolds $\mathcal{N}_{p,\gamma(r)}$ are given by

$$\mathcal{N}_{p,\gamma(r)} = \left\{ z \equiv (x,t) \in \Omega^{1,2}_{p,\gamma(r)} \;\middle|\; \exists c_z \in \mathbb{R}\colon \langle \delta(x), \dot{x}\rangle_0 - \dot{t} = c_z \text{ a.e. on } I \right\}. \tag{53}$$

The constant c_z in (53) can be easily computed, namely integration over I provides

$$c_z = \int_0^1 \langle \delta(x), \dot{x} \rangle_0 - \triangle, \tag{54}$$

so c_z depends solely on x and it will be denoted by c_x.

We are going to see that the functionals $\{J^r\}_{r\in\mathbb{R}}$ are c-precompact for every $c \in \mathbb{R}$. Let $\{z_m\}_{m\in\mathbb{N}}$, $z_m \equiv (x_m, t_m)$, a sequence of curves contained in the sublevel $(J^r)^c \subset \mathcal{N}_{p,\gamma(r)}$, $c \in \mathbb{R}$. Taking into account the definition of $\mathcal{N}_{p,\gamma(r)}$ (cf. (53)), the action functional evaluated on such a sequence is

$$\begin{aligned}\frac{1}{2}\int_0^1 \langle \dot{z}_m, \dot{z}_m \rangle \,\mathrm{d}s &= \frac{1}{2}\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_0 \,\mathrm{d}s + \int_0^1 \langle \delta(x_m), \dot{x}_m \rangle_0 \dot{t}_m \,\mathrm{d}s - \frac{1}{2}\int_0^1 \dot{t}_m^2 \,\mathrm{d}s \\ &= \frac{1}{2}\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_0 \,\mathrm{d}s + \frac{1}{2}\int_0^1 \langle \delta(x_m), \dot{x}_m \rangle_0^2 \,\mathrm{d}s - \frac{1}{2}c_{x_m}^2.\end{aligned}$$

Since $\frac{1}{2}\int_0^1 \langle \dot{z}_m, \dot{z}_m \rangle \,\mathrm{d}s \leqslant c$, from (54) we obtain

$$\begin{aligned}\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_0 \,\mathrm{d}s &\leqslant 2c - \int_0^1 \langle \delta(x_m), \dot{x}_m \rangle_0^2 \,\mathrm{d}s + c_{x_m}^2 \leqslant 2c - 2\triangle \int_0^1 \langle \delta(x_m), \dot{x}_m \rangle_0 \,\mathrm{d}s + \triangle^2 \\ &\leqslant 2c + 2\triangle D_0 \int_0^1 \sqrt{\langle \dot{x}_m, \dot{x}_m \rangle_0} \,\mathrm{d}s + \triangle^2.\end{aligned}$$

This last inequality implies that $\sup_m \int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_0 \,\mathrm{d}s < +\infty$, so by the Ascoli–Arzelà Theorem there exists a subsequence of $\{x_m\}_{m\in\mathbb{N}}$ converging uniformly to a curve on Λ_0. From (54) and the equality

$$\int_0^1 \dot{t}_m^2 \,\mathrm{d}s = \int_0^1 \big(\langle \delta(x_m), \dot{x}_m \rangle_0 - c_{x_m}\big)^2 \,\mathrm{d}s$$

we deduce that $\sup_m \int_0^1 \dot{t}_m^2 \,\mathrm{d}s < +\infty$; so the sequence $\{t_m\}_{m\in\mathbb{N}}$ admits a subsequence uniformly converging to a curve $t : I \to \mathbb{R}$. As a consequence, from the sequence $\{z_m\}_{m\in\mathbb{N}}$ we can extract a subsequence converging uniformly to the curve (x, t).

If we take $q = (x_1, 0)$, a simple calculation shows that functionals F_+ and F_- are independent on the t component of the curve $z = (x, t)$ and are defined in the following way:

$$F_+(z) \equiv F_+(x) = t_0 + \int_0^1 \langle \delta(x), \dot{x} \rangle_0 \,\mathrm{d}s + \sqrt{\int_0^1 \langle \delta(x), \dot{x} \rangle_0^2 \,\mathrm{d}s + \int_0^1 \langle \dot{x}, \dot{x} \rangle_0 \,\mathrm{d}s},$$

$$F_-(z) \equiv F_-(x) = t_0 + \int_0^1 \langle \delta(x), \dot{x} \rangle_0 \,\mathrm{d}s - \sqrt{\int_0^1 \langle \delta(x), \dot{x} \rangle_0^2 \,\mathrm{d}s + \int_0^1 \langle \dot{x}, \dot{x} \rangle_0 \,\mathrm{d}s}.$$

Now we pass to study some physically relevant space–times. We shall prove that the results obtained in this paper can be applied to such space–times.

