Density of Natural Numbers and the Lévy Group

NOBUKI OBATA

Department of Mathematics, Faculty of Science, Nagoya University, Chikusa-ku, Nagoya, 464, Japan

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The density of natural numbers, regarded as an analogue of a probability measure, enjoys an ergodic property under a certain permutation group called the Lévy group. Furthermore, the density is characterized by its invariance under the Lévy group. Finally, rearrangement of uniformly distributed sequences is discussed in connection with the Lévy group.

INTRODUCTION

Let N denote the set of all natural numbers. For a subset S of N the quantity

$$\delta(S) = \lim_{N \to \infty} \frac{1}{N} |S \cap \{1, 2, ..., N\}|,$$

where |·| stands for the cardinality, is called the (asymptotic) density of S if the limit exists. Let $\mathcal{F}$ denote the collection of all subsets of N which admit the density. Then the triple $(N, \mathcal{F}, \delta)$ may be regarded as an analogue of a probability space. Some problems of additive number theory were discussed by M. Kac [5, 6] from this viewpoint.

The main purpose of this paper is to discuss the density along with permutation groups. Let $\text{aut}(N)$ be the group of all permutations of N. The set

$$\mathcal{G} = \left\{ g \in \text{aut}(N); \lim_{N \to \infty} \frac{1}{N} \left| \{1 \leq n \leq N; g(n) > N\} \right| = 0 \right\}$$

forms a subgroup of $\text{aut}(N)$ and is called the Lévy group after T. Hida [3]. The density is shown to be invariant under the Lévy group. Thus, the system $(N, \mathcal{F}, \delta; \mathcal{G})$ is the main object of study in this paper. We first show that the density is ergodic under the Lévy group (Theorem 1). Next we give a characterization of the density by means of the invariance under the Lévy group (Theorems 2 and 3).
The Appendix contains a characterization of the Lévy group in terms of functionals on sequences. As an application we shall prove that any permutation of the Lévy group preserves the property of uniform distribution of sequences. This result is closely related to the work of J. von Neumann [14].

The Lévy group, originally introduced by P. Lévy [9, Part III], has been studied in [11–13] in connection with certain problems of functional analysis.

1. DENSITY OF NATURAL NUMBERS

Let \( \mathbb{N} \) denote the set of all natural numbers. For a subset \( S \) of \( \mathbb{N} \) we put

\[
\delta(S) = \limsup_{N \to \infty} \frac{1}{N} |S \cap \{1, 2, \ldots, N\}|
\]

and

\[
\delta(S) = \liminf_{N \to \infty} \frac{1}{N} |S \cap \{1, 2, \ldots, N\}|,
\]

where \( |\cdot| \) denotes the cardinality. These are called the upper and lower (asymptotic) density of \( S \), respectively. If the two are equal, we refer to their common value as the (asymptotic) density of \( S \) and denote it by \( \delta(S) \).

We denote by \( \mathcal{F} \) the collection of all subsets of \( \mathbb{N} \) which admit the density. The triple \( (\mathbb{N}, \mathcal{F}, \delta) \) is an analogue of a probability space but not quite. In fact, one should note that \( \mathcal{F} \) is not finitely additive. Nevertheless, we have the following

**Proposition 1.1.** If \( S \in \mathcal{F} \), then \( S^c \in \mathcal{F} \) and \( \delta(S^c) = 1 - \delta(S) \).

**Proposition 1.2.** Let \( S_1 \) and \( S_2 \) be members of \( \mathcal{F} \). Then the following four conditions are mutually equivalent:

(i) \( S_1 \cup S_2 \in \mathcal{F} \);  
(ii) \( S_1 \cap S_2 \in \mathcal{F} \);
(iii) \( S_1 - S_2 \in \mathcal{F} \);  
(iv) \( S_2 - S_1 \in \mathcal{F} \).

If one of the four conditions is satisfied, we have

\[
\delta(S_1 \cup S_2) = \delta(S_1) + \delta(S_2) - \delta(S_1 \cap S_2).
\]

The proofs of the above results are quite simple and are omitted. The next result, which might be intuitively apparent and well known, is of great importance to our goal.
PROPOSITION 1.3. Let $A \in \mathcal{F}$. For any $\lambda$, $0 \leq \lambda \leq \delta(A)$, there exists a subset $B \subset A$ such that $\delta(B) = \lambda$.

