A variant of Jensen’s inequality of Mercer’s type for operators with applications

A. Matković a, J. Pečarić b,*, I. Perić c

a Department of Mathematics, Faculty of Natural Sciences, Mathematics and Education, University of Split, Teslina 12, 21000 Split, Croatia
b Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
c Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

Received 24 October 2005; accepted 28 February 2006
Available online 2 May 2006
Submitted by R.A. Brualdi

Abstract

A variant of Jensen’s operator inequality for convex functions, which is a generalization of Mercer’s result, is proved. Obtained result is used to prove a monotonicity property for Mercer’s power means for operators, and a comparison theorem for quasi-arithmetic means for operators.
© 2006 Elsevier Inc. All rights reserved.

AMS classification: 47A63; 47A64

Keywords: Jensen’s operator inequality; Monotonicity; Power means; Quasi-arithmetic means

1. Introduction

For a given \( a < b \), let \( x = (x_1, \ldots, x_k) \) be such that \( a \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq b \) and \( w = (w_1, \ldots, w_k) \) be nonnegative weights such that \( \sum_{j=1}^{k} w_j = 1 \). Mercer [3] proved the following variant of Jensen’s inequality.

* Corresponding author.
E-mail addresses: anita@pmfst.hr (A. Matković), pecaric@hazu.hr (J. Pečarić), iperic@pb.hr (I. Perić).
Theorem A. If \( f \) is a convex function on \([a, b]\) then
\[
f \left( a + b - \sum_{j=1}^{k} w_j x_j \right) \leq f(a) + f(b) - \sum_{j=1}^{k} w_j f(x_j).
\]
For \( a > 0 \) the (weighted) power means \( M_r(x, w) \) are defined as
\[
M_r(x, w) = \begin{cases} 
\left( \frac{1}{k} \sum_{j=1}^{k} w_j x_j^r \right)^{\frac{1}{r}}, & r \neq 0, \\
\exp \left( \frac{1}{k} \sum_{j=1}^{k} w_j \ln x_j \right), & r = 0.
\end{cases}
\]
In [4] Mercer defined the family of functions
\[
Q_r(a, b, x) = \begin{cases} 
[a^r + b^r - M_r(x, w)]^{\frac{1}{r}}, & r \neq 0, \\
\frac{ab}{M_0(x, w)}, & r = 0
\end{cases}
\]
and proved the following.

Theorem B. For \( r < s \), \( Q_r(a, b, x) \leq Q_s(a, b, x) \).

In this paper we consider similar inequalities in a more general setting. To do this we need some well known results. The first one is Löwner–Heinz inequality (see for example [5, p. 9]).

Theorem C. Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \). If \( A \succeq B \), then \( A^p \succeq B^p \) for all \( p \in [0, 1] \).

In [5, p. 220, 232, 250] the following theorems are also proved.

Theorem D. Let \( A, B \) be positive operators on a Hilbert space \( H \) with \( \text{Sp}(A) \subseteq [m_1, M_1] \), and \( \text{Sp}(B) \subseteq [m_2, M_2] \) for some scalars \( M_j > m_j > 0 \) (\( j = 1, 2 \)). If \( A \succeq B \), then the following inequalities hold:

(i) for all \( p > 1 \):
\[
K(m_1, M_1, p) A^p \succeq B^p,
\]
\[
K(m_2, M_2, p) A^p \succeq B^p,
\]
(ii) for all \( p < -1 \):
\[
K(m_1, M_1, p) B^p \succeq A^p,
\]
\[
K(m_2, M_2, p) B^p \succeq A^p,
\]
where a generalized Kantorovich constant \( K(m, M, p) \) is defined by
\[
K(m, M, p) = \frac{(mM^p - Mm^p)(p - 1)}{(p - 1)(M - m)} \left( \frac{M^p - m^p}{Mm^p - Mm^p} \right)^p
\]
for all \( p \in \mathbb{R} \).
Theorem E. Let $A, B$ be selfadjoint operators on a Hilbert space $H$ with $\text{Sp}(B) \subseteq [m, M]$ for some scalars $M > m$. If $A \geq B$, then

$$S(e^{M-m})e^A \geq e^B,$$

where the Specht ratio $S(h)$ for $h > 0$ is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{e \ln h}}}{h-1}(h \neq 1) \text{ and } S(1) = 1.$$

In Section 2 we give the main result of our paper which is an extension of Theorem A to selfadjoint operators and positive linear maps. This variant of Jensen’s inequality for operators holds for arbitrary convex functions, while Davis–Choi–Jensen’s inequality asserts that

$$f(\Phi(A)) \leq \Phi(f(A))$$

holds for an operator convex function $f$ defined on an interval $(-a, a)$, where $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a normalized positive linear map and $A$ is a selfadjoint operator with spectrum in $(-a, a)$ (see [1,2]).

In Section 3 we use that result to prove a monotonicity property of power means of Mercer’s type for operators. In the final section we consider related quasi-arithmetic means for operators.

