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A variant of Jensen's inequality of Mercer's type for operators with applications

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Abstract

A variant of Jensen's operator inequality for convex functions, which is a generalization of Mercer's result, is proved. Obtained result is used to prove a monotonicity property for Mercer's power means for operators, and a comparison theorem for quasi-arithmetic means for operators.

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1. Introduction

For a given $a < b$, let $\mathbf{x} = (x_1, \dots, x_k)$ be such that $a \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k \leqslant b$ and $\mathbf{w} = (w_1, \dots, w_k)$ be nonnegative weights such that $\sum_{j=1}^{k} w_j = 1$. Mercer [3] proved the following variant of Jensen's inequality.

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Theorem A. *If f is a convex function on* $[a, b]$ *then*

$$f\left(a+b-\sum_{j=1}^{k} w_j x_j\right) \leqslant f(a)+f(b)-\sum_{j=1}^{k} w_j f(x_j).$$

For $a > 0$ the (weighted) power means $M_r(\mathbf{x}, \mathbf{w})$ are defined as

$$M_r(\mathbf{x}, \mathbf{w}) = \begin{cases} \left(\sum\limits_{j=1}^{k} w_j x_j^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\sum\limits_{j=1}^{k} w_j \ln x_j\right), & r = 0. \end{cases}$$

In [4] Mercer defined the family of functions

$$Q_r(a, b, \mathbf{x}) = \begin{cases} [a^r + b^r - M_r^r(\mathbf{x}, \mathbf{w})]^{\frac{1}{r}}, & r \neq 0, \\ \dfrac{ab}{M_0(\mathbf{x}, \mathbf{w})}, & r = 0 \end{cases}$$

and proved the following.

Theorem B. *For* $r < s$, $Q_r(a, b, \mathbf{x}) \leqslant Q_s(a, b, \mathbf{x})$.

In this paper we consider similar inequalities in a more general setting. To do this we need some well known results. The first one is Löwner–Heinz inequality (see for example [5, p. 9]).

Theorem C. *Let A and B be positive operators on a Hilbert space H. If* $A \geqslant B$, *then* $A^p \geqslant B^p$ *for all* $p \in [0, 1]$.

In [5, p. 220, 232, 250] the following theorems are also proved.

Theorem D. *Let* A, B *be positive operators on a Hilbert space H with* $Sp(A) \subseteq [m_1, M_1]$, *and* $Sp(B) \subseteq [m_2, M_2]$ *for some scalars* $M_j > m_j > 0$ $(j = 1, 2)$. *If* $A \geqslant B$, *then the following inequalities hold*:

(i) *for all* $p > 1$:

$$K(m_1, M_1, p)A^p \geqslant B^p,$$
$$K(m_2, M_2, p)A^p \geqslant B^p,$$

(ii) *for all* $p < -1$:

$$K(m_1, M_1, p)B^p \geqslant A^p,$$
$$K(m_2, M_2, p)B^p \geqslant A^p,$$

where a generalized Kantorovich constant $K(m, M, p)$ *is defined by*

$$K(m, M, p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p$$

for all $p \in \mathbf{R}$.

Theorem E. *Let A, B be selfadjoint operators on a Hilbert space H with $Sp(B) \subseteq [m, M]$ for some scalars $M > m$. If $A \geqslant B$, then*

$$S(e^{M-m})e^{A} \geqslant e^{B},$$

where the Specht ratio $S(h)$ for $h > 0$ is defined by $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h} (h \neq 1)$ and $S(1) = 1$.

In Section 2 we give the main result of our paper which is an extension of Theorem A to selfadjoint operators and positive linear maps. This variant of Jensen's inequality for operators holds for arbitrary convex functions, while Davis–Choi–Jensen's inequality asserts that

$$f(\Phi(A)) \leqslant \Phi(f(A))$$

holds for an operator convex function f defined on an interval $(-a, a)$, where $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ is a normalized positive linear map and A is a selfadjoint operator with spectrum in $(-a, a)$ (see [1,2]).

In Section 3 we use that result to prove a monotonicity property of power means of Mercer's type for operators. In the final section we consider related quasi-arithmetic means for operators.

2. Main result

In what follows we assume that H and K are Hilbert spaces, $\mathscr{B}(H)$ and $\mathscr{B}(K)$ are C^*-algebras of all bounded operators on the appropriate Hilbert space and $\mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ is the set of all positive linear maps from $\mathscr{B}(H)$ to $\mathscr{B}(K)$. We denote by $C([m, M])$ the set of all real valued continuous functions on an interval $[m, M]$.