Proof. The assertion is trivial if $\lambda = 0$ or $\lambda = \delta(A)$. Suppose that $0 < \lambda < \delta(A)$. We may assume that $A$ does not contain 1 and we put
\[ \alpha(N) = |A \cap \{1, 2, \ldots, N\}|. \]

We shall define $\{N_k\}_{k=1}^\infty$, $\{M_k\}_{k=1}^\infty$, $\{P_k\}_{k=1}^\infty$, and $\{I_k\}_{k=1}^\infty$. First we may find two numbers $N_1$ and $M_1$ uniquely by the conditions:
\[ 0 = \frac{\alpha(1)}{1}, \frac{\alpha(2)}{2}, \ldots, \frac{\alpha(N_1 - 1)}{N_1 - 1} \leq \frac{\alpha(N_1)}{N_1} < \frac{\alpha(N_1 - 1) + 1}{N_1}, \]
\[ \frac{\alpha(N_1)}{N_1} > \frac{\alpha(N_1)}{N_1 + 1} > \cdots > \frac{\alpha(N_1)}{M_1 - 1} \geq \frac{\alpha(N_1)}{M_1}. \]

We put
\[ I_1 = \{N_1 + 1, \ldots, M_1\} \quad \text{and} \quad p_1 = |I_1 \cap A|. \]

Assume that the sequences are determined up to $k - 1$. Then we may find two numbers $N_k$ and $M_k$ uniquely by the conditions:
\[ \frac{\alpha(M_k - 1) - q_{k-1}}{M_k - 1} \geq \frac{\alpha(M_k - 1) + 1 - q_{k-1}}{M_k - 1} \geq \cdots \geq \frac{\alpha(M_k - 1) + 1 - q_{k-1}}{M_k - 1} \geq \frac{\alpha(N_k - 1) - q_{k-1}}{N_k - 1}, \]
\[ \frac{\alpha(N_k) - q_{k-1}}{N_k} > \frac{\alpha(N_k) - q_{k-1}}{N_k + 1} > \cdots > \frac{\alpha(N_k) - q_{k-1}}{M_k - 1} \geq \frac{\alpha(N_k) - q_{k-1}}{M_k}. \]

where $q_{k-1} = p_1 + \cdots + p_{k-1}$. We put
\[ I_k = \{N_k + 1, \ldots, M_k\} \quad \text{and} \quad p_k = |I_k \cap A|. \]

Note that $1 = M_0 < N_1 < M_1 < N_2 < M_2 < \cdots$.

We would like to show that the set
\[ B = A - \bigcup_{k=1}^{\infty} I_k \]

satisfies the desired property. To this end it is sufficient to show that $\delta(B) \geq \lambda$ because $\delta(B) = \lambda$ is obvious by construction. Since
\[ \frac{\beta(M_k - 1)}{M_k - 1} \geq \frac{\beta(M_k)}{M_k} \quad \text{and} \quad \frac{\beta(N_k - 1)}{N_k - 1} \leq \frac{\beta(N_k)}{N_k}. \]
where \( \beta(N) = |B \cap \{1, 2, \ldots, N\}| \), we have

\[
\left\{ N \in \mathbb{N} : \frac{\beta(N)}{N} < \lambda \right\} \subset \bigcup_{k=0}^{\infty} \{ M_k, \ldots, N_{k+1} - 1 \}.
\]

In view of the identity

\[
\beta(N) = \alpha(N) - q_k, \quad M_k \leq N < N_{k+1},
\]

we have

\[
\frac{\beta(N)}{N} - \lambda = \frac{\alpha(N)}{N} - \lambda = \left( \frac{\alpha(N)}{N} - \delta(A) \right) + \left( \delta(A) - \frac{q_k}{N} - \lambda \right).
\]

