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Maximal and minimal entry in the principal eigenvector for the distance matrix of a graph

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ABSTRACT

Let $G = (V, E)$ be a simple, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Also let $D(G)$ be the distance matrix of a graph G (Janežič et al., 2007) [13]. Here we obtain Nordhaus–Gaddum-type result for the spectral radius of distance matrix of a graph.

A sharp upper bound on the maximal entry in the principal eigenvector of an adjacency matrix and signless Laplacian matrix of a simple, connected and undirected graph are investigated in Das (2009) [4] and Papendieck and Recht (2000) [15]. Generally, an upper bound on the maximal entry in the principal eigenvector of a symmetric nonnegative matrix with zero diagonal entries and without zero diagonal entries are investigated in Zhao and Hong (2002) [21] and Das (2009) [4], respectively. In this paper, we obtain an upper bound on minimal entry in the principal eigenvector for the distance matrix of a graph and characterize extremal graphs. Moreover, we present the lower and upper bounds on maximal entry in the principal eigenvector for the distance matrix of a graph and characterize extremal graphs.

1. Introduction

Since the distance matrix and related matrices, based on graph-theoretical distances [13], are rich sources of many graph invariants (topological indices) that have found use in structure-property-activity modeling [7,14,17], it is of interest to study spectra and polynomials of these matrices [9,19].

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For $i \in V$, d_i is the degree of the i-th vertex of G, $i = 1, 2, \ldots, n$. The minimum vertex degree is denoted by δ and the maximum by Δ. The diameter of a connected graph G, d, is the maximum distance between two vertices of G. The distance matrix $D(G)$ of G is an $n \times n$ matrix $(d_{i,j})$ such that $d_{i,j}$ is just the distance (i.e., the number of edges of a shortest path) between the vertices i and j in G [14]. A set S of vertices in a graph G is called independent if no two vertices in S are connected by an edge. The maximum cardinality of such a set is called the independence number of G and denoted by α.

Let $A = (a_{i,j})$ be a symmetric nonnegative irreducible square matrix of order n. The smallest diagonal entry of A is denoted by m and the largest by M. Also let $\mu(A)$ be the spectral radius of matrix A. It is well known that the p-norm of a vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ is defined as follows:

$$\|\mathbf{X}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \quad \text{if } 1 \le p < \infty. \tag{1}$$

Clearly, $D(G)$ is a real symmetric nonnegative irreducible matrix and hence all its eigenvalues are real. The distance eigenvalue of largest magnitude is the distance spectral radius, denoted by μ; it corresponds, by the Perron–Frobenius theorem [8], to a positive distance eigenvector. Balaban et al. [3] proposed the use of μ as a structure-descriptor, and it

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was successfully used to make inferences about the extent of branching and boiling points of alkanes [3,10]. In [5,11,22], some lower and upper bounds on the spectral radius of the distance matrix of a graph are presented. Balasubramanian [2] computed the spectrum of its distance matrix using the Givens–Householder method. In [1], the distance polynomials (that is, the characteristic polynomials of the distance matrices) were computed for several graphs. There exists a vast amount of literature that studies the spectral radius of the distance matrix. We refer the reader to [12,22,20] for surveys and more information. Let $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$ be the positive eigenvector corresponding to the spectral radius $\mu = \mu(D(G))$. We may choose $\mathbf{Y}$ so that $\|\mathbf{Y}\|_p = 1$ $(1 \leq p < \infty)$ and the unique positive vector $\mathbf{Y}$ is called the principal eigenvector of $D(G)$. The principal eigenvector is of interest since it is often used in applications (see [18] for an overview). Let $y^{\max_p}$ and $y^{\min_p}$ be the maximal and the minimal entry of the principal eigenvector of $D(G)$. The path, star and complete graph of order n are denoted by P_n, $K_{1,n-1}$ and K_n, respectively.

