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Stable manifolds for nonuniform polynomial dichotomies ☆

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Abstract

We establish the existence of smooth stable manifolds in Banach spaces for sufficiently small perturbations of a new type of dichotomy that we call *nonuniform polynomial dichotomy*. This new dichotomy is more restrictive in the "nonuniform part" but allow the "uniform part" to obey a polynomial law instead of an exponential (more restrictive) law. We consider two families of perturbations. For one of the families we obtain local Lipschitz stable manifolds and for the other family, assuming more restrictive conditions on the perturbations and its derivatives, we obtain C^1 global stable manifolds. Finally we present an example of a family of nonuniform polynomial dichotomies and apply our results to obtain stable manifolds for some perturbations of this family.

Keywords: Invariant manifolds; Nonautonomous dynamics; Nonuniform polynomial dichotomies

1. Introduction

The existence of invariant manifolds is one of the key features in the theory of nonuniform hyperbolicity. The concept of nonuniform hyperbolicity introduced by Pesin [20–22] (see [1,2] for a description of the current status of the theory) is a generalization of the classical concept of (uniform) hyperbolicity. In the nonuniform hyperbolic context the rates of expansion and

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contraction are allowed to vary from point to point. The stable manifold theorem for nonuniform hyperbolic trajectories obtained by Pesin [20] in the finite-dimensional setting is an elaboration of the classical work of Perron. Since then other proofs were obtained, namely, in [24] Ruelle gave a proof based on the study of perturbations of products of matrices occurring in Oseledets' multiplicative ergodic theorem [18]. The proof given by Pugh and Shub in [23] is based on the classical work of Hadamard and uses graph transform techniques. There exist also versions of the stable manifold theorem for dynamical systems in infinite-dimensional spaces. In [25] Ruelle established a corresponding version in Hilbert spaces under some compactness assumptions, following his approach in [24]. In [17] Mañé considered transformations in Banach spaces under some compactness and invertibility assumptions that includes the case of differentiable maps with compact derivative at each point. The results of Mañé were generalized by Thieullen in [27] for a family of transformations satisfying some asymptotic compactness.

The existence of invariant manifolds is also an important subject in the context of exponential dichotomies introduced by Perron in [19]. There is a substantial amount of literature concerning the existence of stable and unstable manifolds for exponential dichotomies (see for example [26] and the references given there).

As mentioned before, the notion of nonuniform hyperbolicity is a generalization of the concept of (uniform) hyperbolicity. Similarly there is also a concept introduced by Barreira and Valls [9,6] of nonuniform exponential dichotomy that is a weaker (and therefore more general) version of the classical notion of exponential dichotomy.

In the discrete time setting, Barreira and Valls obtained C^1 stable manifolds for nonuniformly exponential dichotomies in finite dimension in [7]. Building on this result Barreira, Silva and Valls were able in [3] to establish the existence of C^k local manifolds for C^k perturbations, using an induction process and considering a more geometric approach based on the linear extension of the dynamics. Assuming some exponential decay of the derivatives along the orbits, the same authors established in [4] the existence of C^1 global manifolds for perturbations of nonuniformly exponential dichotomies in Banach spaces. Continuous time versions of this results were obtained by Barreira and Valls in [5,6,8,11].

In a recent work Barreira and Valls [12] considered a generalization of the concept of nonuniform exponential dichotomies that they call ρ-nonuniform exponential dichotomy, with ρ an increasing differentiable function from $\mathbb{R}_0^+$ into $\mathbb{R}_0^+$ such that

$$\lim_{t\to+\infty}\frac{\log t}{\rho(t)}=0. \tag{1}$$

With this generalization, Barreira and Valls replaced the asymptotic rates e^{ct} that appear in the nonuniform exponential case by the asymptotic rates $e^{c\rho(t)}$. Barreira and Valls established, in a finite-dimensional space, the existence of ρ-nonuniform exponential dichotomies and trichotomies for a general family of nonautonomous linear differential equations $v'=A(t)v$, where $A(t)$ are matrices in some block form. To achieve this Barreira and Valls used an adapted type of Lyapunov exponent.

In this work we consider a different kind of nonuniform dichotomy where the rates of expansion and contraction are allowed to vary polynomially. Naturally, the nonuniform parts must vary at most polynomially (see (2) and (3)). We thus consider a new type of behavior: there are families of nonuniform polynomial dichotomies that are not nonuniform exponential dichotomies and vice versa. Even in the more general case of ρ-nonuniform exponential dichotomy, it is not possible to have a polynomial behavior because, to have a polynomial behavior, $\rho(t)$ would have to increase at most logarithmically, and by (1) this is not possible. Note that our case allows situations for which the Lyapunov exponent defined in [10, Section 8] for Hilbert spaces is zero for all $v \in E_1$ (see Section 2 for the definition of E_1).

The main results of this paper are stable manifolds theorems for perturbations of nonuniform polynomial dichotomies. In Section 3 we get local Lipschitz stable manifolds and in Section 4 we get global C^1 stable manifolds. The reason for the difference in the regularity of the manifolds obtained is that to get C^1 manifolds in the local case we would have to consider perturbations that are zero outside a ball of increasingly small radius and it is not known in the infinite-dimensional setting how to obtain appropriate cutoff functions (see the comment in [11, p. 2]).

The content of the paper is the following: in Section 2 we introduce some notations, the main definitions and also establish a technical lemma used several times; then, respectively in Sections 3 and 4, we obtain a local and a global stable manifold theorem; finally we present in Section 5 examples of families of nonuniform polynomial dichotomies and apply our results to obtain stable manifolds for some perturbations of these families.

2. Notation and preliminaries

Let $B(X)$ be the space of bounded linear operators in the Banach space X. Given a sequence $(A_n)_{n\in\mathbb{N}}$ of invertible operators of $B(X)$ we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\dots A_n & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n. \end{cases}$$

We say that $(A_n)_{n\in\mathbb{N}}$ admits a *nonuniform polynomial dichotomy* if there exist projections $P_n : X \to X$ for $n \in \mathbb{N}$ such that

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n, \quad m,n\in\mathbb{N},$$

and constants $a < 0 \leqslant b$, $\varepsilon \geqslant 0$ and $D \geqslant 1$ such that for every $m \geqslant n$,

$$\big\|\mathcal{A}(m,n)P_n\big\| \leqslant D(m-n+1)^a n^\varepsilon, \tag{2}$$

$$\big\|\mathcal{A}(m,n)^{-1}Q_m\big\| \leqslant D(m-n+1)^{-b} m^\varepsilon, \tag{3}$$

where $Q_m = \mathrm{Id} - P_m$ is the complementary projection. When $\varepsilon = 0$ we say that we have a *uniform polynomial dichotomy* or simply a *polynomial dichotomy*.

In these conditions we define, for each $n \in \mathbb{N}$, the linear subspaces $E_n = P_n(X)$ and $F_n = Q_n(X)$. Without loss of generality, we always identify the spaces $E_n \times F_n$ and $E_n \oplus F_n$ as the same space and we equip these spaces with the norm given by $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in E_n \times F_n$.

We are going to address the problem of existence of stable manifolds of the dynamics given by

$$\mathcal{F}(m,n) = \begin{cases} (A_{m-1}+f_{m-1}) \circ \dots \circ (A_n+f_n) & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n, \end{cases} \tag{4}$$

where $(A_n)_{n\in\mathbb{N}}$ admits a nonuniform polynomial dichotomy and $f_n : X \to X$ are perturbations that verify some conditions to be specified later.

Given $n \in \mathbb{N}$ and $v_n = (\xi, \eta) \in E_n \times F_n$, for each $m > n$ we write

$$v_m = \mathcal{F}(m,n)(v_n) = (x_m, y_m) \in E_m \times F_m.$$

Writing $f_m = (g_m, h_m)$ where $g_m = P_{m+1} f_m$ and $h_m = Q_{m+1} f_m$ for each $m \geqslant n$, the trajectory $(v_m)_{m \geqslant n}$ satisfies the following equations

$$x_m = \mathcal{A}(m,n)\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) g_k(x_k, y_k), \tag{5}$$

$$y_m = \mathcal{A}(m,n)\eta + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) h_k(x_k, y_k). \tag{6}$$

In what follows we are going to use Dirichlet series. For every $\alpha < -1$, we denote by λ_α the sum of the Dirichlet series $\sum_{k=1}^{\infty} k^\alpha$. The following lemma will be used several times.

Lemma 1. *Let $m, n \in \mathbb{N}$ with $m > n$, $a < 0$, $q > 0$ and $\varepsilon \geqslant 0$.*

(a) *If $aq + \varepsilon < -1$, then the following inequality holds*

$$\sum_{k=n}^{m-1} (m-k)^a (k+1)^\varepsilon (k-n+1)^{aq+a} \leqslant 2^{\varepsilon - a} \lambda_{aq+\varepsilon} (m-n+1)^a n^\varepsilon. \tag{7}$$

(b) *If $\varepsilon > 0$, then the following inequality holds*

$$\sum_{k=n}^{m-1} (m-k)^a (k-n+1)^a (k+1)^\varepsilon k^{-3\varepsilon - 1} \leqslant 2^{\varepsilon - a} \lambda_{-2\varepsilon - 1} (m-n+1)^a. \tag{8}$$

Proof. (a) Because the sum of the factors of $(m-k)(k-n+1)$ is constant and $a < 0$, it follows that

$$(m-k)^a (k-n+1)^a \leqslant (m-n)^a \leqslant 2^{-a} (m-n+1)^a, \tag{9}$$

$k = n, \ldots, m-1$. On the other hand, we have

$$(k+1)^\varepsilon = \left(1 + \frac{n}{k-n+1}\right)^\varepsilon (k-n+1)^\varepsilon \leqslant 2^\varepsilon n^\varepsilon (k-n+1)^\varepsilon,$$

$k = n, \ldots, m-1$. Therefore

$$\begin{aligned} \sum_{k=n}^{m-1} (m-k)^a (k+1)^\varepsilon (k-n+1)^{aq+a} &\leqslant 2^{\varepsilon - a} (m-n+1)^a n^\varepsilon \sum_{k=n}^{m-1} (k-n+1)^{aq+\varepsilon} \\ &\leqslant 2^{\varepsilon - a} \lambda_{aq+\varepsilon} (m-n+1)^a n^\varepsilon. \end{aligned}$$

(b) It follows immediately from (9) and $(k+1)^\varepsilon \leqslant 2^\varepsilon k^\varepsilon$. $\square$

3. Local stable manifolds

In this section we assume that there are constants $c > 0$ and $q > 1$ such that the functions f_n in the perturbed dynamics (4) verify the following conditions

$$f_n(0) = 0, \tag{10}$$

$$\left\| f_n(u) - f_n(v) \right\| \leqslant c\|u - v\| \big(\|u\| + \|v\|\big)^q \tag{11}$$

for every $n \in \mathbb{N}$ and $u, v \in X$. Making $v = 0$ in (11) we have

$$\left\| f_n(u) \right\| \leqslant c\|u\|^{q+1} \tag{12}$$

for every $u \in X$.