Moreover,

\[
\delta(A) - \frac{q_k}{N} - \lambda \geq \delta(A) - \frac{q_k}{M_k - 1} \frac{\beta(M_k - 1)}{M_k - 1} = \delta(A) - \frac{\alpha(M_k)}{M_k - 1}.
\]

Therefore,

\[
\frac{\beta(N)}{N} - \lambda \geq \left( \frac{\alpha(N)}{N} - \delta(A) \right) + \left( \delta(A) - \frac{\alpha(M_k)}{M_k - 1} \right),
\]

where \( M_k \leq N < N_{k+1} \). Since \( \lim_{N \to \infty} \alpha(N)/N = \delta(A) \), we conclude that \( \lambda \leq \delta(B) \) as desired. Q.E.D.

2. THE LÉVY GROUP

Let \( \text{aut}(N) \) be the group of all permutations of \( N \) and \( \mathcal{G}(\delta) \) the subgroup of all permutations which preserve the density:

\[
\mathcal{G}(\delta) = \{ g \in \text{aut}(N) ; g \mathcal{F} = \mathcal{F} \text{ and } \delta(g(S)) = \delta(S) \text{ for any } S \in \mathcal{F} \}.
\]

For any \( g \in \text{aut}(N) \) we put

\[
\text{supp } g = \{ n \in \mathbb{N} ; g(n) \neq n \}.
\]

Then \( g \) is a bijection from \( \text{supp } g \) onto itself. In particular, \( \text{supp } g = \text{supp } g^{-1} \). From the inequality

\[
\delta(S_1 \cup S_2) \leq \delta(S_1) + \delta(S_2), \quad S_1, S_2 \in \mathcal{F},
\]

which is verified easily, we see that the set

\[
\mathcal{G}_0 = \{ g \in \text{aut}(N); \delta(\text{supp } g) = 0 \}
\]
forms a subgroup of \( \text{aut}(\mathbb{N}) \). Obviously, the group of all finite permutations, denoted by \( \mathcal{G}_\infty \), is a proper subgroup of \( \mathcal{G}_0 \).

For \( g \in \text{aut}(\mathbb{N}) \) we put

\[
F^+_N(g) = \{ 1 \leq n \leq N; g(n) > N \}.
\]

It is shown that the set

\[
\mathcal{G} = \left\{ g \in \text{aut}(\mathbb{N}); \lim_{N \to \infty} \frac{1}{N} \left| F^+_N(g) \right| = 0 \right\}
\]

becomes a subgroup of \( \text{aut}(\mathbb{N}) \). Following T. Hida [3] we call \( \mathcal{G} \) the Lévy group. The next result was proved in [11].

**Proposition 2.1.** \( \mathcal{G}_0 \subset \mathcal{G} \subset \mathcal{G}(\delta) \).

**Example 2.2.** Let \( 0 = N_0 < N_1 < \cdots \) be an increasing sequence of integers. For any \( g \in \text{aut}(\mathbb{N}) \) which leaves every subset \( \{N_k - 1, \ldots, N_k\} \) stable, we have

\[
\limsup_{N \to \infty} \frac{1}{N} \left| F^+_N(g) \right| \leq \limsup_{k \to \infty} \frac{N_k}{N_k - 1}.
\]

Therefore, \( g \in \mathcal{G} \) whenever \( \lim_{k \to \infty} \frac{N_k}{N_k - 1} = 1 \).

**Remark 2.3.** From the above example it follows that \( \mathcal{G}_0 \) is a proper subgroup of \( \mathcal{G} \). Furthermore, it may be shown that \( \mathcal{G} \) is a proper subgroup of \( \mathcal{G}(\delta) \) and that \( \mathcal{G}_0 \) is a normal subgroup of \( \mathcal{G}(\delta) \).

**Lemma 2.4.** Let \( A = \{ a_1 < a_2 < \cdots \} \) and \( B = \{ b_1 < b_2 < \cdots \} \) be members of \( \mathcal{F} \) with the same density. Put \( A^c = \{ a_1 < a_2 < \cdots \} \) and \( B^c = \{ b_1 < b_2 < \cdots \} \). Assume that \( A, A^c, B, \) and \( B^c \) are infinite sets. A permutation \( g \in \text{aut}(\mathbb{N}) \) defined by

\[
g(a_n) = b_n, \quad g(a'_n) = b'_n, \quad n = 1, 2, \ldots,
\]

belongs to the Lévy group \( \mathcal{G} \).