Papendieck and Recht [15] obtained an upper bound on the maximal entry $y^{\max_p}$ of the principal eigenvector of adjacency matrix of a simple, connected and undirected graph:

$$y^{\max_p} \leq \left(\frac{\rho^{p-2}}{1+\rho^{p-2}}\right)^{1/p}, \tag{2}$$

where ρ is the spectral radius of adjacency matrix. Moreover, the equality holds if and only if $G \cong K_{1,n-1}$.

In [4], we obtained an upper bound on the maximal entry $y^{\max_p}$ of the principal eigenvector of signless Laplacian matrix $Q(G)$ of a simple, connected and undirected graph:

$$y^{\max_p} \leq \left(\frac{(q-\delta)^{p-1}}{q-\Delta+(q-\delta)^{p-1}}\right)^{1/p} \quad p \geq 1, \tag{3}$$

where q is the spectral radius of $Q(G)$, and Δ, δ are the maximum and minimum degrees of G, respectively. Moreover, the equality holds in (3) if and only if G is isomorphic to a super graph of star $K_{1,n-1}$ for $p = 1$ or $G \cong K_{1,n-1}$ for $p > 1$.

In [4], we obtained a sharp upper bound on the maximal entry in the principal eigenvector of symmetric nonnegative irreducible matrix in terms of its order (n), the largest (M) and the smallest (m) diagonal entries of that matrix. The result is as follows:

$$y^{\max_p} \leq \left(\frac{(n-1)^{(p-2)/2}(\mu-m)^{p/2}}{(n-1)^{(p-2)/2}(\mu-m)^{p/2}+(\mu-M)^{p/2}}\right)^{1/p} \quad (p \geq 2), \tag{4}$$

and equality can be attained. Moreover, the upper bound of Zhao and Hong [21] for the maximal entry $y^{\max_p}$ follows as a special case.

The paper is organized as follows. In Section 2, we give a list of lemmas. In Section 3, we obtain Nordhaus–Gaddum-type result for the spectral radius of distance matrix of a graph. In Section 4, we present the upper bound on the minimal entry in the principal eigenvector of the distance matrix of a graph and characterize the graphs which achieve the upper bound. In Section 5, we give the lower and upper bounds on the maximal entry in the principal eigenvector of the distance matrix of a graph and characterize extremal graphs.

2. Bounds on the spectral radius of distance matrix of a graph

In this section we give the lower and upper bounds on the spectral radius of the distance matrix of a graph.

Lemma 2.1 ([11]). *Let G be a simple connected graph with Wiener index W. Then*

$$\mu \geq \frac{2W}{n}$$

with equality holding if and only if $D_1 = D_2 = \cdots = D_n$, where $D_i = \sum_{j=1}^{n} d_{i,j}$, $W = \frac{1}{2}\sum_{i=1}^{n} D_i$.

Corollary 2.2. *Let G be a simple connected graph of order n. Then*

$$\mu \geq n-1 \tag{5}$$

with equality holding in (5) if and only if $G \cong K_n$.

Proof. For each i,

$$D_i \geq n-1.$$

Since $2W = \sum_{i=1}^{n} D_i$, by Lemma 2.1, we get the required result. □

The following upper bound on the spectral radius of the distance matrix of a graph, is obtained in [22]. This result will be needed in Sections 4 and 5.

Lemma 2.3 ([22]). *Let $D(G)$ be the distance matrix with spectral radius μ. Then*

$$\mu \leq \sqrt{\frac{n-1}{n} S}, \tag{6}$$

where $S = 2\sum_{1\leq i<j\leq n} d_{i,j}^2 = \mathrm{Tr}(D^2(G))$. Moreover, the equality holds if and only if $G \cong K_n$.