We show a variant of Jensen's operator inequality which is an extension of Theorem A to selfadjoint operators and positive linear maps.

Theorem 1. *Let $A_1, \ldots, A_k \in \mathscr{B}(H)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, \ldots, \Phi_k \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. If $f \in C([m, M])$ is convex on $[m, M]$, then*

$$f\left(m1_K + M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \leqslant f(m)1_K + f(M)1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)). \tag{1}$$

In fact, to be more specific, the following series of inequalities holds

$$\begin{aligned} f\left(m1_K + M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) &\leqslant \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} \cdot f(M) \\ &\quad + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} \cdot f(m) \\ &\leqslant f(m)1_K + f(M)1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)). \end{aligned} \tag{2}$$

If a function f is concave, then inequalities (1) and (2) are reversed.

Proof. Since f is continuous and convex, the same is also true for the function $g : [m, M] \to \mathbf{R}$ defined by $g(t) = f(m + M - t)$, $t \in [m, M]$. Hence, the following inequalities hold for every $t \in [m, M]$ (see for example [6, p. 2]):

$$f(t) \leqslant \frac{t-m}{M-m} \cdot f(M) + \frac{M-t}{M-m} \cdot f(m),$$
$$g(t) \leqslant \frac{t-m}{M-m} \cdot g(M) + \frac{M-t}{M-m} \cdot g(m).$$

Since $m1_H \leqslant A_j \leqslant M1_H$ for $j = 1, \ldots, k$ and $\sum_{j=1}^k \Phi_j(1_H) = 1_K$, it follows that $m1_K \leqslant \sum_{j=1}^k \Phi_j(A_j) \leqslant M1_K$. Now, using the functional calculus we have

$$g\left(\sum_{j=1}^k \Phi_j(A_j)\right) \leqslant \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M-m} \cdot g(M) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M-m} \cdot g(m)$$

or

$$\begin{aligned} f&\left(m1_K + M1_K - \sum_{j=1}^k \Phi_j(A_j)\right) \\ &\leqslant \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M-m} \cdot f(m) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M-m} \cdot f(M) \\ &= f(m)1_K + f(M)1_K \\ &\quad - \left[\frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M-m} \cdot f(m) + \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M-m} \cdot f(M)\right]. \end{aligned} \tag{3}$$

On the other hand, using the functional calculus we also have

$$f(A_j) \leqslant \frac{A_j - m1_H}{M-m} \cdot f(M) + \frac{M1_H - A_j}{M-m} \cdot f(m).$$

Applying positive linear maps Φ_j and summing, it follows that

$$\sum_{j=1}^k \Phi_j(f(A_j)) \leqslant \frac{\sum_{j=1}^k \Phi_j(A_j) - m1_K}{M-m} \cdot f(M) + \frac{M1_K - \sum_{j=1}^k \Phi_j(A_j)}{M-m} \cdot f(m). \tag{4}$$

Using inequalities (3) and (4), we obtain desired inequalities (1) and (2).

The last statement follows immediately from the fact that if φ is concave then $-\varphi$ is convex. □

3. Applications to Mercer's power means

We suppose that:

(i) $\mathbf{A} = (A_1, \ldots, A_k)$, where $A_j \in \mathscr{B}(H)$ are positive invertible operators with $Sp(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$.
(ii) $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)$, where $\Phi_j \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ are positive linear maps with $\sum_{j=1}^k \Phi_j(1_H) = 1_K$.

(iii) $\Delta(m, M, p) = K(m^p, M^p, \frac{1}{p}) = \frac{p(m^p M - M^p m)}{(1-p)(M^p - m^p)} \left(\frac{(1-p)(M-m)}{m^p M - M^p m}\right)^{\frac{1}{p}}$, for $0 < m < M$ and $p \in \mathbf{R}$, $p \neq 0$. Set: $\Delta(m, M, 0) = \lim_{p\to 0} \Delta(m, M, p) = S\left(\frac{M}{m}\right) = \frac{M-m}{\ln M - \ln m} \exp\left(\frac{m(1+\ln M) - M(1+\ln m)}{M-m}\right)$.