**Proof.** For each \( N \in \mathbb{N} \) we put

\[
\alpha(N) = |A \cap \{1, 2, \ldots, N\}| \quad \text{and} \quad \beta(N) = |B \cap \{1, 2 \cdots N\}|.
\]

By assumption we have

\[
\lim_{N \to \infty} \frac{\alpha(N)}{N} = \delta(A) = \delta(B) = \lim_{N \to \infty} \frac{\beta(N)}{N}.
\]
With the help of the inequalities

\[ a_{\alpha(N)} \leq N < a_{\alpha(N) + 1} \quad \text{and} \quad b_{\beta(N)} \leq N < b_{\beta(N) + 1}, \]

we have

\[ |F_N^+(g)| - |\alpha(N) - \beta(N)|. \]

Therefore

\[ \limsup_{N \to \infty} \frac{1}{N} |F_N^+(g)| = \limsup_{N \to \infty} \left| \frac{\alpha(N) - B(N)}{N} \right| = 0. \]

Hence we have \( g \in \mathcal{G} \).

Q.E.D.

The following result is then immediate.

**PROPOSITION 2.5.** Let \( A \) and \( B \) be members of \( \mathcal{F} \) such that

\[ 0 < \delta(A) = \delta(B) < 1. \]

Then there exists a permutation \( g \in \mathcal{G} \) such that

\[ g(A) = B. \]

3. **The Main Results**

In the previous sections we have shown that the density \( \delta \) is an analogue of a probability measure on \( \mathbb{N} \) which is invariant under the Lévy group \( \mathcal{G} \). From this point of view we shall give two noticeable facts which correspond to the well-known properties of invariant measures on a homogeneous space.

The following result means that the density is ergodic under the Lévy group.

**THEOREM 1.** Assume that \( A \in \mathcal{F} \) is almost invariant under the Lévy group, i.e.,

\[ \delta(A \ominus g(A)) = 0 \quad \text{for all } g \in \mathcal{G}, \]

where \( \ominus \) denotes the symmetric difference. Then \( \delta(A) = 0 \) or 1.

**Proof.** We show the assertion by contradiction. Suppose that

\[ 0 < \delta(A) < 1. \]

Replacing \( A \) with \( A^c \) in the case of \( \frac{1}{2} \leq \delta(A) < 1 \), we may assume that \( 0 < \delta(A) \leq \frac{1}{2} \). With the help of Propositions 1.3 and 2.5 we take a subset \( B \subset A^c \) such that \( \delta(B) = \delta(A) \) and a permutation \( g \in \mathcal{G} \) such that

\[ g(A) = B. \]

then we have

\[ \delta(A \ominus g(A)) = \delta(A \ominus B) = \delta(A \cup B) = \delta(A) + \delta(B) > 0, \]

which completes the proof.

Q.E.D.
Next we give a group-theoretical characterization of the density.

**Theorem 2.** Let \( \gamma \) be a non-negative function defined on \( \mathcal{F} \). Assume that \( \gamma \) satisfies the following three conditions:

(i) (finite additivity) \( \gamma(A \cup B) = \gamma(A) + \gamma(B) \) for any pair \( A, B \in \mathcal{F} \) with \( A \cap B = \emptyset \);

(ii) (invariance) \( \gamma(g(A)) = \gamma(A) \) for any \( A \in \mathcal{F} \) and \( g \in \mathcal{G} \);

(iii) (normalization) \( \gamma(N) = 1 \).

Then \( \gamma \) coincides with the density \( \delta \).