2. Main result

In what follows we assume that $H$ and $K$ are Hilbert spaces, $\mathcal{B}(H)$ and $\mathcal{B}(K)$ are $C^*$-algebras of all bounded operators on the appropriate Hilbert space and $\mathcal{P}[\mathcal{B}(H), \mathcal{B}(K)]$ is the set of all positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. We denote by $C([m, M])$ the set of all real valued continuous functions on an interval $[m, M]$.

We show a variant of Jensen’s operator inequality which is an extension of Theorem A to selfadjoint operators and positive linear maps.

Theorem 1. Let $A_1, \ldots, A_k \in \mathcal{B}(H)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, \ldots, \Phi_k \in \mathcal{P}[\mathcal{B}(H), \mathcal{B}(K)]$ positive linear maps with $\sum_{j=1}^k \Phi_j(1_H) = 1_K$. If $f \in C([m, M])$ is convex on $[m, M]$, then

$$f \left( m 1_K + M 1_K - \sum_{j=1}^k \Phi_j(A_j) \right) \leq f(m)1_K + f(M)1_K - \sum_{j=1}^k \Phi_j(f(A_j)). \quad (1)$$

In fact, to be more specific, the following series of inequalities holds

$$f \left( m 1_K + M 1_K - \sum_{j=1}^k \Phi_j(A_j) \right) \leq \frac{M 1_K - \sum_{j=1}^k \Phi_j(A_j)}{M - m} \cdot f(M)$$

$$+ \frac{\sum_{j=1}^k \Phi_j(A_j) - m 1_K}{M - m} \cdot f(m)$$

$$\leq f(m)1_K + f(M)1_K - \sum_{j=1}^k \Phi_j(f(A_j)). \quad (2)$$

If a function $f$ is concave, then inequalities (1) and (2) are reversed.
Proof. Since \( f \) is continuous and convex, the same is also true for the function \( g : [m, M] \to \mathbb{R} \) defined by \( g(t) = f(m + M - t), \ t \in [m, M] \). Hence, the following inequalities hold for every \( t \in [m, M] \) (see for example [6, p. 2]):

\[
\begin{align*}
    f(t) &\leq \frac{t - m}{M - m} \cdot f(M) + \frac{M - t}{M - m} \cdot f(m), \\
g(t) &\leq \frac{t - m}{M - m} \cdot g(M) + \frac{M - t}{M - m} \cdot g(m).
\end{align*}
\]

Since \( m1_H \leq A_j \leq M1_H \) for \( j = 1, \ldots, k \) and \( \sum_{j=1}^k \Phi_j(1_H) = 1_K \), it follows that \( m1_K \leq \sum_{j=1}^k \Phi_j(A_j) \leq M1_K \). Now, using the functional calculus we have

\[
g \left( \sum_{j=1}^k \Phi_j(A_j) \right) \leq \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M - m} \cdot g(M) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M - m} \cdot g(m)
\]

or

\[
f \left( m1_K + M1_K - \sum_{j=1}^k \Phi_j(A_j) \right) \leq \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M - m} \cdot f(m) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M - m} \cdot f(M)
\]

\[= f(m)1_K + f(M)1_K - \left[ \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M - m} \cdot f(m) + \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M - m} \cdot f(M) \right]. \quad (3)
\]

On the other hand, using the functional calculus we also have

\[
f(A_j) \leq \frac{A_j - m1_H}{M - m} \cdot f(M) + \frac{M1_H - A_j}{M - m} \cdot f(m).
\]

Applying positive linear maps \( \Phi_j \) and summing, it follows that

\[
\sum_{j=1}^k \Phi_j(f(A_j)) \leq \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M - m} \cdot f(m) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M - m} \cdot f(M). \quad (4)
\]

Using inequalities (3) and (4), we obtain desired inequalities (1) and (2).

The last statement follows immediately from the fact that if \( \varphi \) is concave then \( -\varphi \) is convex. \( \square \)

3. Applications to Mercer’s power means

We suppose that:

(i) \( A = (A_1, \ldots, A_k) \), where \( A_j \in \mathcal{B}(H) \) are positive invertible operators with \( \text{Sp}(A_j) \subseteq [m, M] \) for some scalars \( 0 < m < M \).

(ii) \( \Phi = (\Phi_1, \ldots, \Phi_k) \), where \( \Phi_j \in \mathcal{P} [\mathcal{B}(H), \mathcal{B}(K)] \) are positive linear maps with \( \sum_{j=1}^k \Phi_j(1_H) = 1_K \).
(iii) \( \Delta(m, M, p) = K(m^p, M^p, \frac{1}{p}) = \frac{p(m^p M^p - m^p m)}{(1-p)(m^p - m^p m)} \left( \frac{(1-p)(M-m)}{m^p M^p - m^p m} \right)^{1/p} \), for \( 0 < m < M \) and \( p \in \mathbb{R}, \ p \neq 0 \). Set: \( \Delta(m, M, 0) = \lim_{p \to 0} \Delta(m, M, p) = S(M) = \frac{M-m}{\ln M - \ln m} \exp \left( \frac{m(1+\ln M) - M(1+\ln m)}{M-m} \right) \).