We denote by $B_n(r)$ the open ball of E_n centered at zero and with radius $r > 0$. The initial condition at time n will be taken in $B_n(\delta n^{-\beta})$ for some $\delta, \beta > 0$. We denote by $\mathcal{X}_\beta$ the space of sequences $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions $\varphi_n : B_n(\delta n^{-\beta}) \to F_n$ such that

$$\varphi_n(0) = 0, \tag{13}$$

$$\left\|\varphi_n(\xi) - \varphi_n(\bar{\xi})\right\| \leqslant \|\xi - \bar{\xi}\| \tag{14}$$

for every $\xi, \bar{\xi} \in B_n(\delta n^{-\beta})$. For each $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}_\beta$ we define

$$\|\varphi\|' = \sup\left\{ \frac{\|\varphi_n(\xi)\|}{\|\xi\|} \colon n \in \mathbb{N} \text{ and } \xi \in B_n\big(\delta n^{-\beta}\big) \setminus \{0\} \right\}.$$

Clearly $\|\varphi\|' \leqslant 1$, and given $m \in \mathbb{N}$ and $\xi \neq 0$, we have

$$\left\|\varphi_n(\xi)\right\| \leqslant \delta n^{-\beta} \frac{\|\varphi_n(\xi)\|}{\|\xi\|} \leqslant \delta\|\varphi\|' \leqslant \delta$$

for every $\varphi \in \mathcal{X}_\beta$. This readily implies that $\mathcal{X}_\beta$ is a complete metric space with the distance induced by $\|\cdot\|'$.

We also consider the space $\mathcal{X}_\beta^*$ of sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n : E_n \to F_n$ such that the sequence $(\varphi_n | B_n(\delta n^{-\beta}))_{n\in\mathbb{N}}$ is in $\mathcal{X}_\beta$ and, for each $n \in \mathbb{N}$,

$$\varphi_n(\xi) = \varphi_n\left(\frac{\delta n^{-\beta}\xi}{\|\xi\|}\right) \quad \text{whenever } \xi \notin B_n\big(\delta n^{-\beta}\big).$$

There is a one-to-one correspondence between the sequences in $\mathcal{X}_\beta$ and in $\mathcal{X}_\beta^*$ because for each sequence of functions $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}_\beta$ there is a unique extension $\bar{\varphi} = (\bar{\varphi}_n)_{n\in\mathbb{N}}$ such that each $\bar{\varphi}_n$ is a Lipschitz extension of φ_n to $\overline{B_n(\delta n^{-\beta})}$. Clearly $\mathcal{X}_\beta^*$ is also a complete metric space with the metric induced by $\mathcal{X}_\beta^* \ni \varphi \mapsto \|\varphi | \mathcal{X}_\beta\|$. Furthermore, one can easily verify that given $\varphi \in \mathcal{X}_\beta^*$ and $n \in \mathbb{N}$ we have

$$\left\|\varphi_n(x) - \varphi_n(y)\right\| \leqslant 2\|x - y\| \quad \text{for every } x, y \in E. \tag{15}$$

Given $\beta, \delta > 0$ and $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}_\beta$, for each $n \in \mathbb{N}$, we consider the graph

$$\mathcal{V}_{n,\delta,\beta} = \{(\xi, \varphi_n(\xi)) \colon \xi \in B_n(\delta n^{-\beta})\}. \tag{16}$$

We now present the main result of this section.

Theorem 1 *(Local stable manifolds). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of invertible bounded linear operators, acting on a Banach space X, that admits a nonuniform polynomial dichotomy satisfying* (2) *and* (3) *for some $D \geqslant 1$, $a < 0 \leqslant b$ and $\varepsilon > 0$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, acting on X, that verifies* (10) *and* (11) *for some $c > 0$ and some $q > 1$. If*

$$aq + \varepsilon + 1 < 0 \quad \textit{and} \quad a + \beta < 0$$

hold with $\beta = \varepsilon(1 + 2/q)$, then, for every $C > D$, choosing δ sufficiently small, there is a unique $\varphi \in \mathcal{X}_\beta$ such that

$$\mathcal{F}(m,n)(\mathcal{V}_{n,\frac{\delta}{C},\beta+\varepsilon}) \subset \mathcal{V}_{m,\delta,\beta} \quad \textit{for every } m \geqslant n, \tag{17}$$

with $\mathcal{F}(m,n)$ given by (4) *and $\mathcal{V}_{n,\frac{\delta}{C},\beta+\varepsilon}$ and $\mathcal{V}_{m,\delta,\beta}$ given by* (16).

Furthermore, for every $m \geqslant n$ and $\xi, \bar{\xi} \in B_n(\delta n^{-(\beta+\varepsilon)}/C)$ we have

$$\|\mathcal{F}(m,n)(\xi, \varphi_n(\xi)) - \mathcal{F}(m,n)(\bar{\xi}, \varphi_n(\bar{\xi}))\| \leqslant 2C(m - n + 1)^a n^\varepsilon \|\xi - \bar{\xi}\|. \tag{18}$$

We call each $\mathcal{V}_{n,\delta,\beta}$ a *stable manifold*.

In view of the forward invariance in (17), each trajectory starting $\mathcal{V}_{n,\frac{\delta}{C},\beta+\varepsilon}$ must be in $\mathcal{V}_{m,\delta,\beta}$. Thus, for every $(\xi, \varphi_n(\xi)) \in \mathcal{V}_{n,\frac{\delta}{C},\beta+\varepsilon}$, using Eqs. (5) and (6), we have to prove that

$$x_m(\xi) = \mathcal{A}(m,n)\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) g_k(x_k(\xi), \varphi_k(x_k(\xi))), \tag{19}$$

$$\varphi_m(x_m(\xi)) = \mathcal{A}(m,n)\varphi_n(\xi) + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) h_k(x_k(\xi), \varphi_k(x_k(\xi))), \tag{20}$$

and

$$\|x_m(\xi)\| \leqslant \delta m^{-\beta} \tag{21}$$

for every $\xi \in B_n(\delta n^{-(\beta+\varepsilon)}/C)$ and every $m > n$, where

$$\mathcal{F}(m,n)(\xi, \varphi_n(\xi)) = (x_m(\xi), \varphi_m(x_m(\xi))) \in E_m \times F_m.$$

The idea of the proof of Theorem 1 is to solve Eqs. (19) and (20) separately using in each case the Banach fixed point theorem. For this we first establish, using the Banach fixed point theorem on a suitable space $\mathcal{B}$, that for every $\varphi \in \mathcal{X}_\beta$ there is a unique sequence of functions $x^\varphi = (x_m)_{m\geqslant n} \in \mathcal{B}$ that verifies (19) and (21). To prove Eq. (20), we first prove that this equation is equivalent to another one and we solve this second equation applying the Banach fixed point theorem in the space $\mathcal{X}_\beta^*$.

Let $\mathcal{B} = \mathcal{B}_{n,\beta}$ be the space of all sequences $x = (x_m)_{m \geqslant n}$ of functions

$$x_m : B_n\big(\delta n^{-\beta}\big) \to E_m$$

such that for every $m \geqslant n$ and every $\xi, \bar{\xi} \in B_n(\delta n^{-\beta})$ we have

$$x_n(\xi) = \xi, \quad x_m(0) = 0, \tag{22}$$

$$\big\| x_m(\xi) - x_m(\bar{\xi}) \big\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi - \bar{\xi}\|, \tag{23}$$

for some constant $C > D$. Making $\bar{\xi} = 0$ in (23) we obtain the following estimate

$$\big\| x_m(\xi) \big\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi\| \leqslant C\delta(m-n+1)^a n^{\varepsilon-\beta} \tag{24}$$

for every $m \geqslant n$ and every $\xi \in B_n(\delta n^{-\beta})$.

This space $\mathcal{B}$ allows to estimate the speed of decay of the stable component of the solution along the graphs given by φ. In fact, if $\xi \in B_n(\delta n^{-(\beta+\varepsilon)}/C)$, then, (21) holds because $a + \beta < 0$:

$$\big\| x_m(\xi) \big\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi\| \leqslant \delta(m-n+1)^a n^{-\beta} \leqslant \delta m^{-\beta}.$$

For every $x \in \mathcal{B}$, we define

$$\|x\|' = \sup\left\{ \frac{\|x_m(\xi)\|}{(m-n+1)^a n^\varepsilon} \colon m \geqslant n,\ \xi \in B_n\big(\delta n^{-\beta}\big) \right\} \tag{25}$$

and with the metric induced by (25), $\mathcal{B}$ is a complete metric space.

Given $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}_\beta^*$ and $x = (x_m)_{m \geqslant n} \in \mathcal{B}$ we write

$$\varphi_m^* = \varphi_m \circ x_m \quad \text{and} \quad f_m^*(\xi) = f_m\big(x_m(\xi), \varphi_m^*(\xi)\big).$$

Lemma 2. *Given $\delta > 0$ sufficiently small, for each $\varphi \in \mathcal{X}_\beta^*$ and $n \in \mathbb{N}$ there exists a unique sequence $x = x^\varphi \in \mathcal{B}$ satisfying Eq.* (19) *for every $m \geqslant n$ and $\xi \in B_n(\delta n^{-\beta})$.*

Proof. We define an operator J in $\mathcal{B}$ by $(Jx)_n(\xi) = \xi$, and for each $m > n$ by

$$(Jx)_m(\xi) = \mathcal{A}(m,n)\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) g_k\big(x_k(\xi), \varphi_k\big(x_k(\xi)\big)\big).$$

One can easily verify from (22), (13) and (10) that $(Jx)_m(0) = 0$ for every $m \geqslant n$.

From the definition of the operator it follows that

$$\begin{aligned}
&\big\| (Jx)_m(\xi) - (Jx)_m(\bar{\xi}) \big\| \\
&\quad \leqslant \big\| \mathcal{A}(m,n)P_n\xi - \mathcal{A}(m,n)P_n\bar{\xi} \big\| + \sum_{k=n}^{m-1} \big\| \mathcal{A}(m,k+1)P_{k+1} \big\| \cdot \big\| f_k^*(\xi) - f_k^*(\bar{\xi}) \big\|.
\end{aligned} \tag{26}$$

Using (2) we have

$$\|\mathcal{A}(m,n)P_n\xi - \mathcal{A}(m,n)P_n\bar{\xi}\| \leqslant \|\mathcal{A}(m,n)P_n\| \|\xi-\bar{\xi}\| \leqslant D(m-n+1)^a n^\varepsilon \|\xi-\bar{\xi}\|$$

and from (2), (11), (15), (23) and (7) we also have

$$\begin{aligned}
&\sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|f_k^*(\xi) - f_k^*(\bar{\xi})\| \\
&\quad\leqslant cD\sum_{k=n}^{m-1}(m-k)^a(k+1)^\varepsilon\big(\|x_k(\xi)-x_k(\bar{\xi})\| + \|\varphi_k(x_k(\xi)) - \varphi_k(x_k(\bar{\xi}))\|\big) \\
&\qquad\times\big(\|x_k(\xi)\| + \|\varphi_k(x_k(\xi))\| + \|x_k(\bar{\xi})\| + \|\varphi_k(x_k(\bar{\xi}))\|\big)^q \\
&\quad\leqslant cD\sum_{k=n}^{m-1}(m-k)^a(k+1)^\varepsilon\big(3\|x_k(\xi)-x_k(\bar{\xi})\|\big)\big(3\|x_k(\xi)\| + 3\|x_k(\bar{\xi})\|\big)^q \\
&\quad\leqslant c2^q(3C)^{q+1}D\delta^q n^{\varepsilon(q+1)-\beta q}\|\xi-\bar{\xi}\|\sum_{k=n}^{m-1}(m-k)^a(k+1)^\varepsilon(k-n+1)^{aq+a} \\
&\quad\leqslant 2^{q+\varepsilon-a}c(3C)^{q+1}D\delta^q\lambda_{aq+\varepsilon}(m-n+1)^a n^{\varepsilon(q+2)-\beta q}\|\xi-\bar{\xi}\|.
\end{aligned}$$

Choosing δ sufficiently small, it follows from (26) that

$$\|(Jx)_m(\xi) - (Jx)_m(\bar{\xi})\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi-\bar{\xi}\|$$

and this implies the inclusion $J(\mathcal{B}) \subset \mathcal{B}$.