**Proof.** We first show that \( \gamma(A) = \delta(A) \) for every \( A \in \mathcal{F} \) such that \( \delta(A) \) is rational and \( 0 < \delta(A) < 1 \). Suppose that \( \delta(A) = q/p \), where \( p \geq 2 \) and \( q \geq 1 \) are integers. With the help of Proposition 1.3 we may find disjoint subsets \( A_1, \ldots, A_p \in \mathcal{F} \) with the same density \( 1/p \) such that \( N = A_1 \cup \cdots \cup A_p \) and \( A = A_1 \cup \cdots \cup A_q \). By Proposition 2.5 there exist permutations \( g_1, \ldots, g_q \in \mathcal{G} \) such that \( g_1(A_1) = \cdots = g_q(A_q) = A_1 \). Then it follows from conditions (i)–(iii) that

\[
1 = \gamma(N) = \gamma(A_1) + \cdots + \gamma(A_p)
\]

\[
= \gamma(A_1) + \gamma(g_1(A_1)) + \cdots + \gamma(g_q(A_q))
\]

\[
= \gamma(A_1) + \gamma(A_1) + \cdots + \gamma(A_1) = p\gamma(A_1).
\]

Therefore \( \gamma(A_1) = 1/p \) and

\[
\gamma(A) = \gamma(A_1) + \cdots + \gamma(A_q)
\]

\[
= \gamma(A_1) + \gamma(g_2(A_1)) + \cdots + \gamma(g_q(A_q))
\]

\[
= q\gamma(A_1) = q/p = \delta(A).
\]

Next we show that \( \delta(A) \leq \gamma(A) \) for any \( A \in \mathcal{F} \). If \( \delta(A) = 0 \) the assertion is obvious, so we assume that \( \delta(A) > 0 \). Take a sequence \( \{r_n\}_{n=1}^\infty \) of rational numbers such that \( 0 < r_1 < r_2 < \cdots < \delta(A) \). With the help of Proposition 1.3, for each \( n \geq 1 \) we choose a subset \( A_n \in \mathcal{F} \) such that \( A_n \subseteq A \) and \( \delta(A_n) = r_n \). From the result of the previous paragraph it follows that

\[
r_n = \delta(A_n) = \gamma(A_n) \leq \gamma(A),
\]

where the last inequality follows from condition (i) and the non-negativity of \( \gamma \). Hence, \( \delta(A) \leq \gamma(A) \) as desired.

In order to prove the final assertion we have only to note that \( \gamma(A^c) = 1 - \gamma(A) \) for any \( A \in \mathcal{F} \).

Q.E.D.
Finally we discuss a possibility of weakening the assumption in the above statement. Let $G$ be the field generated by $F$; namely, $G$ is the smallest family of subsets of $N$ which includes $F$ and contains the complement of each of its members and the union of each finite collection of its members. Note that $G$ is stable under $F$. In fact, $gF$ is a field including $F$, namely, $gF \supset F$. Since $gG$ is arbitrary, we have $gG = G$.

Following [1, Chap. III] we recall some standard notions. A real-valued function $\mu$ defined on the field $F$ is called finitely additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any pair $A, B \in F$ with $A \cap B = \emptyset$. Let $\mu$ be a finitely additive function defined on $F$. For every $A \in F$ the total variation of $\mu$ on $A$, denoted by $|\mu|(A)$, is defined as

$$|\mu|(A) = \sup \sum_{k=1}^{n} |\mu(A_k)|,$$

where the supremum is taken over all partitions of $A$ into finitely many subsets $\{A_k\} \in F$. We say that the function $\mu$ is of bounded variation if $|\mu|(N) < \infty$. It is known that $|\mu|$ is a non-negative, finitely additive function if $\mu$ is of bounded variation.

**Theorem 3.** Let $\gamma$ be a real-valued function defined on $F$. Assume that $\gamma$ can be extended to a real-valued function $\tilde{\gamma}$ on $F$ which admits the following properties:

(i) $\tilde{\gamma}$ is finitely additive;

(ii) $\tilde{\gamma}$ is of bounded variation;

(iii) $\tilde{\gamma}$ is invariant under $G$.

Then $\gamma = c\delta$ with $c = \gamma(N)$.