We define, for any \( r \in \mathbb{R} \)

\[
\tilde{M}_r(A, \Phi) := \begin{cases} 
[m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)]^{\frac{1}{r}}, & r \neq 0, \\
\exp \left( (\ln m) 1_K + (\ln M) 1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j)) \right), & r = 0.
\end{cases}
\]

Observe that, since \( 0 < m 1_H \leq A_j \leq M 1_H \), it follows that:

- \( 0 < m^r 1_H \leq A_j^r \leq M^r 1_H \) holds for all \( r > 0 \),
- \( 0 < M^r 1_H \leq A_j^r \leq m^r 1_H \) holds for all \( r < 0 \),
- \( (\ln m) 1_H \leq \ln(A_j) \leq (\ln M) 1_H \ (j = 1, \ldots, k) \).

Applying positive linear maps \( \Phi_j \) and summing, it follows that:

- \( 0 < m^r 1_K \leq \sum_{j=1}^{k} \Phi_j(A_j^r) \leq M^r 1_K \), for all \( r > 0 \),
- \( 0 < M^r 1_K \leq \sum_{j=1}^{k} \Phi_j(A_j^r) \leq m^r 1_K \), for all \( r < 0 \),
- \( (\ln m) 1_K \leq \sum_{j=1}^{k} \Phi_j(\ln(A_j)) \leq (\ln M) 1_K \),

since \( \sum_{j=1}^{k} \Phi_j(1_H) = 1_K \). Hence, \( \tilde{M}_r(A, \Phi) \) is well defined.

Furthermore, we define, for any \( r, s \in \mathbb{R} \)

\[
S(r, s, A, \Phi) := \begin{cases} 
\left[ \frac{M^r 1_K - S_r}{M^r - m^r} \cdot \frac{S_r - m^r 1_K}{M^r - m^r} \cdot M^s \right]^{\frac{1}{r}}, & r \neq 0, s \neq 0, \\
\exp \left( \frac{M^r 1_K - S_r}{M^r - m^r} \cdot \ln M + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot \ln m \right), & r \neq 0, s = 0, \\
\left[ \frac{(\ln M) 1_K - S_0}{\ln M - \ln m} \cdot M^s + \frac{S_0 - (\ln m) 1}{\ln M - \ln m} \cdot M^s \right]^{\frac{1}{r}}, & r = 0, s \neq 0,
\end{cases}
\]

where \( S_r = \sum_{j=1}^{k} \Phi_j(A_j^r) \) and \( S_0 = \sum_{j=1}^{k} \Phi_j(\ln(A_j)) \). It is easy to see that \( S(r, s, A, \Phi) \) is also well defined.

**Theorem 2.** Let \( r, s \in \mathbb{R}, \ r < s \).

(i) If either \( r \leq -1 \) or \( s \geq 1 \),

\[
\tilde{M}_r(A, \Phi) \leq \tilde{M}_s(A, \Phi).
\]

(ii) If \(-1 < r \) and \( s < 1 \),

\[
\tilde{M}_r(A, \Phi) \leq \Delta(m, M, s) \cdot \tilde{M}_s(A, \Phi).
\]

**Proof.** (i) **Step 1**: Suppose that \( 0 < r < s \) and \( s \geq 1 \).

Applying the inequality (1) to the convex function \( f(t) = t^\frac{1}{r} \) (note that \( \frac{1}{r} > 1 \) here) and replacing \( A_j, m \) and \( M \) with \( A_j^r, m^r \) and \( M^r \), respectively, we have
\[
\left[ m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j (A^r_j) \right]^{\frac{1}{s}} \leq m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j (A^s_j). \tag{5}
\]

Raising both sides to the power \( \frac{1}{s} \) (0 < \( \frac{1}{s} \) ≤ 1), it follows from Theorem C that
\[
\widetilde{M}_r(\mathbf{A}, \Phi) \leq \widetilde{M}_s(\mathbf{A}, \Phi).
\]

**Step 2:** Suppose that \( r < 0 \) and \( s \geq 1 \).
Applying the inequality (1) to the convex function \( f(t) = t^s \) (note that \( \frac{s}{r} < 0 \) here) and proceeding in the same way as in **Step 1**, we have
\[
\widetilde{M}_r(\mathbf{A}, \Phi) \leq \widetilde{M}_s(\mathbf{A}, \Phi).
\]