3. Nordhaus–Gaddum-type results for the spectral radius of the distance matrix of a graph

Let Γ be the class of graphs $H = (V, E)$ such that H is connected graph of diameter d ($3 \leq d \leq 4$) with $|V(H)| \geq d + 2$, having the following property. Let P_{d+1} be the $(d + 1)$-vertex path contained in H. Then for any vertex $i \in V(H) \setminus V(P_{d+1})$ and for any vertex $j \in V(H), j \neq i$, it should be either $d(i, j) = 1$ or $d(i, j) = 2$. In [6], two examples of H in Γ are depicted in Fig. 1. The following lower bound for $W(G) + W(\overline{G})$ in [6]:

Lemma 3.1 ([6]). *Let G be a connected graph on $n \geq 4$ vertices, diameter d, and with a connected complement $\overline{G}$. Then*

$$W(G) + W(\overline{G}) \geq \frac{3}{2}n(n-1) + \frac{1}{6}(d-2)(d-1)d \tag{7}$$

with equality holding in (7) if and only if G is a graph of diameter 2 or $G \cong P_n$ or $G \in \Gamma$ and $\overline{G}$ is a graph of diameter 2.

Now we give the lower bound for $\mu(G) + \mu(\overline{G})$.

Theorem 3.2. *Let G be a connected graph on $n \geq 4$ vertices, diameter d, and with a connected complement $\overline{G}$. Then*

$$\mu(G) + \mu(\overline{G}) \geq 3(n-1) + \frac{1}{3n}(d-2)(d-1)d \tag{8}$$

with equality holding in (8) if and only if G and $\overline{G}$ are both regular graphs of diameter 2.

Proof. Using Lemmas 2.1 and 3.1, we get the required result (8). Moreover, one can see easily that the equality holds in (8) if and only if G and $\overline{G}$ are both regular graphs of diameter 2. □

Corollary 3.3. *Let G be a connected graph on $n \geq 4$ vertices with a connected complement $\overline{G}$. Then*

$$\mu(G) + \mu(\overline{G}) \geq 3(n-1) \tag{9}$$

with equality holding in (9) if and only if G and $\overline{G}$ are both regular graphs of diameter 2.

Proof. The proof follows directly from Theorem 3.2. □

4. Upper bound on the minimal entry in the principal eigenvector of the distance matrix of a graph

Denote by $CI(n, \alpha) = K_{n-\alpha} \nabla \overline{K}_\alpha$ ($1 \leq \alpha \leq n-1$), the join of $K_{n-\alpha}$ with $\overline{K}_\alpha$. One can see easily that the spectral radius of the distance matrix of $CI(n, \alpha)$ is given by

$$\mu = \frac{1}{2}\left(n + \alpha - 3 + \sqrt{(n-\alpha+1)^2 + 4\alpha(\alpha-1)}\right)$$

corresponding eigenvector $\mathbf{Y} = (\underbrace{a, \ldots, a}_{\alpha}; \underbrace{b, \ldots, b}_{n-\alpha})^T$ such that $\|\mathbf{Y}\|_p = 1$ ($1 \leq p < \infty$), where

$$a = \left[\frac{(\mu - n + \alpha + 1)^p}{(n-\alpha)\alpha^p + \alpha\,(\mu - n + \alpha + 1)^p}\right]^{1/p} \tag{10}$$

and

$$b = \left[\frac{(\mu - 2\alpha + 2)^p}{(n-\alpha)\,(\mu - 2\alpha + 2)^p + \alpha(n-\alpha)^p}\right]^{1/p}. \tag{11}$$

By the Perron–Frobenius theorem, the distance spectral radius μ has a positive eigenvector $\mathbf{Y}$. Ruzieh et al. [16] studied that how the entries in this eigenvector are related. Now we obtain the following upper bound on the minimal entry in the principal eigenvector of the distance matrix of a graph and characterize extremal graphs.