We now show that J is a contraction for the metric induced by (25). Let $x, y \in \mathcal{B}$. Then

$$\|(Jx)_m(\xi) - (Jy)_m(\xi)\| \leqslant \sum_{k=n}^{m-1}\|\mathcal{A}(m,k+1)P_{k+1}\|\alpha_k,$$

where $\alpha_k = \|f_k(x_k(\xi), \varphi_k(x_k(\xi))) - f_k(y_k(\xi), \varphi_k(y_k(\xi)))\|$. By (11) and (15) we have

$$\begin{aligned}
\alpha_k &\leqslant c\big(\|x_k(\xi)-y_k(\xi)\| + \|\varphi_k(x_k(\xi)) - \varphi_k(y_k(\xi))\|\big) \\
&\quad\times\big(\|x_k(\xi)\| + \|\varphi_k(x_k(\xi))\| + \|y_k(\xi)\| + \|\varphi_k(y_k(\xi))\|\big)^q \\
&\leqslant 3^{q+1}c\|x_k(\xi)-y_k(\xi)\|\big(\|x_k(\xi)\| + \|y_k(\xi)\|\big)^q
\end{aligned}$$

and, using (24), it follows that

$$\begin{aligned}
\alpha_k &\leqslant 2^q3^{q+1}cC^q\delta^q(k-n+1)^{aq}n^{(\varepsilon-\beta)q}\|x_k(\xi)-y_k(\xi)\| \\
&\leqslant 2^q3^{q+1}cC^q\delta^q(k-n+1)^{aq+a}n^{\varepsilon(q+1)-\beta q}\|x-y\|'.
\end{aligned} \tag{27}$$

Hence, from (2) and (27), we have

$$
\begin{aligned}
&\big\|(Jx)_m(\xi)-(Jy)_m(\xi)\big\| \\
&\quad\leqslant 2^q 3^{q+1} c C^q D \delta^q n^{\varepsilon(q+1)-\beta q}\|x-y\|' \sum_{k=n}^{m-1}(m-k)^a(k+1)^{\varepsilon}(k-n+1)^{aq+a} \\
&\quad\leqslant 2^{q-a+\varepsilon} 3^{q+1} c C^q D \delta^q \lambda_{aq+\varepsilon}(m-n+1)^a n^{\varepsilon(q+2)-\beta q}\|x-y\|'
\end{aligned}
$$

and choosing δ sufficiently small it follows that

$$
\big\|(Jx)_m(\xi)-(Jy)_m(\xi)\big\| \leqslant \mu(m-n+1)^a n^{\varepsilon}\|x-y\|'
$$

with $\mu<1$. Therefore,

$$
\|Jx-Jy\|' \leqslant \mu\|x-y\|',
$$

and J is a contraction in $\mathcal{B}$ provided that δ is sufficiently small. Because $\mathcal{B}$ is complete, by the Banach fixed point theorem, the map J has a unique fixed point x^{φ} in $\mathcal{B}$, which is thus the desired sequence. □

We now represent by $(x^{\varphi}_{n,k})_{k\geqslant n}\in\mathcal{B}_{n,\beta}$ the unique sequence given by Lemma 2.

Lemma 3. *Given $\delta>0$ sufficiently small and $\varphi\in\mathcal{X}^*_{\beta}$ the following properties hold*:

(1) *If for every $n\in\mathbb{N}$, $m\geqslant n$ and $\xi\in B_n(\delta n^{-\beta})$ the identity* (20) *holds, then*

$$
\varphi_n(\xi) = -\sum_{k=n}^{\infty}\mathcal{A}(k+1,n)^{-1}h_k\big(x^{\varphi}_{n,k}(\xi),\varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big) \tag{28}
$$

for every $n\in\mathbb{N}$, $m\geqslant n$ and $\xi\in B_n(\delta n^{-\beta})$.

(2) *If for every $n\in\mathbb{N}$, $m\geqslant n$ and $\xi\in B_n(\delta n^{-\beta})$ Eq.* (28) *holds, then* (20) *holds for every $\xi\in B_n(\delta n^{-(\beta+\varepsilon)}/C)$.*

Proof. First we prove that the series in (28) is convergent. From (3), (12), (15) and (24), we conclude that

$$
\begin{aligned}
&\sum_{k=n}^{\infty}\big\|\mathcal{A}(k+1,n)^{-1}h_k\big(x^{\varphi}_{n,k}(\xi),\varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big)\big\| \\
&\quad\leqslant \sum_{k=n}^{\infty}\big\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\big\|\,\big\|f_k\big(x^{\varphi}_{n,k}(\xi),\varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big)\big\| \\
&\quad\leqslant \sum_{k=n}^{\infty} D(k-n+2)^{-b}(k+1)^{\varepsilon}c\big(\big\|x_{n,k}(\xi)\big\|+\big\|\varphi_k\big(x_{n,k}(\xi)\big)\big\|\big)^{q+1} \\
&\quad\leqslant cD\sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^{\varepsilon}\big(3C\delta(k-n+1)^a n^{\varepsilon-\beta}\big)^{q+1}
\end{aligned}
$$

$$\leqslant c(3C\delta)^{q+1} D n^{(\varepsilon-\beta)(q+1)} \sum_{k=n}^{\infty} (k-n+1)^{aq+a-b}(k+1)^{\varepsilon}$$

$$\leqslant 2^{\varepsilon} c(3C\delta)^{q+1} D n^{\varepsilon(q+2)-\beta(q+1)} \sum_{k=n}^{\infty} (k-n+1)^{aq+a-b+\varepsilon} < \infty.$$

Let us suppose that (20) holds. Then by $\mathcal{A}(m,n)^{-1}\mathcal{A}(m,k+1) = \mathcal{A}(k+1,n)^{-1}$, Eq. (20) can be written in the following equivalent form

$$\varphi_n(\xi) = \mathcal{A}(m,n)^{-1}\varphi_m\big(x^{\varphi}_{n,m}(\xi)\big) - \sum_{k=n}^{m-1} \mathcal{A}(k+1,n)^{-1} h_k\big(x^{\varphi}_{n,k}(\xi), \varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big). \tag{29}$$

Using (3), (15) and (24), we have

$$\begin{aligned} \big\|\mathcal{A}(m,n)^{-1}\varphi_m\big(x^{\varphi}_{n,m}(\xi)\big)\big\| &= \big\|\mathcal{A}(m,n)^{-1} Q_m \varphi_m\big(x^{\varphi}_{n,m}(\xi)\big)\big\| \\ &\leqslant 2D(m-n+1)^{-b} m^{\varepsilon} \big\|x^{\varphi}_{n,m}(\xi)\big\| \\ &\leqslant 2D(m-n+1)^{-b} m^{\varepsilon} C\delta(m-n+1)^{a} n^{\varepsilon-\beta} \\ &\leqslant 2CD\delta(m-n+1)^{a-b+\varepsilon} n^{2\varepsilon-\beta} \end{aligned}$$

and this converges to zero when $m \to \infty$. Hence, letting $m \to \infty$ in (29) we obtain the identity (28).

We now assume that for every $n \in \mathbb{N}$, $m \geqslant n$ and $\xi \in B_n(\delta n^{-\beta})$ the identity (28) holds. If $\xi \in B_n(\delta n^{-(\beta+\varepsilon)}/C)$ then, since $a+\beta < 0$, we get

$$\big\|x_{n,m}(\xi)\big\| \leqslant C(m-n+1)^{a} n^{\varepsilon}\|\xi\| \leqslant \delta(m-n+1)^{a} n^{-\beta} \leqslant \delta m^{-\beta}. \tag{30}$$

Then

$$\mathcal{A}(m,n)\varphi_n(\xi) = -\sum_{k=n}^{\infty} \mathcal{A}(m,k+1) h_k\big(x^{\varphi}_{n,k}(\xi), \varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big),$$

and thus it follows from (28), the uniqueness of the sequences x^{φ} and (30) that

$$\begin{aligned} &\mathcal{A}(m,n)\varphi_n(\xi) + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) h_k\big(x^{\varphi}_{n,k}(\xi), \varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big) \\ &\quad = -\sum_{k=m}^{\infty} \mathcal{A}(m,k+1) h_k\big(x^{\varphi}_{n,k}(\xi), \varphi_k\big(x^{\varphi}_{n,k}(\xi)\big)\big) \\ &\quad = -\sum_{k=m}^{\infty} \mathcal{A}(m,k+1) h_k\big(x^{\varphi}_{m,k}\big(x^{\varphi}_{n,m}(\xi)\big), \varphi_k\big(x^{\varphi}_{m,k}\big(x^{\varphi}_{n,m}(\xi)\big)\big)\big) \\ &\quad = \varphi_m\big(x^{\varphi}_{n,m}(\xi)\big). \end{aligned}$$

This proves the lemma. □

Lemma 4. *Given $\delta > 0$ sufficiently small, for each $\varphi, \psi \in \mathcal{X}_\beta^*$, $n \in \mathbb{N}$, $m \geqslant n$ and $\xi \in B_n(\delta n^{-\beta})$ we have*

$$\left\| x_m^\varphi(\xi) - x_m^\psi(\xi) \right\| \leqslant \frac{C}{2}(m-n+1)^a \|\xi\| \|\varphi - \psi\|'. \tag{31}$$