**Step 3:** Suppose that \( r = 0 \) and \( s \geq 1 \).
Applying the inequality (1) to the convex function \( f(t) = t^s \cdot t \) and replacing \( A_j, m \) and \( M \) with \( \ln(A_j), \ln m \) and \( \ln M \), respectively, we have
\[
\exp \left( s \left( (\ln m) 1_K + (\ln M) 1_K - \sum_{j=1}^{k} \Phi_j (\ln(A_j)) \right) \right)
\leq \exp(s \ln m) 1_K + \exp(s \ln M) 1_K - \sum_{j=1}^{k} \Phi_j (\exp(s \ln(A_j)))
\]
\[
= m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j (A^s_j)
\tag{6}
\]
or
\[
[\widetilde{M}_0(\mathbf{A}, \Phi)]^s \leq [\widetilde{M}_s(\mathbf{A}, \Phi)]^s.
\]
Raising both sides to the power \( \frac{1}{s} \) (0 < \( \frac{1}{s} \) ≤ 1), it follows from Theorem C that
\[
\widetilde{M}_0(\mathbf{A}, \Phi) \leq \widetilde{M}_s(\mathbf{A}, \Phi).
\]

**Step 4:** Suppose that \( r < s < 0 \) and \( r \leq -1 \).
Applying the inequality (1) to the convex function \( f(t) = t^\frac{r}{s} \) (note that \( \frac{r}{s} > 1 \) here) and replacing \( A_j, m \) and \( M \) with \( A^s_j, m^s \) and \( M^s \), respectively, we have
\[
\left[ m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j (A^s_j) \right]^{\frac{r}{s}} \leq m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j (A^r_j). \tag{7}
\]
Raising both sides to the power \(-\frac{1}{r} \) (0 < \(-\frac{1}{r} \) ≤ 1), it follows from Theorem C that
\[
[\widetilde{M}_s(\mathbf{A}, \Phi)]^{-1} \leq [\widetilde{M}_r(\mathbf{A}, \Phi)]^{-1}.
\]
Hence, we have
\[
\widetilde{M}_r(\mathbf{A}, \Phi) \leq \widetilde{M}_s(\mathbf{A}, \Phi).
\]

**Step 5:** Suppose that \( s > 0 \) and \( r \leq -1 \).
Applying the inequality (1) to the convex function \( f(t) = t^\frac{r}{s} \) (note that \( \frac{r}{s} < 0 \) here) and proceeding in the same way as in Step 4, we have
\[
\tilde{M}_r(A, \Phi) \leq \tilde{M}_s(A, \Phi).
\]

**Step 6:** Suppose that \( s = 0 \) and \( r \leq -1 \).
Applying the inequality (1) to the convex function \( f(t) = \exp(r \cdot t) \) and replacing \( A_j, m \) and \( M \) with \( \ln(A_j) \), \( \ln m \) and \( \ln M \), respectively, we have
\[
\exp \left( r \left( \ln m \right)_K + \left( \ln M \right)_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \right)
\leq \exp(r \ln m)_K + \exp(r \ln M)_K - \sum_{j=1}^k \Phi_j(\exp(r \ln(A_j)))

= m^r \left( \ln m \right)_K + M^r \left( \ln M \right)_K - \sum_{j=1}^k \Phi_j(A_j^r) \tag{8}
\]
or
\[
[\tilde{M}_0(A, \Phi)]^{-1} \leq [\tilde{M}_r(A, \Phi)]^{-1}.
\]
Raising both sides to the power \(-\frac{1}{r} \) (\( 0 < \frac{1}{r} \leq 1 \)), it follows from Theorem C that
\[
[\tilde{M}_0(A, \Phi)]^{-1} \leq [\tilde{M}_r(A, \Phi)]^{-1}.
\]
Hence, we have
\[
\tilde{M}_r(A, \Phi) \leq \tilde{M}_0(A, \Phi).
\]

(ii) **Step 1:** Suppose that \( 0 < r < s < 1 \).
In the same way as in (i) Step 1 we obtain inequality (5). Observe that, since \( m^s \left( \ln m \right)_K \leq \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s \left( \ln M \right)_K \), it follows that \( m^s \left( \ln m \right)_K \leq m^s \left( \ln m \right)_K + M^s \left( \ln M \right)_K - \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s \left( \ln M \right)_K \). Raising both sides of (5) to the power \( \frac{1}{s} \) (\( \frac{1}{s} > 1 \)), it follows from Theorem D (i) that
\[
\tilde{M}_r(A, \Phi) \leq K \left( m^s, M^s, \frac{1}{s} \right) \tilde{M}_s(A, \Phi).
\]

**Step 2:** Suppose that \( 0 < r < s < 1 \).
In the same way as in (i) Step 3 we obtain inequality (6). With the same observation as in (ii) Step 1 and raising both sides of (6) to the power \( \frac{1}{s} \) (\( \frac{1}{s} > 1 \)), it follows from Theorem D (i) that
\[
\tilde{M}_0(A, \Phi) \leq K \left( m^s, M^s, \frac{1}{s} \right) \tilde{M}_s(A, \Phi).
\]