Theorem 4.1. *Let G be a simple, connected and undirected graph. Also let $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$ be the p-norm normalized principal eigenvector corresponding to spectral radius μ of $D(G)$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. If $p \geq 1$, then*

$$\begin{aligned} y^{\min_p} &= y_n \\ &\leq \min\left\{\left[\frac{(\mu-n+\alpha+1)^p}{(n-\alpha)\alpha^p+\alpha\,(\mu-n+\alpha+1)^p}\right]^{1/p}, \left[\frac{(\mu-2\alpha+2)^p}{(n-\alpha)\,(\mu-2\alpha+2)^p+\alpha(n-\alpha)^p}\right]^{1/p}\right\}, \end{aligned} \tag{12}$$

where α is the independence number of G. Moreover, the equality holds in (12) if and only if $G \cong CI(n, \alpha)$.

Proof. Since G is a simple connected graph on n vertices with independence number α, we can assume that $V(G) = A \cup B$, where $A = \{1, 2, \ldots, \alpha\}$ and $B = \{\alpha+1, \alpha+2, \ldots, n\}$ such that no two vertices are adjacent in the set A. Since $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$ is an eigenvector of $D(G)$ corresponding to the spectral radius μ, we have $D(G)\mathbf{Y} = \mu\mathbf{Y}$. We can assume that $y_i = \min_{k\in A} y_k$, and $y_j = \min_{k\in B} y_k$. For $i \in A$,

$$\begin{aligned} \mu y_i &= \sum_{k=1,k\neq i}^{\alpha} d_{i,k}y_k + \sum_{k=\alpha+1}^{n} d_{i,k}y_k \\ &\geq 2(\alpha-1)y_i + (n-\alpha)y_j, \end{aligned}$$

that is,

$$y_i \geq \frac{(n-\alpha)}{[\mu-2(\alpha-1)]}y_j. \tag{13}$$

For $j \in B$,

$$\begin{aligned} \mu y_j &= \sum_{k=1}^{\alpha} d_{j,k}y_k + \sum_{k=\alpha+1,k\neq j}^{n} d_{j,k}y_k \\ &\geq \alpha y_i + (n-\alpha-1)y_j, \end{aligned}$$

that is,

$$y_j \geq \frac{\alpha}{[\mu-(n-\alpha-1)]}y_i. \tag{14}$$

It follows from normalization that

$$\alpha y_i^p + (n-\alpha)y_j^p \leq 1. \tag{15}$$

From (13) and (15), we get

$$\alpha\left[\frac{(n-\alpha)y_j}{\mu-2(\alpha-1)}\right]^p + (n-\alpha)y_j^p \leq 1,$$

$$\text{i.e.,}\quad y_j \leq \left[\frac{(\mu-2\alpha+2)^p}{(n-\alpha)\,(\mu-2\alpha+2)^p+\alpha(n-\alpha)^p}\right]^{1/p}. \tag{16}$$

From (14) and (15), we get

$$\alpha y_i^p + (n-\alpha)\left[\frac{\alpha y_i}{\mu-(n-\alpha-1)}\right]^p \leq 1,$$

$$\text{i.e.,}\quad y_i \leq \left[\frac{(\mu-n+\alpha+1)^p}{(n-\alpha)\alpha^p+\alpha\,(\mu-n+\alpha+1)^p}\right]^{1/p}. \tag{17}$$

Thus, we complete the first part of the proof.

Now suppose that equality holds in (12). Then all inequalities in the above argument must be equalities. From equality in (13), we get

$$\begin{aligned} &y_k = y_j, \quad ik \in E(G), \quad \text{for all } k \in B \\ &\text{and}\quad y_k = y_i, \quad \text{for all } k \in A. \end{aligned} \tag{18}$$

From equality in (14), we get

$$\begin{aligned} &y_k = y_j, \quad \text{for all } k \in B \\ &\text{and}\quad y_k = y_i, \quad jk \in E(G), \quad \text{for all } k \in A. \end{aligned} \tag{19}$$

Thus each vertex in A is adjacent to all the vertices on the other set B and each vertex in B is adjacent to all the remaining vertices in $V(G)$. Hence G is isomorphic to $CI(n,\alpha)$.