**Proof.** For $A \in F$ we put

$$\mu^+(A) = \frac{1}{2}(|\tilde{\gamma}|(A) + \tilde{\gamma}(A)), \quad \mu^-(A) = \frac{1}{2}(|\tilde{\gamma}|(A) - \tilde{\gamma}(A)).$$

Then $\mu^+$ and $\mu^-$ become non-negative, finitely additive functions defined on $F$ [1, Theorem III.1.8]. Moreover, they are invariant under $G$ by condition (iii). It then follows from Theorem 2 that there exist non-negative constants $c_1$ and $c_2$ such that $\mu^+(A) = c_1\delta(A)$ and $\mu^-(A) = c_2\delta(A)$ for any $A \in F$. Therefore we have

$$\gamma(A) = \tilde{\gamma}(A) = \mu^+(A) - \mu^-(A) = c\delta(A), \quad A \in F,$$

where $c = c_1 - c_2 = \gamma(N)$. Q.E.D.
APPENDIX: REARRANGEMENT OF UNIFORMLY DISTRIBUTED SEQUENCES

We begin with a characterization of the Lévy group. Let $l^\infty$ be the Banach space of all bounded real sequences $a = (a_n)_{n=1}^{\infty}$ equipped with the norm $\|a\|_{\infty} = \sup |a_n|$. Put

$$L^+(a) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n, \quad L^-(a) = \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n,$$

where $a = (a_n)_{n=1}^{\infty} \in l^\infty$. Obviously, $L^+(-a) = -L^-(a)$. The group $\text{aut}(\mathbb{N})$ acts on $l^\infty$ as a coordinate permutation, i.e., by means of the maps:

$$a = (a_n)_{n=1}^{\infty} \mapsto ga = (a_{g^{-1}(n)})_{n=1}^{\infty}, \quad g \in \text{aut}(\mathbb{N}).$$

The following result was proved in [11].

**Proposition A.1.** The Lévy group is the maximal permutation group which keeps $L^+$ (or $L^-$) invariant.

Let $\mathcal{D}$ be the space of all $a \in l^\infty$ such that $L^+(a) = L^-(a)$. Then $\mathcal{D}$ becomes a closed subspace of $l^\infty$ and the functional

$$L(a) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n, \quad a = (a_n)_{n=1}^{\infty} \in \mathcal{D},$$

is continuous and linear. We denote by $\mathcal{G}(L, \mathcal{D})$ the group of all permutations which leave $L$ invariant:

$$\mathcal{G}(L, \mathcal{D}) = \{ g \in \text{aut}(\mathbb{N}); g \mathcal{D} = \mathcal{D}, L(ga) = L(a) \text{ for all } a \in \mathcal{D} \}.$$

The next result was also proved in [11].

**Proposition A.2.** $\mathcal{G} \subset \mathcal{G}(L, \mathcal{D}) \subset \mathcal{G}(\delta)$.

A sequence $(x_n)_{n=1}^{\infty}$, $0 \leq x_n < 1$, is called uniformly distributed on the interval $[0, 1)$ if

$$\lim_{N \to \infty} \frac{1}{N} |\{ 1 \leq n \leq N; a \leq x_n < b \}| = b - a$$

for any pair $a, b$ of real numbers with $0 \leq a < b \leq 1$. Obviously, this property depends upon the arrangement of the sequence.

**Proposition A.3.** Assume that $x = (x_n)_{n=1}^{\infty}$ is uniformly distributed on $[0, 1)$. Then, for any $g \in \mathcal{G}$, the rearranged sequence $gx$ is also uniformly distributed on $[0, 1)$. 

Proof. Let \( f \) be a real-valued continuous function defined on the interval \([0, 1]\) and put
\[
a = (a_n)_{n=1}^{\infty}, \quad a_n = f(x_n).
\]
By virtue of Weyl's theorem \([15]\), we see that \( a \in \mathcal{D} \) and that
\[
L(a) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{0}^{1} f(x) \, dx.
\]
It follows from Proposition A.2 that
\[
L(ga) = L(a) = \int_{0}^{1} f(x) \, dx,
\]
for every \( g \in \mathcal{G} \). Hence, using Weyl's theorem again, we conclude that the rearranged sequence \( g x \) is uniformly distributed. Q.E.D.

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