**Step 3:** Suppose that \( -1 < r < s < 0 \).
Applying reversed inequality (1) to the concave function \( f(t) = t^\frac{r}{s} \) (note that \( 0 < \frac{r}{s} < 1 \) here) and replacing \( A_j, m \) and \( M \) with \( A_j^r, m^r \) and \( M^r \), respectively, we obtain reversed inequality (5). Observe that, since \( M^s \left( \ln M \right)_K \leq \sum_{j=1}^k \Phi_j(A_j^s) \leq m^s \left( \ln m \right)_K \), it follows that \( M^s \left( \ln M \right)_K \leq M^s \left( \ln M \right)_K + M^s \left( \ln M \right)_K - \sum_{j=1}^k \Phi_j(A_j^s) \leq m^s \left( \ln m \right)_K \). Raising both sides of reversed (5) to the power \( \frac{1}{s} \) (\( \frac{1}{s} < -1 \)), it follows from Theorem D (ii) that
Since \( K(M, m, p) = K(m, M, p) \) (see [5, p. 77]), we have
\[ \tilde{M}_r(A, \Phi) \leq K \left( M^s, m^s, \frac{1}{s} \right) \tilde{M}_s(A, \Phi). \]

**Step 4:** Suppose that \(-1 < r < s = 0\).
Applying the inequality (1) to the convex function \( f(t) = \frac{1}{r} \ln t \) and replacing \( A_j, m \) and \( M \) with \( A'_j, M' \) and \( m' \), respectively, we obtain
\[
\frac{1}{r} \ln \left( m'^r 1_K + M'^r 1_K - \sum_{j=1}^{k} \Phi_j(A'_j) \right) \leq (\ln m) 1_K + (\ln M) 1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j)).
\]
Observing that both sides have spectra in \([\ln m, \ln M]\), it follows from Theorem E that
\[ \tilde{M}_r(A, \Phi) \leq \tilde{M}_0(A, \Phi). \]

**Step 5:** Suppose that \(-1 < r < 0 < s < 1\).
In the same way as in (i) Step 2 we obtain inequality (5). With the same observation as in (ii) Step 1 and raising both sides of (5) to the power \( \frac{1}{s} \left( \frac{1}{s} > 1 \right) \), it follows from Theorem D (i) that
\[ \tilde{M}_r(A, \Phi) \leq K \left( m^s, M^s, \frac{1}{s} \right) \tilde{M}_s(A, \Phi). \]

If we use inequalities (2) instead of the inequality (1), then we have the following results:

**Theorem 3.** Let \( r, s \in \mathbb{R}, r < s \).

(i) If \( s \geq 1 \), then
\[ \tilde{M}_r(A, \Phi) \leq S(r, s, A, \Phi) \leq \tilde{M}_s(A, \Phi). \]
If \( r \leq -1 \), then
\[ \tilde{M}_r(A, \Phi) \leq S(s, r, A, \Phi) \leq \tilde{M}_s(A, \Phi). \]

(ii) If \(-1 < r < s < 1\), then
\[
\frac{1}{\Delta(m, M, s)} \cdot \tilde{M}_r(A, \Phi) \leq S(r, s, A, \Phi) \leq \Delta(m, M, s) \cdot \tilde{M}_s(A, \Phi).
\]

**Proof.** (i) Step 1: Suppose that \( 0 < r < s \) and \( s \geq 1 \).
Applying inequalities (2) to the convex function \( f(t) = t^\frac{r}{s} \) (note that \( \frac{r}{s} \geq 1 \) here) and replacing \( A_j, m \) and \( M \) with \( A'_j, m' \) and \( M' \), respectively, we have
\[
\left[ m'^r 1_K + M'^r 1_K - \sum_{j=1}^{k} \Phi_j(A'_j) \right]^\frac{1}{s} \leq \frac{M'^r 1_K - S_r}{M'^r - m'} \cdot M^s + \frac{S_r - m'^r 1_K}{M'^r - m'} \cdot m^s
\]
\[
\leq m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s).
\]

Raising these inequalities to the power $\frac{1}{s} (0 < \frac{1}{s} \leq 1)$, it follows from Theorem C that

$$\tilde{M}_r(\mathbf{A}, \Phi) \leq S(r, s, \mathbf{A}, \Phi) \leq \tilde{M}_s(\mathbf{A}, \Phi).$$

**Step 2:** Suppose that $r < 0$ and $s \geq 1$.

Applying inequalities (2) to the convex function $f(t) = t^\frac{r}{s}$ (note that $\frac{r}{s} < 0$ here) and proceeding in the same way as in **Step 1**, we have

$$\tilde{M}_r(\mathbf{A}, \Phi) \leq S(r, s, \mathbf{A}, \Phi) \leq \tilde{M}_s(\mathbf{A}, \Phi).$$