Conversely, one can see easily that (12) holds for $CI(n,\alpha)$. □

From Theorem 4.1, we get the following upper bound on $y^{\min_p}$.

Theorem 4.2. *Let G be a simple, connected and undirected graph. Also let $\mathbf{Y}=(y_1,y_2,\ldots,y_n)^T$ be the p-norm normalized principal eigenvector corresponding to spectral radius μ of $D(G)$ and $y_1\geq y_2\geq\cdots\geq y_n$. If $p\geq 1$, then*

$$y^{\min_p}=y_n\leq \min\left\{\left[\frac{\left(\sqrt{\frac{n-1}{n}S}-n+\alpha+1\right)^p}{(n-\alpha)\alpha^p+\alpha\left(\sqrt{\frac{n-1}{n}S}-n+\alpha+1\right)^p}\right]^{1/p},\left[\frac{\left(\sqrt{\frac{n-1}{n}S}-2\alpha+2\right)^p}{(n-\alpha)\left(\sqrt{\frac{n-1}{n}S}-2\alpha+2\right)^p+\alpha(n-\alpha)^p}\right]^{1/p}\right\},\tag{20}$$

where $S=2\sum_{1\leq i<j\leq n}d_{i,j}^2=\mathrm{Tr}(D^2(G))$ and α is the independence number of G. Moreover, the equality holds in (20) if and only if $G\cong K_n$.

Proof. Since $p\geq 1$, by (6), we have

$$\alpha+\frac{(n-\alpha)\alpha^p}{(\mu-n+\alpha+1)^p}\geq\alpha+\frac{(n-\alpha)\alpha^p}{\left(\sqrt{\frac{n-1}{n}S}-n+\alpha+1\right)^p}$$

and

$$n-\alpha+\frac{\alpha(n-\alpha)^p}{(\mu-2\alpha+2)^p}\geq n-\alpha+\frac{\alpha(n-\alpha)^p}{\left(\sqrt{\frac{n-1}{n}S}-2\alpha+2\right)^p}.$$

Using above results in Theorem 4.1, we obtain the result in (20). Moreover, the equality holds in (20) if and only if $G\cong K_n$, by Lemma 2.3 and Theorem 4.1. This completes the proof of this theorem. □

5. Lower and upper bounds on the maximal entry in the principal eigenvector of the distance matrix of a graph

In this section we obtain the lower and upper bounds on the maximal entry in the principal eigenvector of the distance matrix of a simple connected graph, and characterize the extremal graphs.

Theorem 5.1. *Let G be a simple, connected and undirected graph. Also let $\mathbf{Y}=(y_1,y_2,\ldots,y_n)^T$ be the p-norm normalized principal eigenvector corresponding to spectral radius μ of $D(G)$ and $y_1\geq y_2\geq\cdots\geq y_n$. If $p\geq 2$, then*

$$\left[\frac{\mu^{p-2}}{\mu^{p-2}+((n-1)d-(d-1)\delta)^{p-1}}\right]^{1/p}\leq y^{\max_p}=y_1\leq\left[\frac{d(\mu-n+2)^{p-1}}{\mu+d(\mu-n+2)^{p-1}}\right]^{1/p},\tag{21}$$

where d,δ are the diameter and the minimum degree of G, respectively. Moreover, both sides of the equality hold in (21) if and only if $G\cong K_n$. If $p=1$, then

$$\frac{1}{\mu+1}\leq y^{\max_p}=y_1\leq\frac{d}{\mu+d}\tag{22}$$

with equality holding (both sides) in (22) if and only if $G\cong K_n$.