Proof. Putting

$$\gamma_k = \left\| f_k\left(x_k^\varphi(\xi), \varphi_k\left(x_k^\varphi(\xi)\right)\right) - f_k\left(x_k^\psi(\xi), \varphi_k\left(x_k^\psi(\xi)\right)\right) \right\|, \tag{32}$$

by (11) it follows that

$$\begin{aligned}
\gamma_k &\leqslant c\left(\left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \left\|\varphi_k\left(x_k^\varphi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\|\right) \\
&\quad \times \left(\left\|x_k^\varphi(\xi)\right\| + \left\|\varphi_k\left(x_k^\varphi(\xi)\right)\right\| + \left\|x_k^\psi(\xi)\right\| + \left\|\psi_k\left(x_k^\psi(\xi)\right)\right\|\right)^q \\
&\leqslant c\left(\left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \left\|\varphi_k\left(x_k^\varphi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\|\right)\left(3\left\|x_k^\varphi(\xi)\right\| + 3\left\|x_k^\psi(\xi)\right\|\right)^q \\
&\leqslant c(6C\delta)^q\left(\left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \left\|\varphi_k\left(x_k^\varphi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\|\right)(k-n+1)^{aq} n^{(\varepsilon-\beta)q}
\end{aligned}$$

and because

$$\begin{aligned}
\left\|\varphi_k\left(x_k^\varphi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\| &\leqslant \left\|\varphi_k\left(x_k^\varphi(\xi)\right) - \varphi_k\left(x_k^\psi(\xi)\right)\right\| + \left\|\varphi_k\left(x_k^\psi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\| \\
&\leqslant \left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \left\|\varphi_k\left(x_k^\psi(\xi)\right) - \psi_k\left(x_k^\psi(\xi)\right)\right\| \\
&\leqslant \left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \|\varphi - \psi\|'\left\|x_k^\psi(\xi)\right\|
\end{aligned}$$

we have

$$\gamma_k \leqslant c(6C\delta)^q\left(2\left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \|\varphi - \psi\|'\left\|x_k^\psi(\xi)\right\|\right)(k-n+1)^{aq} n^{(\varepsilon-\beta)q}. \tag{33}$$

Using (33), we are going to prove this lemma by induction on m. For $m = n$ the result follows immediately because $x_n^\varphi(\xi) = \xi = x_n^\psi(\xi)$.

Suppose that (31) is true for $n, \ldots, m-1$. Then for $k = n, \ldots, m-1$ we have

$$\begin{aligned}
&2\left\|x_k^\varphi(\xi) - x_k^\psi(\xi)\right\| + \|\varphi - \psi\|'\left\|x_k^\psi(\xi)\right\| \\
&\quad \leqslant C(k-n+1)^a \|\xi\| \, \|\varphi - \psi\|' + C(k-n+1)^a n^\varepsilon \|\xi\| \|\varphi - \psi\|' \\
&\quad \leqslant 2C(k-n+1)^a n^\varepsilon \|\xi\| \|\varphi - \psi\|'
\end{aligned}$$

and this implies by (33) that

$$\gamma_k \leqslant 2cC^{q+1}(6\delta)^q \|\xi\| \|\varphi - \psi\|' (k-n+1)^{aq+a} n^{\varepsilon(q+1)-\beta q}.$$

Hence we have

$$
\begin{aligned}
&\left\|x_m^\varphi(\xi)-x_m^\psi(\xi)\right\| \\
&\quad\leqslant \sum_{k=n}^{m-1}\left\|\mathcal{A}(m,k+1)P_{k+1}\right\|\gamma_k \\
&\quad\leqslant 2cC^{q+1}D(6\delta)^q n^{\varepsilon(q+1)-\beta q}\|\xi\|\|\varphi-\psi\|' \sum_{k=n}^{m-1}(m-k)^a(k+1)^\varepsilon(k-n+1)^{aq+a} \\
&\quad\leqslant 2^{1+\varepsilon-a}cC^{q+1}D(6\delta)^q\lambda_{aq+\varepsilon}(m-n+1)^a n^{\varepsilon(q+2)-\beta q}\|\varphi-\psi\|'\|\xi\|.
\end{aligned}
$$

Choosing $\delta>0$ sufficiently small and using the fact that $\varepsilon(q+2)-\beta q=0$ we obtain (31). □

Lemma 5. *Given $\delta>0$ sufficiently small there is a unique $\varphi\in\mathcal{X}_\beta^*$ such that*

$$
\varphi_n(\xi)=-\sum_{k=n}^{\infty}\mathcal{A}(k+1,n)^{-1}h_k\bigl(x_k^\varphi(\xi),\varphi_k\bigl(x_k^\varphi(\xi)\bigr)\bigr)
$$

for every $n\in\mathbb{N}$ and every $\xi\in B_n(\delta n^{-\beta})$.

Proof. We consider the operator Φ defined for each $\varphi\in\mathcal{X}_\beta^*$ by

$$
(\Phi\varphi)_n(\xi)=-\sum_{k=n}^{\infty}\mathcal{A}(k+1,n)^{-1}h_k\bigl(x_k^\varphi(\xi),\varphi_k\bigl(x_k^\varphi(\xi)\bigr)\bigr), \tag{34}
$$

where $x^\varphi=(x_k^\varphi)_{k\geqslant n}$ is the unique sequence given by Lemma 2. Since $x^\varphi\in\mathcal{B}_\beta$ we have $x_m^\varphi(0)=0$, $m\geqslant n$. It follows from (10), (22), (13) and (34) that $(\Phi\varphi)_n(0)=0$ for each $n\in\mathbb{N}$. Furthermore, given $\xi,\bar\xi\in B_n(\delta n^{-\beta})$, by (3) and (11), we have

$$
\begin{aligned}
&\left\|(\Phi\varphi)_n(\xi)-(\Phi\varphi)_n(\bar\xi)\right\| \\
&\quad\leqslant \sum_{k=n}^{\infty}\left\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\right\|\cdot\left\|f_k^*(\xi)-f_k^*(\bar\xi)\right\| \\
&\quad\leqslant cD\sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^\varepsilon\bigl(3\left\|x_k(\xi)-x_k(\bar\xi)\right\|\bigr)\bigl(3\left\|x_k(\xi)\right\|+3\left\|x_k(\bar\xi)\right\|\bigr)^q \\
&\quad\leqslant 3cC^{q+1}D(6\delta)^q n^{\varepsilon(q+1)-\beta q}\|\xi-\bar\xi\|\sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^\varepsilon(k-n+1)^{aq+a} \\
&\quad\leqslant 2^{q+\varepsilon}c(3C)^{q+1}D\delta^q n^{\varepsilon(q+2)-\beta q}\|\xi-\bar\xi\|\sum_{k=n}^{\infty}(k-n+1)^{aq+a-b+\varepsilon} \\
&\quad\leqslant 2^\varepsilon 3cC^{q+1}D(6\delta)^q\lambda_{aq+a-b+\varepsilon}\|\xi-\bar\xi\|.
\end{aligned}
$$

Hence, choosing $\delta>0$ sufficiently small (independently of φ, n and ξ) we have (14). Therefore $\Phi(\mathcal{X}_\beta^*)\subset\mathcal{X}_\beta^*$.

We now show that Φ is a contraction. Given $\varphi, \psi \in \mathcal{X}_\beta^*$ and $n \in \mathbb{N}$, let x^φ and x^ψ be the unique sequences given by Lemma 2 respectively for φ and ψ. By (33) (see (32) for the definition of γ_k) and Lemma 4 we have

$$\gamma_k \leqslant c2C^{q+1}(6\delta)^q \|\xi\| \|\varphi - \psi\|'(k-n+1)^{aq+a} n^{\varepsilon(q+1)-\beta q}$$

and this inequality and (3) imply

$$\begin{aligned}
&\big\|(\Phi\varphi)_n(\xi) - (\Phi\psi)_n(\xi)\big\| \\
&\quad \leqslant \sum_{k=n}^{\infty} \big\|\mathcal{A}(k+1,n)^{-1} Q_{k+1}\big\| \gamma_k \\
&\quad \leqslant 2cC^{q+1} D(6\delta)^q \|\xi\| \|\varphi - \psi\|' n^{\varepsilon(q+1)-\beta q} \sum_{k=n}^{\infty} (k-n+2)^{-b}(k+1)^{\varepsilon}(k-n+1)^{aq+a} \\
&\quad \leqslant 2^{1+\varepsilon} cC^{q+1} D(6\delta)^q \|\xi\| \|\varphi - \psi\|' n^{\varepsilon(q+2)-\beta q} \sum_{k=n}^{\infty} (k-n+1)^{aq+a-b+\varepsilon} \\
&\quad \leqslant 2^{1+\varepsilon} cC^{q+1} D(6\delta)^q \|\xi\| \lambda_{aq+a-b+\varepsilon} \|\varphi - \psi\|'.
\end{aligned}$$

Hence, choosing $\delta > 0$ sufficiently small, we have

$$\big\|(\Phi\varphi)_n(\xi) - (\Phi\psi)_n(\xi)\big\| \leqslant \mu \|\xi\| \|\varphi - \psi\|'$$

for some $\mu < 1$. This implies that

$$\|\Phi\varphi - \Phi\psi\|' \leqslant \mu \|\varphi - \psi\|'$$

and this means that Φ is a contraction in $\mathcal{X}_\beta^*$. Therefore the map Φ has a unique fixed point φ in $\mathcal{X}_\beta^*$ that is the desired sequence. □

We are now in conditions to prove Theorem 1.

Proof of Theorem 1. By Lemma 2, for each $\varphi \in \mathcal{X}_\beta^*$ there is a unique sequence $x^\varphi \in \mathcal{B}$ satisfying (19). It remains to solve (20) with $x = x^\varphi$. By Lemma 3, this is equivalent to solve (28). Finally, by Lemma 5, there is a unique solution of (28). This establishes the existence of the stable manifolds for $\delta > 0$ sufficiently small. Moreover, for each $n \in \mathbb{N}$, $m \geqslant n$ and $\xi, \bar{\xi} \in B_n(\delta n^{-(\beta+\varepsilon)}/C)$ it follows from (14) that

$$\begin{aligned}
\big\|\mathcal{F}(m,n)\big(\xi, \varphi_n(\xi)\big) - \mathcal{F}(m,n)\big(\bar{\xi}, \varphi_n(\bar{\xi})\big)\big\| &\leqslant \big\|x_m(\xi) - x_m(\bar{\xi})\big\| + \big\|\varphi_m^*(\xi) - \varphi_m^*(\bar{\xi})\big\| \\
&\leqslant 2\big\|x_m(\xi) - x_m(\bar{\xi})\big\| \\
&\leqslant 2C(m-n+1)^a n^{\varepsilon} \|\xi - \bar{\xi}\|.
\end{aligned}$$

Hence we obtain (18) and the theorem is proved. □

4. Global stable manifolds

We say that a function is of class $C^{1,1}$ if it is of class C^1 and its derivative is Lipschitz.

We will consider that the perturbations f_n in (4) are of class $C^{1,1}$ and we assume that there exists $\delta > 0$ such that, for every $n \in \mathbb{N}$ and $u, v \in X$,

$$f_n(0) = 0, \quad d_0 f_n = 0, \tag{35}$$

$$\|d_u f_n\| \leqslant \delta n^{-3\varepsilon-1}, \tag{36}$$

$$\|d_u f_n - d_v f_n\| \leqslant \delta n^{-3\varepsilon-1} \|u - v\|, \tag{37}$$

with the same ε as in (2) and (3).

In this section, for technical reasons, we have to assume that $\varepsilon > 0$. It is easy to verify that all the results in this section remain true in the case $\varepsilon = 0$ if we replace the exponent in (36) and (37) by $-1 - \gamma$ with $\gamma > 0$.