**Step 3:** Suppose that $r = 0$ and $s \geq 1$.

Applying inequalities (2) to the convex function $f(t) = \exp(s \cdot t)$ and replacing $A_j$, $m$ and $M$ with $\ln(A_j)$, $\ln m$ and $\ln M$, respectively, we have

$$\exp \left( s \left( (\ln m) 1_K + (\ln M) 1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j)) \right) \right) \leq \frac{(\ln M) 1_K - S_0}{\ln M - \ln m} \cdot \exp(s \ln M) + \frac{S_0 - (\ln m) 1_K}{\ln M - \ln m} \cdot \exp(s \ln m) \leq \exp(s \ln m) 1_K + \exp(s \ln M) 1_K - \sum_{j=1}^{k} \Phi_j(\exp(s \ln(A_j)))$$

$$= m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s)$$

or

$$[\tilde{M}_0(\mathbf{A}, \Phi)]^s \leq [S(0, s, \mathbf{A}, \Phi)]^s \leq [\tilde{M}_s(\mathbf{A}, \Phi)]^s.$$

Raising these inequalities to the power $\frac{1}{s} (0 < \frac{1}{s} \leq 1)$, it follows from Theorem C that

$$\tilde{M}_0(\mathbf{A}, \Phi) \leq S(0, s, \mathbf{A}, \Phi) \leq \tilde{M}_s(\mathbf{A}, \Phi).$$

**Step 4:** Suppose that $r < s < 0$ and $r \leq -1$.

Applying inequalities (2) to the convex function $f(t) = t^\frac{r}{s}$ (note that $\frac{r}{s} \geq 1$ here) and replacing $A_j$, $m$ and $M$ with $A_j^s$, $m^s$ and $M^s$, respectively, we have

$$\left[ m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s) \right]^{\frac{r}{s}} \leq \frac{M^s 1_K - S_r}{M^s - m^s} \cdot Mr + \frac{S_r - m^s 1_K}{M^s - m^s} \cdot Mr \leq Mr 1_K + Mr 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s).$$

Raising these inequalities to the power $-\frac{1}{r} (0 < -\frac{1}{r} \leq 1)$, it follows from Theorem C that

$$[\tilde{M}_s(\mathbf{A}, \Phi)]^{-1} \leq [S(s, r, \mathbf{A}, \Phi)]^{-1} \leq [\tilde{M}_r(\mathbf{A}, \Phi)]^{-1}.$$

Hence, we have

$$\tilde{M}_r(\mathbf{A}, \Phi) \leq S(s, r, \mathbf{A}, \Phi) \leq \tilde{M}_s(\mathbf{A}, \Phi).$$
Step 5: Suppose that \( s > 0 \) and \( r \leq -1 \).
Applying inequalities (2) to the convex function \( f(t) = t^\frac{r}{s} \) (note that \( \frac{r}{s} < 0 \) here) and proceeding in the same way as in Step 4, we have
\[
\tilde{M}_r(A, \Phi) \leq S(s, r, A, \Phi) \leq \tilde{M}_s(A, \Phi).
\]

Step 6: Suppose that \( s = 0 \) and \( r \leq -1 \).
Applying inequalities (2) to the convex function \( f(t) = \exp(r \cdot t) \) and replacing \( A_j, m \) and \( M \) with \( \ln(A_j), \ln m \) and \( \ln M \), respectively, we have
\[
\begin{aligned}
\exp \left( r \left( (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \right) \right) \\
&\leq \frac{(\ln M)1_K - S_0}{\ln M - \ln m} \cdot \exp(r \ln M) + \frac{S_0 - (\ln m)1_K}{\ln M - \ln m} \cdot \exp(r \ln m) \\
&\leq \exp(r \ln m)1_K + \exp(r \ln M)1_K - \sum_{j=1}^k \Phi_j(\exp(r \ln(A_j))) \\
&= m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r)
\end{aligned}
\]
or
\[
[\tilde{M}_0(A, \Phi)]^r \leq [S(0, r, A, \Phi)]^r \leq [\tilde{M}_r(A, \Phi)]^r.
\]
Raising these inequalities to the power \( -\frac{1}{r} \) (\( 0 < \frac{1}{r} \leq 1 \)), it follows from Theorem C that
\[
[\tilde{M}_0(A, \Phi)]^{-1} \leq [S(0, r, A, \Phi)]^{-1} \leq [\tilde{M}_r(A, \Phi)]^{-1}.
\]
Hence, we have
\[
\tilde{M}_r(A, \Phi) \leq S(0, r, A, \Phi) \leq \tilde{M}_0(A, \Phi).
\]

(ii) Step 1: Suppose that \( 0 < r < s < 1 \).
In the same way as in (i) Step 1 we obtain inequalities (9). Observe that, since \( m^r 1_K \leq \sum_{j=1}^k \Phi_j(A_j^r) \leq M^r 1_K \) and \( m^s 1_K \leq \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s 1_K \), it follows that \( m^s 1_K \leq [m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r)]^\frac{1}{r} \leq M^s 1_K \) and \( m^s 1_K \leq m^s 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s 1_K \).
Raising inequalities (9) to the power \( \frac{1}{s} \) (\( \frac{1}{s} > 1 \)), it follows from Theorem D (i) that
\[
\begin{aligned}
K \left( m^s, M^s, \frac{1}{s} \right)^{-1} &\left[ m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r) \right]^{\frac{1}{r}} \\
&\leq \left[ \frac{M^r 1_K - S_r}{M^r - m^r} \cdot M^s + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot m^s \right]^{\frac{1}{s}} \\
&\leq K \left( m^s, M^s, \frac{1}{s} \right) \left[ m^s 1_K + M^s 1_K - \sum_{j=1}^k \Phi_j(A_j^s) \right]^{\frac{1}{s}},
\end{aligned}
\]
or
\[ \Delta(m, M, s)^{-1} \widetilde{M}_r(A, \Phi) \leq S(r, s, A, \Phi) \leq \Delta(m, M, s) \widetilde{M}_s(A, \Phi). \]