Proof. First we give the proof for the lower bound. It follows from normalization that

$$\sum_{k=2}^{n}y_k^p=1-y_1^p.\tag{23}$$

We have

$$D(G)\mathbf{Y}=\mu\mathbf{Y}.\tag{24}$$

From the first equation of (24),

$$\mu y_1 = \sum_{k=2}^{n} d_{1,k} y_k \geq \sum_{k=2}^{n} y_k \quad \text{as } d_{1,k} \geq 1. \tag{25}$$

For $p = 1$, we get

$$y_1 \geq \frac{1}{\mu + 1} \quad \text{as } \sum_{i=1}^{n} y_i = 1.$$

Otherwise, $p > 1$. Multiplying both sides in (25) by y_2^{p-1}, we get

$$\mu y_1 y_2^{p-1} \geq \sum_{k=2}^{n} y_2^{p-1} y_k \geq \sum_{k=2}^{n} y_k^p \quad \text{as } y_2 \geq y_3 \geq \cdots \geq y_n. \tag{26}$$

Using (23) in (26), we get

$$y_2^{p-1} \geq \frac{1 - y_1^p}{\mu y_1},$$

that is,

$$y_2 \geq \left(\frac{1 - y_1^p}{\mu y_1} \right)^{1/(p-1)}. \tag{27}$$

From second equation of (24) we have

$$\mu y_2 = \sum_{k=1, k \neq 2}^{n} d_{2,k} y_k.$$

Since d_2 is the degree of the vertex v_2 and d is the diameter of G, the above equation becomes

$$\mu y_2 \leq d_2 y_1 + (n - d_2 - 1) d y_1,$$
$$\text{i.e.,} \quad \mu y_2 \leq [(n-1)d - (d-1)\delta]\, y_1 \quad \text{as } d_2 \geq \delta. \tag{28}$$

From (27) and (28), we get

$$\mu \left(\frac{1 - y_1^p}{\mu y_1} \right)^{\frac{1}{p-1}} \leq [(n-1)d - (d-1)\delta]\, y_1,$$
$$\text{i.e.,} \quad \mu^{p-2} \leq \left[\mu^{p-2} + ((n-1)d - (d-1)\delta)^{p-1} \right] y_1^p,$$

which gives the lower bound in (21).

Now suppose that equality holds in (21). Then all inequalities in the above argument must be equalities. First we assume that $p = 1$. From equality (25), we get

$$d_{1,k} = 1, \quad k = 2, 3, \ldots, n \text{ and hence } \Delta = n - 1.$$

Also we have

$$\sum_{i=2}^{n} y_i = \mu y_1 \geq \mu y_i = \sum_{k=1, k \neq i}^{n} d_{i,k} y_k, \quad \text{for any } i,\ i \neq 1,$$

that is, we must have

$$d_{i,k} = 1, \quad k = 1, 2, \ldots, n;\ k \neq i, \text{ for any } i,\ i \neq 1.$$

Hence G is isomorphic to complete graph K_n.

Next we assume that $p > 1$. From equality in (25) and (26), we get

$$1 = d_{1,2} = d_{1,3} = \cdots = d_{1,n} \quad \text{and} \quad y_2 = y_3 = \cdots = y_n.$$

From equality in (28), we get

$$y_1 = y_3 = y_4 = \cdots = y_n, \qquad d \leq 2 \quad \text{and} \quad d_2 = \delta.$$

From above results, we have

$$y_1 = y_2 = \cdots = y_n \quad \text{and hence } \mu = n - 1.$$

Thus $G \cong K_n$, by Corollary 2.2.

Conversely, let G be a complete graph K_n. For $G = K_n$, $\mu = n-1$, $d = 1$, $\delta = n-1$ and $y = \left(\left(\frac{1}{n}\right)^{1/p}, \left(\frac{1}{n}\right)^{1/p}, \ldots, \left(\frac{1}{n}\right)^{1/p}\right)^T$.

The upper bound is obtained step-by-step analogous to the proof of Theorem 3.1 from [4]. This completes the proof of this theorem. □

From Theorem 5.1, we get the following lower and upper bounds on $y^{\max_p}$.