Let $\mathcal{X}$ be the space of sequences of $C^{1,1}$ functions $\varphi_n : E_n \to F_n$ such that, for every $n \in \mathbb{N}$ and $x, y \in E_n$,

$$\varphi_n(0) = 0, \quad d_0\varphi_n = 0, \tag{38}$$

$$\|d_x\varphi_n\| \leqslant 1, \tag{39}$$

$$\|d_x\varphi_n - d_y\varphi_n\| \leqslant \|x - y\|. \tag{40}$$

Given $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}$ we consider the graphs

$$\mathcal{V}_n = \big\{\big(\xi, \varphi_n(\xi)\big)\colon \xi \in E_n\big\} \tag{41}$$

that we call *global stable manifolds*. We have the following result.

Theorem 2 *(Global stable manifolds). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of bounded invertible linear operators, acting on a Banach space X, that admits a nonuniform polynomial dichotomy satisfying* (2) *and* (3) *for some $D \geqslant 1$, $a < 0 \leqslant b$ and $\varepsilon > 0$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, acting on X, that verify* (35), (36) *and* (37) *for some $\delta > 0$. If*

$$a + \varepsilon < b, \tag{42}$$

then, choosing $\delta > 0$ sufficiently small, there exists a unique sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}$ such that

(a) *$\mathcal{F}(m,n)(\mathcal{V}_n) = \mathcal{V}_m$ for every $m \geqslant n$, where $\mathcal{F}(m,n)$ is given by* (4) *and $\mathcal{V}_n$ and $\mathcal{V}_m$ are given by* (41);
(b) *$\mathcal{V}_n$ is a $C^{1,1}$ manifold with $T_0\mathcal{V}_n = E_n$ for each $n \in \mathbb{N}$;*
(c) *there exists $K > 0$ such that for $m \geqslant n$ and $\xi, \bar{\xi} \in E_n$ we have*

$$\big\|\mathcal{F}(m,n)(v) - \mathcal{F}(m,n)(\bar{v})\big\| \leqslant K(m-n+1)^a n^{\varepsilon} \|\xi - \bar{\xi}\|, \tag{43}$$

$$\big\|d_v\mathcal{F}(m,n) - d_{\bar{v}}\mathcal{F}(m,n)\big\| \leqslant K(m-n+1)^a n^{2\varepsilon} \|\xi - \bar{\xi}\|, \tag{44}$$

where $v = (\xi, \varphi_n(\xi))$ and $\bar{v} = (\bar{\xi}, \varphi_n(\bar{\xi}))$.

The idea of the proof of this theorem follows the same lines of the proof of Theorem 1, although with different spaces of sequences of functions. As in Theorem 1, we have to solve the following equations

$$x_m(\xi) = \mathcal{A}(m,n)\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) g_k\big(x_k(\xi), \varphi_k\big(x_k(\xi)\big)\big), \tag{45}$$

$$\varphi_m\big(x_m(\xi)\big) = \mathcal{A}(m,n)\varphi_n(\xi) + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) h_k\big(x_k(\xi), \varphi_k\big(x_k(\xi)\big)\big), \tag{46}$$

for every $\xi \in E_n$ and every $m > n$, where

$$\mathcal{F}(m,n)\big(\xi, \varphi_n(\xi)\big) = \big(x_m(\xi), \varphi_m\big(x_m(\xi)\big)\big) \in E_m \times F_m,$$

and prove that these solutions verify (43) and (44).

Given $n \in \mathbb{N}$ and $C > D$ (note that $C > 1$), let $\mathcal{B} = \mathcal{B}_n$ be the space of sequences of $C^{1,1}$ functions $x_m : E_n \to E_m$ that, for every $m \geqslant n$ and $\xi, \bar{\xi} \in E_n$, satisfy the following conditions

$$x_n(\xi) = \xi, \quad x_m(0) = 0, \tag{47}$$

$$\|d_\xi x_m\| \leqslant C(m-n+1)^a n^\varepsilon, \tag{48}$$

$$\|d_\xi x_m - d_{\bar{\xi}} x_m\| \leqslant C(m-n+1)^a n^{2\varepsilon} \|\xi - \bar{\xi}\|. \tag{49}$$

From the mean value theorem, (47) and (48) it follows that

$$\big\|x_m(\xi)\big\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi\| \tag{50}$$

for every $\xi \in E_n$. This allows us to equip $\mathcal{B}$ with the metric induced by

$$\|x\|' = \sup\left\{ \frac{\|x_m(\xi)\|(m-n+1)^{-a} n^{-\varepsilon}}{\|\xi\|} \colon m \geqslant n,\ \xi \in E_n \setminus \{0\} \right\} \tag{51}$$

for $x = (x_m)_{m \geqslant n} \in \mathcal{B}$. Note that $\|x\|' \leqslant C$ for each $x \in \mathcal{B}$.

Proposition 1. *The space $\mathcal{B}$ is a complete metric space with the metric induced by* (51).

Proof. Let $(x_k)_k = ((x_{m,k})_{m \geqslant n})_k$ be a Cauchy sequence in $\mathcal{B}$ with respect to the metric induced by (51). Then for each $m \geqslant n$ the sequence $(x_{m,k}|_{B_n(r)})_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the supremum norm in the space of bounded functions from $B_n(r)$ into F_m. Here $B_n(r)$ is the open ball of E_n centered at 0 and with radius r. Therefore, for each $m \geqslant n$, there exists a function $y_m : B_n(r) \to F_m$ such that $(x_{m,k}|_{B_n(r)})_{k \in \mathbb{N}}$ converges to y_m in the space of bounded functions from $B_n(r)$ into F_m equipped with the supremum norm.

For every $\xi \in B_n(r)$, from (50), (48) and (49) we get

$$\left\|x_{m,k}(\xi)\right\| \leqslant C(m-n+1)^a n^\varepsilon r$$
$$\|d_\xi x_{m,k}\| \leqslant C(m-n+1)^a n^\varepsilon,$$
$$\|d_\xi x_{m,k} - d_{\bar{\xi}} x_{m,k}\| \leqslant C(m-n+1)^a n^{2\varepsilon} \|\xi - \bar{\xi}\|.$$

Putting $b = C(m-n+1)^a n^\varepsilon \max\{n^\varepsilon, r\}$, it follows that, for each $m \geqslant n$ and $k \in \mathbb{N}$, $x_{m,k} \in C_b^{1,1}(B_n(r), F_m)$. Here $C_b^{1,1}(B_n(r), F_m)$ is the space of $C^{1,1}$ functions from $B_n(r)$ into F_m such that the norm defined by

$$\|u\|_{1,1} = \max\bigl\{\|u\|_\infty, \|du\|_\infty, L(du)\bigr\},$$

is less or equal than b, $\|\cdot\|_\infty$ is the supremum norm and

$$L(u) = \sup\left\{\frac{\|u(x) - u(y)\|}{\|x-y\|}\colon x, y \in X \text{ with } x \neq y\right\}.$$

It follows from the generalization of Henry's lemma (see [15, p. 151]) given by Elbialy [14] (for related results see also [16] and [13]) that $y_m \in C_b^{1,1}(B_n(r), F_m)$ and

$$(d_\xi x_{m,k})_{k\in\mathbb{N}} \text{ converges pointwise to } d_\xi \bar{y}_m \text{ when } k \to \infty \tag{52}$$

for every $\xi \in B_n(r)$. By the uniqueness of each function y_m in the ball $B_n(r)$, we can obtain a function $\bar{y}_m \in C^{1,1}(E_m, F_m)$ such that $\bar{y}_m|_{B_n(r)} = y_m$ for each $r > 0$. Using (52) we can easily verify that $(\bar{y}_m)_m \in \mathcal{B}$. Moreover, since $(x_k)_k$ is a Cauchy sequence, for each $\kappa > 0$ there exists $p \in \mathbb{N}$ such that if $k, m > p$ then

$$\left\|x_{q,k}(\xi) - x_{q,m}(\xi)\right\| \leqslant \kappa(q-n+1)^a n^\varepsilon \|\xi\| \tag{53}$$

for every $q \geqslant n$ and $\xi \in E_m$. Letting $m \to \infty$ in (53) we obtain

$$\left\|x_{q,k}(\xi) - \bar{y}_q(\xi)\right\| \leqslant \kappa(q-n+1)^a n^\varepsilon \|\xi\|,$$

and thus $(x_k)_k$ converges to $\bar{y} = (\bar{y}_q)_{q\geqslant n}$ in the space $\mathcal{B}$. □

Given $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}$ and $x = (x_m)_{m\geqslant n} \in \mathcal{B}$ we write

$$\varphi_m^* = \varphi_m \circ x_m \quad \text{and} \quad f_m^*(\xi) = f_m\bigl(x_m(\xi), \varphi_m^*(\xi)\bigr).$$

Lemma 6. *For each $\varphi \in \mathcal{X}$, $n \in \mathbb{N}$, $x \in \mathcal{B}$, $\xi, \bar{\xi} \in E_n$, and $m \geqslant n$ we have*

$$\left\|x_m(\xi) - x_m(\bar{\xi})\right\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi - \bar{\xi}\|, \tag{54}$$
$$\left\|d_\xi \varphi_m^*\right\| \leqslant C(m-n+1)^a n^\varepsilon, \tag{55}$$
$$\left\|d_\xi f_m^*\right\| \leqslant 2C\delta m^{-3\varepsilon-1}(m-n+1)^a n^\varepsilon, \tag{56}$$
$$\left\|\varphi_m^*(\xi) - \varphi_m^*(\bar{\xi})\right\| \leqslant C(m-n+1)^a n^\varepsilon \|\xi - \bar{\xi}\|, \tag{57}$$
$$\left\|d_\xi \varphi_m^* - d_{\bar{\xi}} \varphi_m^*\right\| \leqslant 2C^2(m-n+1)^a n^{2\varepsilon} \|\xi - \bar{\xi}\|, \tag{58}$$
$$\left\|d_\xi f_m^* - d_{\bar{\xi}} f_m^*\right\| \leqslant 7C^2\delta m^{-3\varepsilon-1}(m-n+1)^a n^{2\varepsilon} \|\xi - \bar{\xi}\|. \tag{59}$$

Proof. By the mean value theorem and (48) we have

$$\left\|x_m(\xi)-x_m(\bar\xi)\right\| \leqslant \sup_{r\in[0,1]} \|d_{\xi+r(\bar\xi-\xi)}x_m\|\cdot\|\xi-\bar\xi\| \leqslant C(m-n+1)^a n^\varepsilon\|\xi-\bar\xi\|$$

and this establishes (54).