5.1. Schwarzschild space–time

The Schwarzschild space–time is the solution of Einstein equations, representing the spherically symmetric empty space–time outside a spherically symmetric massive body (see [8]). It is defined as follows: let m be a positive constant (the mass of the body) and

$$\Lambda_0 = \{x \in \mathbb{R}^3 \mid |x| > 2m\}$$

($|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$), the Schwarzschild space–time is the manifold $\Lambda = \Lambda_0 \times \mathbb{R}$ endowed with the metric

$$\mathrm{d}s^2 = \frac{1}{\beta(r)}\,\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) - \beta(r)\,\mathrm{d}t^2, \tag{55}$$

where (r,θ,ϕ) are the polar coordinate on $\mathbb{R}^3$ and $\beta(r) = 1 - \frac{2m}{r}$. Hence Λ is a static standard stationary Lorentz manifold and $\frac{\partial}{\partial t}$ is a timelike Killing vector field. It is well known that Λ_0 endowed with the conformal metric $\frac{\mathrm{d}s^2}{\beta}$ is complete. So if $p = (\bar{x}, \bar{t})$ and $q = (x_0, 0)$ are two points on Λ and γ is the vertical line through q, then for each $r \in \mathbb{R}$, the functional $(J^r)^c$ is c-precompact for all $c \in \mathbb{R}$. Therefore Theorem 1.9 holds in the Schwarzschild space–time.

5.2. Reissner–Nordström space–time

The Reissner–Nordström space–time describes the space–time outside a spherically symmetric massive body carrying an electric charge (see [8]). There exist coordinates in which the metric has the form (55) with

$$\beta(r) = 1 - \frac{2m}{r} + \frac{e^2}{m^2}, \tag{56}$$

where m is the mass and e the charge of the body. As in the Schwarzschild, whenever the electric charge e satisfies the condition $e^2 < m^2$, Theorem 1.9 holds outside the first event horizon, that is on the manifold $\{x \in \mathbb{R}^3\colon\ |x| > m + \sqrt{m^2 - e^2}\} \times \mathbb{R}$ endowed with the static metric (55), with β given by (56).

5.3. Kerr space–time

Finally we give an outline of the Kerr space–time outside the *stationary limit surface*. It is the stationary gravitational field outside a rotating massive object which cover the so-called stationary limit surface. In mathematical terms, if m is the mass of the body, ma is its angular moment as measured from infinity, (r,θ,ϕ) are the usual polar coordinate in $\mathbb{R}^3$ and $m^2 > a^2$, the Kerr space–time outside the stationary limit surface is the Lorentzian manifold $\{x \in \mathbb{R}^3\colon\ |x| > m + \sqrt{m^2 - a^2\cos^2\theta}\} \times \mathbb{R}$ endowed with the stationary metric

$$\mathrm{d}s^2 = \rho^2\left(\frac{\mathrm{d}r^2}{D} + \mathrm{d}\theta^2\right) + \big(r^2 + a^2\big)\sin^2\theta\,\mathrm{d}\phi^2 - \mathrm{d}t^2 + \frac{2mr}{\rho^2}\big(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t\big)^2,$$

where $\rho^2 = \rho^2(r,\theta) = r^2 + a^2\cos^2\theta$ and $D = D(r) = r^2 - 2mr + a^2$. If we suppose that the surface of the rotating body is very close to the stationary limit surface, $r = m + \sqrt{m^2 - a^2\cos^2\theta}$, and the coefficient a is small, an analogue of Theorem 1.9 can be proved, provided that the notion of manifold with smooth *light-convex boundary* is introduced. In this case, we should assume that the timelike Killing vector field is tangent to the boundary at each of its points and we should replace the functional F_+ by a family of perturbed functionals satisfying the c-precompactess condition (see [3] or [9] for the notion of light-convex boundary and for the analogue of Theorem 1.9 in the context of standard stationary Lorentzian manifolds).

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