Theorem 5.2. *Let G be a simple, connected and undirected graph. Also let $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$ be the p-norm normalized principal eigenvector corresponding to spectral radius μ of $D(G)$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. If $p \geq 2$, then*

$$\left(\frac{(n-1)^{p-2}}{(n-1)^{p-2} + ((n-1)d - (d-1)\delta)^{p-1}}\right)^{1/p} \leq y^{\max_p} = y_1 \leq \left(\frac{d\left(\sqrt{\frac{n-1}{n}S} - n + 2\right)^{p-1}}{d\left(\sqrt{\frac{n-1}{n}S} - n + 2\right)^{p-1} + \sqrt{\frac{n-1}{n}S}}\right)^{1/p}, \tag{29}$$

where $S = 2\sum_{1\leq i<j\leq n} d_{i,j}^2 = \mathrm{Tr}(D^2(G))$. Moreover, the equality holds (both sides) in (29) if and only if $G \cong K_n$. For $p = 1$,

$$\frac{1}{\sqrt{\frac{n-1}{n}S} + 1} \leq y^{\max_p} = y_1 \leq \frac{d}{n-1+d}$$

with equality holding (both sides) if and only if $G \cong K_n$.

Proof. First, suppose that $p = 1$. From (22), we have

$$\begin{aligned} y^{\max_p} &= y_1 \geq \frac{1}{\mu + 1} \\ &\geq \frac{1}{\sqrt{\frac{n-1}{n}S} + 1} \quad \text{by (6)}. \end{aligned} \tag{30}$$

Moreover, the equality holds in (30) if and only if $G \cong K_n$, by Lemma 2.3 and Theorem 5.1.

Again from (22), we have

$$\begin{aligned} y^{\max_p} &= y_1 \leq \frac{d}{\mu + d} \\ &\leq \frac{d}{n-1+d} \quad \text{by (5)}. \end{aligned} \tag{31}$$

Moreover, the equality holds in (31) if and only if $G \cong K_n$, by Corollary 2.2 and Theorem 5.1.

Next suppose that $p \geq 2$. Let us consider a function

$$f(x) = \frac{((n-1)d - (d-1)\delta)^{p-1}}{x^{p-2}}, \quad x \geq n-1.$$

One can see easily that $f(x)$ is a decreasing function on $[n-1, \infty)$. By Corollary 2.2, we have $\mu \geq n-1$. Thus we have

$$\frac{((n-1)d - (d-1)\delta)^{p-1}}{\mu^{p-2}} = f(\mu) \leq \frac{((n-1)d - (d-1)\delta)^{p-1}}{(n-1)^{p-2}}.$$

By Theorem 5.1, we get the lower bound in (29). Moreover, the left hand side equality holds in (29) if and only if $G \cong K_n$, by Corollary 2.2 and Theorem 5.1.

Again, let us consider a function

$$g(x) = \frac{x}{d(x-n+2)^{p-1}}, \quad x \geq n-1.$$

We have

$$g'(x) = -\frac{(p-2)x + n - 2}{(x-n+2)^p} < 0 \quad \text{for } x \geq n-1.$$

Thus $g(x)$ is a decreasing function on $[n-1, \infty)$. By Lemma 2.3, we have $\mu \leq \sqrt{\frac{n-1}{n}S}$, where $S = 2\sum_{1\leq i<j\leq n} d_{i,j}^2 = \mathrm{Tr}(D^2(G))$. Hence

$$\frac{\mu}{d(\mu-n+2)^{p-1}} = g(\mu) \geq \frac{\sqrt{\frac{n-1}{n}S}}{d\left(\sqrt{\frac{n-1}{n}S} - n + 2\right)^{p-1}}.$$

By Theorem 5.1, we get the upper bound in (29). Moreover, the right hand side equality holds in (29) if and only if $G \cong K_n$, by Lemma 2.3 and Theorem 5.1. This completes the proof of this theorem. □

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