Eq. (55) follows immediately from (39) and (48). In fact

$$\left\|d_\xi\varphi_m^*\right\| \leqslant \|d_{x_m(\xi)}\varphi_m\|\cdot\|d_\xi x_m\| \leqslant C(m-n+1)^a n^\varepsilon.$$

By (36), (48) and (55) we have

$$\begin{aligned}\left\|d_\xi f_m^*\right\| &\leqslant \left\|\partial^{1,0}_{x_m(\xi),\varphi_m^*(\xi)} f_m\right\|\cdot\|d_\xi x_m\| + \left\|\partial^{0,1}_{x_m(\xi),\varphi_m^*(\xi)} f_m\right\|\cdot\left\|d_\xi\varphi_m^*\right\| \\ &\leqslant 2C\delta m^{-3\varepsilon-1}(m-n+1)^a n^\varepsilon\end{aligned}$$

and this proves (56).

Using the mean value theorem and (55) we obtain (57).

To prove (58) we note that by (40) and (54) we get

$$\|d_{x_m(\xi)}\varphi_m - d_{x_m(\bar\xi)}\varphi_m\| \leqslant \left\|x_m(\xi)-x_m(\bar\xi)\right\| \leqslant C(m-n+1)^a n^\varepsilon\|\xi-\bar\xi\|,$$

and, since $C>1$, it follows from (48), (39) and (49) that

$$\begin{aligned}\left\|d_\xi\varphi_m^* - d_{\bar\xi}\varphi_m^*\right\| &\leqslant \|d_{x_m(\xi)}\varphi_m - d_{x_m(\bar\xi)}\varphi_m\|\cdot\|d_\xi x_m\| + \|d_{x_m(\bar\xi)}\varphi_m\|\cdot\|d_\xi x_m - d_{\bar\xi}x_m\| \\ &\leqslant C^2(m-n+1)^{2a}n^{2\varepsilon}\|\xi-\bar\xi\| + C(m-n+1)^a n^{2\varepsilon}\|\xi-\bar\xi\| \\ &\leqslant 2C^2(m-n+1)^a n^{2\varepsilon}\|\xi-\bar\xi\|.\end{aligned}$$

Finally we are going to prove (59). Writing

$$F_{\lambda,\mu} = \left\|\partial^{\lambda,\mu}_{x_m(\xi),\varphi_m^*(\xi)} f_m - \partial^{\lambda,\mu}_{x_m(\bar\xi),\varphi_m^*(\bar\xi)} f_m\right\|$$

where $\lambda,\mu\in\{0,1\}$ and $\lambda+\mu=1$, we have

$$\begin{aligned}\left\|d_\xi f_m^* - d_{\bar\xi} f_m^*\right\| &\leqslant F_{1,0}\|d_\xi x_m\| + F_{0,1}\left\|d_\xi\varphi_m^*\right\| + \left\|\partial^{1,0}_{x_m(\bar\xi),\varphi_m^*(\bar\xi)} f_m\right\|\cdot\|d_\xi x_m - d_{\bar\xi}x_m\| \\ &\quad + \left\|\partial^{0,1}_{x_m(\bar\xi),\varphi_m^*(\bar\xi)} f_m\right\|\cdot\left\|d_\xi\varphi_m^* - d_{\bar\xi}\varphi_m^*\right\|.\end{aligned} \tag{60}$$

Using (37), (54) and (57) we obtain

$$\begin{aligned}F_{\lambda,\mu} &\leqslant \delta m^{-3\varepsilon-1}\left\|x_m(\xi)-x_m(\bar\xi)\right\| + \delta m^{-3\varepsilon-1}\left\|\varphi_m^*(\xi)-\varphi_m^*(\bar\xi)\right\| \\ &\leqslant 2C\delta m^{-3\varepsilon-1}(m-n+1)^a n^\varepsilon\|\xi-\bar\xi\|.\end{aligned}$$

By the former inequalities, (48), (55), (36), (49) and (58) it follows from (60) that

$$\begin{aligned}\|d_\xi f_m^* - d_{\bar\xi} f_m^*\| &\leqslant 4C^2\delta m^{-3\varepsilon-1}(m-n+1)^{2a}n^{2\varepsilon}\|\xi-\bar\xi\| \\ &\quad + \big(C+2C^2\big)\delta m^{-3\varepsilon-1}(m-n+1)^a n^{2\varepsilon}\|\xi-\bar\xi\|.\end{aligned}$$

This yields (59) and the lemma is proved. □

Lemma 7. *Given $\delta > 0$ sufficiently small, for each $\varphi \in \mathcal{X}$ and $n \in \mathbb{N}$ there exists a unique sequence $x = x^\varphi \in \mathcal{B}$ satisfying* (45) *for every $m > n$ and $\xi \in E_n$.*

Proof. We consider an operator J in $\mathcal{B}$ defined by $(Jx)_n(\xi) = \xi$ and by

$$(Jx)_m(\xi) = \mathcal{A}(m,n)\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)g_k\big(x_k(\xi), \varphi_k\big(x_k(\xi)\big)\big)$$

for $m > n$. For every $m \geqslant n$, one can easily verify that $(Jx)_m$ is a function of class C^1 and, from (47), (38) and (35), that $(Jx)_m(0) = 0$.

From

$$d_\xi(Jx)_m = \mathcal{A}(m,n)P_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}d_\xi f_k^*$$

it follows from (2), (56) and (8) that

$$\begin{aligned}\|d_\xi(Jx)_m\| &\leqslant \|\mathcal{A}(m,n)P_n\| + \sum_{k=n}^{m-1}\|\mathcal{A}(m,k+1)P_{k+1}\|\,\|d_\xi f_k^*\| \\ &\leqslant D(m-n+1)^a n^\varepsilon + \sum_{k=n}^{m-1} D(m-k)^a(k+1)^\varepsilon 2C\delta(k-n+1)^a n^\varepsilon k^{-3\varepsilon-1} \\ &\leqslant D(m-n+1)^a n^\varepsilon + 2CD\delta n^\varepsilon \sum_{k=n}^{m-1}(m-k)^a(k-n+1)^a(k+1)^\varepsilon k^{-3\varepsilon-1} \\ &\leqslant D(m-n+1)^a n^\varepsilon + 2^{1+\varepsilon-a}CD\delta\lambda_{-2\varepsilon-1}(m-n+1)^a n^\varepsilon.\end{aligned}$$

Choosing δ sufficiently small (independently of φ, x, n, m and ξ) and since $C > D$ we have

$$\|d_\xi(Jx)_m\| \leqslant C(m-n+1)^a n^\varepsilon. \tag{61}$$

Proceeding in a similar manner, by (2), (59) and (8) it follows that

$$\begin{aligned}\|d_\xi(Jx)_m - d_{\bar\xi}(Jx)_m\| &\leqslant \sum_{k=n}^{m-1}\|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|d_\xi f_k^* - d_{\bar\xi} f_k^*\| \\ &\leqslant \sum_{k=n}^{m-1} D(m-k)^a(k+1)^\varepsilon 7C^2\delta k^{-3\varepsilon-1}(k-n+1)^a n^{2\varepsilon}\|\xi-\bar\xi\|\end{aligned}$$

$$\leqslant 7\,2^{\varepsilon-a}C^2D\delta\lambda_{-2\varepsilon-1}(m-n+1)^a n^{2\varepsilon}\|\xi-\bar{\xi}\|$$

and choosing δ sufficiently small (independently of φ, x, n, m, ξ and $\bar{\xi}$) we have

$$\left\|d_\xi(Jx)_m-d_{\bar{\xi}}(Jx)_m\right\|\leqslant C(m-n+1)^a n^{2\varepsilon}\|\xi-\bar{\xi}\|. \tag{62}$$

Therefore from (61) and (62) we conclude that $J(\mathcal{B})\subset\mathcal{B}$.

We now show that J is a contraction in $\mathcal{B}$ with the metric induced by (51). Let $x,y\in\mathcal{B}$. By (36), the mean value theorem and (39), for $k\geqslant n$ we have

$$\begin{aligned}\alpha_k &:= \left\|f_k\big(x_k(\xi),\varphi_k\big(x_k(\xi)\big)\big)-f_k\big(y_k(\xi),\varphi_k\big(y_k(\xi)\big)\big)\right\|\\ &\leqslant \delta k^{-3\varepsilon-1}\big(\left\|x_k(\xi)-y_k(\xi)\right\|+\left\|\varphi_k\big(x_k(\xi)\big)-\varphi_k\big(y_k(\xi)\big)\right\|\big)\\ &\leqslant 2\delta k^{-3\varepsilon-1}\left\|x_k(\xi)-y_k(\xi)\right\|\\ &\leqslant 2\delta k^{-3\varepsilon-1}(k-n+1)^a n^\varepsilon\|\xi\|\|x-y\|'.\end{aligned}$$

Using (2), the former inequality and (8) we obtain

$$\begin{aligned}\left\|(Jx)_m(\xi)-(Jy)_m(\xi)\right\| &\leqslant \sum_{k=n}^{m-1}\left\|\mathcal{A}(m,k+1)P_{k+1}\right\|\alpha_k\\ &\leqslant \sum_{k=n}^{m-1}D(m-k)^a(k+1)^\varepsilon 2\delta k^{-3\varepsilon-1}(k-n+1)^a n^\varepsilon\|\xi\|\|x-y\|'\\ &\leqslant \delta\theta'(m-n+1)^a n^\varepsilon\|\xi\|\|x-y\|'\end{aligned}$$

with $\theta'=2^{1+\varepsilon-a}D\lambda_{-2\varepsilon-1}$. Therefore,

$$\|Jx-Jy\|'\leqslant\delta\theta'\|x-y\|',$$

and for $\delta>0$ sufficiently small J is a contraction in $\mathcal{B}$. By Proposition 1, the map J has a unique fixed point x^φ in $\mathcal{B}$, which is thus the desired sequence. □

For the next lemma we need to represent by $(x^\varphi_{n,k})_{k\geqslant n}\in\mathcal{B}_n$ the unique sequence given by Lemma 7.