**Step 2:** Suppose that \( 0 = r < s < 1 \).

In the same way as in (i) **Step 3** we obtain inequalities (10). Observe that, since \((\ln m)1_K \leq (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \leq (\ln M)1_K\) and \(m^s1_K \leq \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s1_K\), it follows that
\[
m^s1_K \leq \exp \left( s \left( (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \right) \right) \leq M^s1_K
\]
and \(m^s1_K \leq m^s1_K + M^s1_K - \sum_{j=1}^k \Phi_j(A_j^s) \leq M^s1_K\). Raising inequalities (10) to the power \(\frac{1}{s}(\frac{1}{s} > 1)\), it follows from Theorem D (i) that
\[ \Delta(m, M, s)^{-1} \widetilde{M}_0(A, \Phi) \leq S(0, s, A, \Phi) \leq \Delta(m, M, s) \widetilde{M}_s(A, \Phi). \]

**Step 3:** Suppose that \(-1 < r < s < 0\).

Applying reversed inequalities (2) to the concave function \(f(t) = t^\frac{1}{r}\) (note that \(0 < \frac{s}{r} < 1\) here) and replacing \(A_j, m\) and \(M\) with \(A_j^r, m^r\) and \(M^r\), respectively, we obtain reversed (9). With the same observation as in **Step 1** and raising reversed (9) to the power \(\frac{1}{s}(\frac{1}{s} < -1)\), it follows from Theorem D (ii) that
\[ \Delta(m, M, s)^{-1} \widetilde{M}_r(A, \Phi) \leq S(r, s, A, \Phi) \leq \Delta(m, M, s) \widetilde{M}_s(A, \Phi). \]

**Step 4:** Suppose that \(-1 < r < s = 0\).

Applying inequalities (2) to the convex function \(f(t) = \frac{1}{r} \ln t\) (note that \(\frac{1}{r} < 0\) here) and replacing \(A_j, m\) and \(M\) with \(A_j^r, m^r\) and \(M^r\), respectively, we obtain
\[
\frac{1}{r} \ln \left( m^r1_K + M^r1_K - \sum_{j=1}^k \Phi_j(A_j^r) \right) \\
\leq \frac{M^r1_K - S_r}{M^r - m^r} \cdot \ln M + \frac{S_r - m^r}{M^r - m^r} \cdot \ln m \\
\leq (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)).
\]

Observe that, since \(r < 0, M^r1_K \leq m^r1_K + M^r1_K - \sum_{j=1}^k \Phi_j(A_j^r) \leq m^r1_K\) and \((\ln m)1_K \leq \sum_{j=1}^k \Phi_j(\ln(A_j)) \leq (\ln M)1_K\), it follows that
\[
\ln m \leq \frac{1}{r} \ln \left( m^r1_K + M^r1_K - \sum_{j=1}^k \Phi_j(A_j^r) \right) \leq \ln M
\]
and \((\ln m)1_K \leq (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \leq (\ln M)1_K\). Now, it follows from Theorem E that
\[ S(e^{\ln M - \ln m})^{-1} \widetilde{M}_r(A, \Phi) \leq S(r, 0, A, \Phi) \leq S(e^{\ln M - \ln m}) \widetilde{M}_0(A, \Phi). \]
Step 5: Suppose that \(-1 < r < 0 < s < 1\).
Applying inequalities (2) to the convex function \(f(t) = t^r\) (note that \(t_r < 0\) here) and replacing \(A_j, m\) and \(M\) with \(A_j', m'\) and \(M'\), respectively, we obtain inequalities (9). Proceeding in the same way as in Step 1, we have
\[
\Delta(m, M, s)^{-1} \tilde{M}_r(A, \Phi) \leq S(r, s, A, \Phi) \leq \Delta(m, M, s) \tilde{M}_s(A, \Phi). \quad \square
\]

Remark 1. Some considerations in Theorems 2 and 3 can be shortened using obvious properties \(\tilde{M}_s(A^{-1}, \Phi) = \tilde{M}_s(A, \Phi)^{-1}\) and \(S(-s, -r, A^{-1}, \Phi) = S(s, r, A, \Phi)^{-1}\), where \(A^{-1} = (A_1^{-1}, \ldots, A_k^{-1})\).