Lemma 8. *Given $\delta>0$ sufficiently small, for each $\varphi\in\mathcal{X}$ the following properties are equivalent*:

(1) *for every $n\in\mathbb{N}$, $m>n$, and $\xi\in E_n$, the identity* (46) *holds with $x=(x^\varphi_{n,k})_{k\geqslant n}$*;
(2) *for every $n\in\mathbb{N}$, and $\xi\in E_n$ we have*

$$\varphi_n(\xi)=-\sum_{k=n}^{\infty}\mathcal{A}(k+1,n)^{-1}h_k\big(x^\varphi_{n,k}(\xi),\varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big). \tag{63}$$

Proof. We first show that the series in (63) converges. By the mean value theorem, using (36), (39) and (50) we obtain

$$\begin{aligned}\big\| f_k\big(x^\varphi_{n,k}(\xi), \varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big)\big\| &\leqslant \delta k^{-3\varepsilon-1}\big(\big\|x^\varphi_{n,k}(\xi)\big\| + \big\|\varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big\|\big)\\ &\leqslant 2\delta k^{-3\varepsilon-1}\big\|x^\varphi_{n,k}(\xi)\big\|\\ &\leqslant 2\delta C k^{-3\varepsilon-1}(k-n+1)^a n^\varepsilon \|\xi\|.\end{aligned}$$

From (3), we conclude that

$$\begin{aligned}&\sum_{k=n}^{\infty}\big\|\mathcal{A}(k+1,n)^{-1}h_k\big(x^\varphi_{n,k}(\xi), \varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big)\big\|\\ &\quad= \sum_{k=n}^{\infty}\big\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}f_k\big(x^\varphi_{n,k}(\xi), \varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big)\big\|\\ &\quad\leqslant \sum_{k=n}^{\infty} D(k-n+2)^{-b}(k+1)^\varepsilon 2\delta C k^{-3\varepsilon-1}(k-n+1)^a n^\varepsilon\|\xi\|\\ &\quad\leqslant 2^{1+\varepsilon}\delta C D n^\varepsilon\|\xi\|\sum_{k=n}^{\infty}(k-n+1)^{a-b}k^{-2\varepsilon-1}\\ &\quad\leqslant 2^{1+\varepsilon}\delta C D n^\varepsilon\|\xi\|\sum_{k=n}^{\infty}(k-n+1)^{a-b-2\varepsilon-1} < \infty.\end{aligned}$$

If the first property holds, the identity $\mathcal{A}(m,n)^{-1}\mathcal{A}(m,k+1) = \mathcal{A}(k+1,n)^{-1}$ allows to write (46) in the following equivalent form

$$\varphi_n(\xi) = \mathcal{A}(m,n)^{-1}\varphi_m\big(x^\varphi_{n,m}(\xi)\big) - \sum_{k=n}^{m-1}\mathcal{A}(k+1,n)^{-1}h_k\big(x^\varphi_{n,k}(\xi), \varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big). \tag{64}$$

By (3) and (39), it follows that

$$\begin{aligned}\big\|\mathcal{A}(m,n)^{-1}\varphi_m\big(x^\varphi_{n,m}(\xi)\big)\big\| &= \big\|\mathcal{A}(m,n)^{-1}Q_m\varphi_m\big(x^\varphi_{n,m}(\xi)\big)\big\|\\ &\leqslant D(m-n+1)^{-b}m^\varepsilon\big\|x^\varphi_{n,m}(\xi)\big\|\\ &\leqslant D(m-n-1)^{-b}m^\varepsilon C(m-n+1)^a n^\varepsilon\|\xi\|\\ &\leqslant CD(m-n+1)^{a-b+\varepsilon}n^{2\varepsilon}\|\xi\|.\end{aligned}$$

Therefore by (42) we have $\|\mathcal{A}(m,n)^{-1}\varphi_m(x^\varphi_{n,m}(\xi))\| \to 0$ when $m\to\infty$ and letting $m\to\infty$ in (64) we obtain (63).

We now assume that (63) holds. Then

$$\mathcal{A}(m,n)\varphi_n(\xi) = -\sum_{k=n}^{\infty}\mathcal{A}(m,k+1)h_k\big(x^\varphi_{n,k}(\xi), \varphi_k\big(x^\varphi_{n,k}(\xi)\big)\big),$$

and thus

$$\begin{aligned}
&\mathcal{A}(m,n)\varphi_n(\xi)+\sum_{k=n}^{m-1}\mathcal{A}(m,k+1)h_k\bigl(x_{n,k}^{\varphi}(\xi),\varphi_k\bigl(x_{n,k}^{\varphi}(\xi)\bigr)\bigr)\\
&\quad=-\sum_{k=m}^{\infty}\mathcal{A}(m,k+1)h_k\bigl(x_{n,k}^{\varphi}(\xi),\varphi_k\bigl(x_{n,k}^{\varphi}(\xi)\bigr)\bigr). \qquad (65)
\end{aligned}$$

It follows from (63) that the right-hand side of (65) is $\varphi_m(x_{n,m}^{\varphi}(\xi))$, and Eq. (46) is satisfied with $x=x^{\varphi}$. $\square$

We now equip the space $\mathcal{X}$ with metric induced by

$$\|\varphi\|'=\sup\left\{\frac{\|\varphi_n(x)\|}{\|x\|}\colon n\in\mathbb{N} \text{ and } x\in E_n\setminus\{0\}\right\}. \qquad (66)$$

Proposition 2. *The space $\mathcal{X}$ is a complete metric space with the metric induced by* (66).

The proof of Proposition 2 is completely analogous to the proof of Proposition 1 and thus it is omitted.

Lemma 9. *Given $\delta>0$ sufficiently small, for each $\varphi,\psi\in\mathcal{X}$, $n\in\mathbb{N}$, $m\geqslant n$, and $\xi\in E_n$ we have*

$$\bigl\|x_m^{\varphi}(\xi)-x_m^{\psi}(\xi)\bigr\|\leqslant\frac{C}{2}(m-n+1)^a n^{\varepsilon}\|\xi\|\cdot\|\varphi-\psi\|'.$$

Proof. Putting

$$\gamma_k=\bigl\|f_k\bigl(x_k^{\varphi}(\xi),\varphi_k\bigl(x_k^{\varphi}(\xi)\bigr)\bigr)-f_k\bigl(x_k^{\psi}(\xi),\psi_k\bigl(x_k^{\psi}(\xi)\bigr)\bigr)\bigr\|, \qquad (67)$$

by the mean value theorem and (36) we obtain

$$\gamma_k\leqslant\delta k^{-3\varepsilon-1}\bigl(\bigl\|x_k^{\varphi}(\xi)-x_k^{\psi}(\xi)\bigr\|+\bigl\|\varphi_k\bigl(x_k^{\varphi}(\xi)\bigr)-\psi_k\bigl(x_k^{\psi}(\xi)\bigr)\bigr\|\bigr).$$

Using again the mean value theorem, (39) and (50) we have

$$\begin{aligned}
\bigl\|\varphi_k\bigl(x_k^{\varphi}(\xi)\bigr)-\psi_k\bigl(x_k^{\psi}(\xi)\bigr)\bigr\| &\leqslant\bigl\|\varphi_k\bigl(x_k^{\varphi}(\xi)\bigr)-\psi_k\bigl(x_k^{\varphi}(\xi)\bigr)\bigr\|+\bigl\|\psi_k\bigl(x_k^{\varphi}(\xi)\bigr)-\psi_k\bigl(x_k^{\psi}(\xi)\bigr)\bigr\|\\
&\leqslant\|\varphi-\psi\|'\cdot\bigl\|x_k^{\varphi}(\xi)\bigr\|+\bigl\|x_k^{\varphi}(\xi)-x_k^{\psi}(\xi)\bigr\|\\
&\leqslant C(k-n+1)^a n^{\varepsilon}\|\varphi-\psi\|'\|\xi\|+\bigl\|x_k^{\varphi}(\xi)-x_k^{\psi}(\xi)\bigr\|
\end{aligned}$$

and this implies that

$$\gamma_k\leqslant\delta k^{-3\varepsilon-1}\bigl(2\bigl\|x_k^{\varphi}(\xi)-x_k^{\psi}(\xi)\bigr\|+C(k-n+1)^a n^{\varepsilon}\|\varphi-\psi\|'\|\xi\|\bigr). \qquad (68)$$

We are now going to prove the lemma by induction. For $k=n$ the result follows immediately. Suppose that the result is true for $k=n,\dots,m-1$. Then for $k=n,\dots,m-1$ we get from (68) and the induction hypothesis that

$$\begin{aligned}\gamma_k &\leqslant \delta k^{-3\varepsilon-1}\big(C(k-n+1)^a n^\varepsilon \|\varphi-\psi\|'\|\xi\| + C(k-n+1)^a n^\varepsilon \|\varphi-\psi\|'\|\xi\|\big)\\ &\leqslant 2\delta C(k-n+1)^a n^\varepsilon k^{-3\varepsilon-1}\|\varphi-\psi\|'\|\xi\|.\end{aligned}$$

Hence by (2), the last inequality and (8) it follows that

$$\begin{aligned}\big\|x_m^\varphi(\xi)-x_m^\psi(\xi)\big\| &\leqslant \sum_{k=n}^{m-1}\big\|\mathcal{A}(m,k+1)P_{k+1}\big\|\gamma_k\\ &\leqslant \sum_{k=n}^{m-1} D(m-k)^a(k+1)^\varepsilon 2\delta C(k-n+1)^a n^\varepsilon k^{-3\varepsilon-1}\|\varphi-\psi\|'\|\xi\|\\ &\leqslant 2^{1+\varepsilon-a}CD\delta\lambda_{-2\varepsilon-1}(m-n+1)^a n^\varepsilon\|\varphi-\psi\|'\|\xi\|.\end{aligned}$$

Choosing δ such that $2^{1+\varepsilon-a}D\delta\lambda_{-2\varepsilon-1} < 1/2$ we have

$$\big\|x_m^\varphi(\xi)-x_m^\psi(\xi)\big\| \leqslant \frac{C}{2}(m-n+1)^a n^\varepsilon\|\varphi-\psi\|'\|\xi\|$$

and this concludes the proof of the lemma. □

Lemma 10. *Given $\delta > 0$ sufficiently small, there is a unique $\varphi \in \mathcal{X}$ such that* (63) *holds for every $n \in \mathbb{N}$ and $\xi \in E_n$.*

Proof. We consider the operator Φ defined for each $\varphi \in \mathcal{X}$ by

$$(\Phi\varphi)_n(\xi) = -\sum_{k=n}^{\infty}\mathcal{A}(k+1,n)^{-1}h_k\big(x_k^\varphi(\xi),\varphi_k\big(x_k^\varphi(\xi)\big)\big),$$

where $x^\varphi = (x_k^\varphi)_{k\geqslant n}$ is the unique sequence given by Lemma 7. One can easily verify that each function $(\Phi\varphi)_n$ is of class C^1. Since $x^\varphi \in \mathcal{B}$ we have $x_m^\varphi(0) = 0$, $m \geqslant n$. It follows from (38) and (35) that $(\Phi\varphi)_n(0) = 0$ for each $n \in \mathbb{N}$. We also have $d_0(\Phi\varphi)_n = 0$ for each $n \in \mathbb{N}$ because $d_0 f_k = 0$, $k \in \mathbb{N}$, $\varphi_m(0) = 0$ and $x_m^\varphi(0) = 0$, $m \geqslant n$.

Furthermore, by (3) and (56) we have

$$\begin{aligned}\big\|d_\xi(\Phi\varphi)_n\big\| &\leqslant \sum_{k=n}^{\infty}\big\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\big\|\cdot\big\|d_\xi f_k^*\big\|\\ &\leqslant 2CD\delta\sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^\varepsilon k^{-3\varepsilon-1}(k-n+1)^a n^\varepsilon\\ &\leqslant 2^{1+\varepsilon}CD\delta\sum_{k=n}^{\infty}(k-n+2)^{-b}(k-n+1)^a k^{-\varepsilon-1}\\ &\leqslant 2^{1+\varepsilon}CD\delta\sum_{k=n}^{\infty}(k-n+1)^{a-b-\varepsilon-1} < \infty.\end{aligned}$$

Hence, for $\delta > 0$ sufficiently small (independently of φ, n and ξ) we have $\|d_\xi(\Phi\varphi)_n\| \leqslant 1$ for every $n \in \mathbb{N}$.

On the other hand, given $\xi, \bar\xi \in E_n$ it follows from (59) that

$$\begin{aligned}
\left\|d_\xi(\Phi\varphi)_n - d_{\bar\xi}(\Phi\varphi)_n\right\| &\leqslant \sum_{k=n}^{\infty}\left\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\right\| \cdot \left\|d_\xi f_k^* - d_{\bar\xi} f_k^*\right\| \\
&\leqslant 7C^2 D\delta n^{2\varepsilon}\|\xi - \bar\xi\| \sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^{\varepsilon}k^{-3\varepsilon-1}(k-n+1)^a \\
&\leqslant 7\,2^{\varepsilon}C^2 D\delta\|\xi - \bar\xi\| \sum_{k=n}^{\infty}(k-n+1)^{a-b-1} \\
&\leqslant \|\xi - \bar\xi\|
\end{aligned}$$

provided that $\delta > 0$ is sufficiently small (independently of φ, n, ξ and $\bar\xi$). Hence, $\Phi(\mathfrak{X}) \subset \mathfrak{X}$.

We now show that Φ is a contraction. Given $\varphi, \psi \in \mathfrak{X}$ and $n \in \mathbb{N}$, let x^φ and x^ψ be the unique sequences given by Lemma 7 respectively for φ and ψ. By (68) (see (67) for the definition of γ_k) and Lemma 9 we have

$$\begin{aligned}
&\left\|(\Phi\varphi)_n(\xi) - (\Phi\psi)_n(\xi)\right\| \\
&\quad\leqslant \sum_{k=n}^{\infty}\left\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\right\|\gamma_k \\
&\quad\leqslant \sum_{k=n}^{\infty}\left\|\mathcal{A}(k+1,n)^{-1}Q_{k+1}\right\|\delta k^{-3\varepsilon-1} \\
&\qquad\times\left(C(k-n+1)^a\|\xi\|\cdot\|\varphi-\psi\|' + C(k-n+1)^a n^{\varepsilon}\|\xi\|\cdot\|\varphi-\psi\|'\right) \\
&\quad\leqslant 2CD\delta\|\xi\|\cdot\|\varphi-\psi\|' \sum_{k=n}^{\infty}(k-n+2)^{-b}(k+1)^{\varepsilon}k^{-3\varepsilon-1}(k-n+1)^a n^{\varepsilon} \\
&\quad\leqslant 2^{1+\varepsilon}CD\delta\|\xi\|\cdot\|\varphi-\psi\|' \sum_{k=n}^{\infty}(k-n+1)^{a-b-\varepsilon-1}.
\end{aligned}$$

Therefore, choosing $\delta > 0$ such that $\lambda := 2^{1+\varepsilon}CD\delta\lambda_{a-b-\varepsilon-1} < 1$, it follows that

$$\|\Phi\varphi - \Phi\psi\|' \leqslant \lambda\|\varphi - \psi\|',$$

and with this λ we obtain a contraction in $\mathfrak{X}$. By Proposition 2, the map Φ has a unique fixed point φ in $\mathfrak{X}$ that is the desired sequence. □

Proof of Theorem 2. By Lemma 7, for each $\varphi \in \mathfrak{X}$ there is a unique sequence $x^\varphi \in \mathfrak{B}$ satisfying identity (45). It remains to solve the identity (46) with $x = x^\varphi$. By Lemma 8, this is the same as solving (63). Finally, by Lemma 10, there is a unique solution of (63). This establishes the existence of the stable manifolds for $\delta > 0$ sufficiently small. Since the functions φ_n are of class

$C^{1,1}$ the graphs $\mathcal{V}_n$ are $C^{1,1}$ manifolds. Finally, for each $n \in \mathbb{N}$, $m \geqslant n$, $\xi, \bar{\xi} \in E_n$, it follows from (54) and (57) that

$$\begin{aligned}\left\|\mathcal{F}(m,n)(v) - \mathcal{F}(m,n)(\bar{v})\right\| &\leqslant \left\|x_m(\xi) - x_m(\bar{\xi})\right\| + \left\|\varphi_m^*(\xi) - \varphi_m^*(\bar{\xi})\right\| \\ &\leqslant 2C(m-n+1)^a n^{\varepsilon}\|\xi - \bar{\xi}\|\end{aligned}$$

and from (49) and (58) that

$$\begin{aligned}\left\|d_v\mathcal{F}(m,n) - d_{\bar{v}}\mathcal{F}(m,n)\right\| &\leqslant \|d_\xi x_m - d_{\bar{\xi}} x_m\| + \left\|d_\xi \varphi_m^* - d_{\bar{\xi}}\varphi_m^*\right\| \\ &\leqslant \left(C + 2C^2\right)(m-n+1)^a n^{2\varepsilon}\|\xi - \bar{\xi}\|,\end{aligned}$$

where $v = (\xi, \varphi_n(\xi))$ and $\bar{v} = (\bar{\xi}, \varphi_n(\bar{\xi}))$. This completes the proof of the theorem. □

5. Examples

In this section we are going to give examples of nonuniform polynomial dichotomies and two families of perturbations, the first one verifying our assumptions in Section 3 and the second one verifying the assumptions of Section 4.

Let $a < 0 \leqslant b$ and $\varepsilon \geqslant 0$. We are going to construct a sequence of linear operators $A_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by diagonal matrices

$$A_n = \begin{bmatrix} a_n & 0 \\ 0 & b_n \end{bmatrix}$$

with positive entries in the diagonal and that verify (2) and (3) with $D = 1$ and projections given by $P_n(x,y) = (x,0)$ and $Q_n(x,y) = (0,y)$. Because $\|\mathcal{A}(2,1)P_1\| = a_1$, from (2) we must have $a_1 \leqslant 2^a$. Therefore we put $a_1 = 2^a$.

Using again (2) we must have

$$\left\|\mathcal{A}(3,2)P_2\right\| \leqslant 2^a 2^{\varepsilon} \quad \text{and} \quad \left\|\mathcal{A}(3,1)P_1\right\| \leqslant 3^a.$$

Since $\|\mathcal{A}(3,2)P_2\| = a_2$ and $\|\mathcal{A}(3,1)P_1\| = a_2 a_1$, we put

$$a_2 = \min\left\{2^a 2^{\varepsilon}, \frac{3^a}{a_1}\right\}.$$

Using the same arguments we put

$$a_3 = \min\left\{2^a 3^{\varepsilon}, \frac{3^a 2^{\varepsilon}}{a_2}, \frac{4^a}{a_2 a_1}\right\}.$$

Hence the values of a_n are given recursively by $a_1 = 2^a$ and

$$a_{n+1} = \min\left\{2^a(n+1)^{\varepsilon}, \frac{3^a n^{\varepsilon}}{a_n}, \dots, \frac{n^a 2^{\varepsilon}}{a_n \dots a_2}, \frac{(n+1)^a}{a_n \dots a_1}\right\}.$$

Using similar arguments, from (3) we set $b_1 = 2^b 2^{-\varepsilon}$ and

$$b_{n+1} = \max\left\{\frac{2^b}{(n+1)^\varepsilon}, \frac{3^b}{(n+1)^\varepsilon b_n}, \dots, \frac{n^b}{(n+1)^\varepsilon b_n \dots b_2}, \frac{(n+1)^b}{(n+1)^\varepsilon b_n \dots b_1}\right\}.$$

It follows immediately from the construction that (A_n) admits a nonuniform polynomial dichotomy. For instance, if $\varepsilon = -a = b$ we can easily verify that

$$A_n = \begin{bmatrix} (\frac{n+1}{n})^a & 0 \\ 0 & 1 \end{bmatrix}.$$

For the local case considered in Section 3 we have the perturbations $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_n(x, y) = \left(x^{q+1}, y^{q+1}\right) \quad \text{or} \quad f_n(x, y) = \left(y^{q+1}, x^{q+1}\right)$$

with $q \in \mathbb{N}$. Obviously we have $f_n(0, 0) = (0, 0)$ and from inequalities

$$\sqrt{a^2 + b^2} \leqslant |a| + |b| \leqslant \sqrt{2}\sqrt{a^2 + b^2}$$

and

$$\left|a^{q+1} - b^{q+1}\right| = \left|(a-b)\sum_{k=1}^{q} a^{q-k} b^k\right| \leqslant |a-b| \sum_{k=1}^{q} |a|^{q-k}|b|^k \leqslant |a-b|\left(|a| + |b|\right)^q$$

it follows that

$$\begin{aligned}
\left\| f_n(x, y) - f_n(u, v)\right\| &= \sqrt{\left(x^{q+1} - u^{q+1}\right)^2 + \left(y^{q+1} - v^{q+1}\right)^2} \\
&\leqslant \left|x^{q+1} - u^{q+1}\right| + \left|y^{q+1} - v^{q+1}\right| \\
&\leqslant |x-u|\left(|x| + |u|\right)^q + |y-v|\left(|y| + |v|\right)^q \\
&\leqslant \left(|x-u| + |y-v|\right)\left(|x| + |u| + |y| + |v|\right)^q \\
&\leqslant \sqrt{2}\left\|(x, y) - (u, v)\right\|\left(|x| + |u| + |y| + |v|\right)^q \\
&\leqslant 2^{(q+1)/2}\left\|(x, y) - (u, v)\right\|\left(\left\|(x, y)\right\| + \left\|(u, v)\right\|\right)^q.
\end{aligned}$$

This proves that the family of perturbations f_n satisfy (10) and (11). Therefore if we have $aq + \varepsilon + 1 < 0$ and $a + \beta < 0$, using Theorem 1 with (A_n) and (f_n) as above, we get local Lipschitz stable manifolds for the dynamics defined by (4).

Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(t) = t^2 \mathrm{e}^{-t^2}$. It is easy to see that

$$\left|g'(t)\right| = \left|2t\left(1 - t^2\right)\mathrm{e}^{-t^2}\right| \leqslant 1 \tag{69}$$

and

$$\left|g'(t) - g'(s)\right| \leqslant 2|t - s| \tag{70}$$

for every $t \in \mathbb{R}$. Then if $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ is the function defined by

$$f_n(x, y) = \frac{\delta}{2} n^{-3\varepsilon-1}\big(g(x), g(y)\big) \quad \text{or} \quad f_n(x, y) = \frac{\delta}{2} n^{-3\varepsilon-1}\big(g(y), g(x)\big),$$

from (69) and (70) it follows immediately that the functions f_n satisfy (35), (36) and (37). A closer look in the proofs of Lemmas 7, 9 and 10, allows us to conclude, taking $C = 2D$, that it is sufficient to have

$$\delta < \frac{1}{14\, 2^{\varepsilon-a} D^3 \lambda_{-\varepsilon-1}},$$

and in our example, since $D = 1$ and $\lambda_{-\varepsilon-1} < 1 + 1/\varepsilon$, it is enough to have

$$\delta < \frac{1}{14\, 2^{\varepsilon-a}(1 + 1/\varepsilon)}.$$

With these (f_n) and with (A_n) as above, using Theorem 2 we conclude that the dynamics given by (4) has global stable C^1 manifolds when $a + \varepsilon < b$ and $\varepsilon > 0$.

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