Remark 2. Since obviously \(S(r, r, A, \Phi) = \tilde{M}_r(A, \Phi)\), inequalities in Theorem 3 (i) give us
\[
S(r, r, A, \Phi) \leq S(r, s, A, \Phi) \leq S(s, s, A, \Phi), r < s, s \geq 1
\]
and
\[
S(r, r, A, \Phi) \leq S(s, r, A, \Phi) \leq S(s, s, A, \Phi), r < s, r \leq -1.
\]
An open problem is to give the list of inequalities comparing “mixed means” \(S(r, s, A, \Phi)\) in remaining cases.

4. Quasi-arithmetic means of Mercer’s type

Let \(A\) and \(\Phi\) be as in the previous section. Let \(\varphi, \psi \in C([m, M])\) be strictly monotonic functions on an interval \([m, M]\). We define
\[
\tilde{M}_\varphi(A, \Phi) = \varphi^{-1}\left(\varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))\right).
\]
Observe that, since \(m1_H \leq A_j \leq M1_H\), it follows that
- \(\varphi(m)1_H \leq \varphi(A_j) \leq \varphi(M)1_H\) if \(\varphi\) is increasing,
- \(\varphi(M)1_H \leq \varphi(A_j) \leq \varphi(m)1_H\) if \(\varphi\) is decreasing.

Applying positive linear maps \(\Phi_j\) and summing, it follows that
- \(\varphi(m)1_K \leq \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \leq \varphi(M)1_K\) if \(\varphi\) is increasing,
- \(\varphi(M)1_K \leq \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \leq \varphi(m)1_K\) if \(\varphi\) is decreasing,

since \(\sum_{j=1}^{k} \Phi_j(1_H) = 1_K\). Hence, \(\tilde{M}_\varphi(A, \Phi)\) is well defined.

A function \(f \in C([m, M])\) is said to be operator increasing if \(f\) is operator monotone, i.e., if \(A \leq B\) implies \(f(A) \leq f(B)\), for all selfadjoint operators \(A\) and \(B\) on a Hilbert space \(H\) with \(Sp(A), Sp(B) \subseteq [m, M]\). A function \(f \in C([m, M])\) is said to be operator decreasing if \(-f\) is operator monotone.
Theorem 4. Under the above hypotheses, we have

(i) if either \( \psi \circ \varphi^{-1} \) is convex and \( \psi^{-1} \) is operator increasing, or \( \psi \circ \varphi^{-1} \) is concave and \( \psi^{-1} \) is operator decreasing, then

\[
\tilde{M}_\psi(A, \Phi) \leq \tilde{M}_\psi(A, \Phi).
\] (11)

In fact, to be more specific, we have the following series of inequalities

\[
\tilde{M}_\psi(A, \Phi)
\leq \psi^{-1} \left( \frac{\varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \cdot \psi(M) 
+ \sum_{j=1}^{k} \frac{\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M) - \varphi(m)} \cdot \psi(m) \right)
\leq \tilde{M}_\psi(A, \Phi)
\] (12)

(ii) if either \( \psi \circ \varphi^{-1} \) is concave and \( \psi^{-1} \) is operator increasing, or \( \psi \circ \varphi^{-1} \) is convex and \( \psi^{-1} \) is operator decreasing, then inequalities (11) and (12) are reversed.

Proof. Suppose that \( \psi \circ \varphi^{-1} \) is convex. If in Theorem 1 we let \( f = \psi \circ \varphi^{-1} \) and replace \( A_j, m \) and \( M \) with \( \varphi(A_j), \varphi(m) \) and \( \varphi(M) \), respectively, then we obtain

\[
(\psi \circ \varphi^{-1}) \left( \left( \varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \right) \right)
\leq \frac{\varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \cdot (\psi \circ \varphi^{-1})(\varphi(M))
+ \sum_{j=1}^{k} \frac{\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M) - \varphi(m)} \cdot (\psi \circ \varphi^{-1})(\varphi(m))
\leq (\psi \circ \varphi^{-1})(\varphi(m))1_K + (\psi \circ \varphi^{-1})(\varphi(M))1_K - \sum_{j=1}^{k} \Phi_j((\psi \circ \varphi^{-1})(\varphi(A_j))).
\]

or

\[
\psi \left( \varphi^{-1} \left( \varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \right) \right)
\leq \frac{\varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)} \cdot \psi(M) + \sum_{j=1}^{k} \frac{\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M) - \varphi(m)} \cdot \psi(m)
\leq \psi(m)1_K + \psi(M)1_K - \sum_{j=1}^{k} \Phi_j(\psi(A_j)).
\] (13)
If $\psi \circ \varphi^{-1}$ is concave then we obtain the reverse of inequalities (13).

If $\psi^{-1}$ is operator increasing, then (13) implies (12). If $\psi^{-1}$ is operator decreasing, then the reverse of (13) implies (12). Analogously, we get the reverse of (12) in the cases when $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator decreasing, or $\psi \circ \varphi^{-1}$ is concave and $\psi^{-1}$ is operator increasing. □

References