We define, for any $r \in \mathbf{R}$

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) := \begin{cases} [m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)]^{\frac{1}{r}}, & r \neq 0, \\ \exp\left((\ln m)1_K + (\ln M)1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j))\right), & r = 0. \end{cases}$$

Observe that, since $0 < m1_H \leqslant A_j \leqslant M1_H$, it follows that:

- $0 < m^r 1_H \leqslant A_j^r \leqslant M^r 1_H$ holds for all $r > 0$,
- $0 < M^r 1_H \leqslant A_j^r \leqslant m^r 1_H$ holds for all $r < 0$,
- $(\ln m)1_H \leqslant \ln(A_j) \leqslant (\ln M)1_H$ $(j = 1, \dots, k)$.

Applying positive linear maps Φ_j and summing, it follows that:

- $0 < m^r 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^r) \leqslant M^r 1_K$, for all $r > 0$,
- $0 < M^r 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^r) \leqslant m^r 1_K$, for all $r < 0$,
- $(\ln m)1_K \leqslant \sum_{j=1}^{k} \Phi_j(\ln(A_j)) \leqslant (\ln M)1_K$,

since $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. Hence, $\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})$ is well defined.

Furthermore, we define, for any $r, s \in \mathbf{R}$

$$S(r, s, \mathbf{A}, \mathbf{\Phi}) := \begin{cases} \left[\frac{M^r 1_K - S_r}{M^r - m^r} \cdot M^s + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\frac{M^r 1_K - S_r}{M^r - m^r} \cdot \ln M + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot \ln m\right), & r \neq 0, s = 0, \\ \left[\frac{(\ln M)1_K - S_0}{\ln M - \ln m} \cdot M^s + \frac{S_0 - (\ln m)1_K}{\ln M - \ln m} \cdot m^s\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

where $S_r = \sum_{j=1}^{k} \Phi_j(A_j^r)$ and $S_0 = \sum_{j=1}^{k} \Phi_j(\ln(A_j))$. It is easy to see that $S(r, s, \mathbf{A}, \mathbf{\Phi})$ is also well defined.

Theorem 2. *Let $r, s \in \mathbf{R}$, $r < s$.*

(i) *If either $r \leqslant -1$ or $s \geqslant 1$, then*

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

(ii) *If $-1 < r$ and $s < 1$, then*

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \Delta(m, M, s) \cdot \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Proof. (i) *Step 1:* Suppose that $0 < r < s$ and $s \geqslant 1$.

Applying the inequality (1) to the convex function $f(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} > 1$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we have

$$\left[m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)\right]^{\frac{s}{r}} \leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s). \tag{5}$$

Raising both sides to the power $\frac{1}{s}$ $(0 < \frac{1}{s} \leqslant 1)$, it follows from Theorem C that

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 2: Suppose that $r < 0$ and $s \geqslant 1$.

Applying the inequality (1) to the convex function $f(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} < 0$ here) and proceeding in the same way as in *Step 1*, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 3: Suppose that $r = 0$ and $s \geqslant 1$.

Applying the inequality (1) to the convex function $f(t) = \exp(s \cdot t)$ and replacing A_j, m and M with $\ln(A_j)$, $\ln m$ and $\ln M$, respectively, we have

$$\begin{aligned} &\exp\left(s\left((\ln m)1_K + (\ln M)1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j))\right)\right) \\ &\quad \leqslant \exp(s \ln m)1_K + \exp(s \ln M)1_K - \sum_{j=1}^{k} \Phi_j(\exp(s \ln(A_j))) \\ &\quad = m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s) \end{aligned} \tag{6}$$

or

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^s \leqslant [\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})]^s.$$

Raising both sides to the power $\frac{1}{s}$ $(0 < \frac{1}{s} \leqslant 1)$, it follows from Theorem C that

$$\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 4: Suppose that $r < s < 0$ and $r \leqslant -1$.

Applying the inequality (1) to the convex function $f(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} > 1$ here) and replacing A_j, m and M with A_j^s, m^s and M^s, respectively, we have

$$\left[m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s))\right]^{\frac{r}{s}} \leqslant m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r). \tag{7}$$

Raising both sides to the power $-\frac{1}{r}$ $(0 < -\frac{1}{r} \leqslant 1)$, it follows from Theorem C that

$$[\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^{-1}.$$

Hence, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 5: Suppose that $s > 0$ and $r \leqslant -1$.

Applying the inequality (1) to the convex function $f(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} < 0$ here) and proceeding in the same way as in *Step 4*, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 6: Suppose that $s = 0$ and $r \leqslant -1$.

Applying the inequality (1) to the convex function $f(t) = \exp(r \cdot t)$ and replacing A_j, m and M with $\ln(A_j)$, $\ln m$ and $\ln M$, respectively, we have

$$\begin{aligned} &\exp\left(r\left((\ln m)1_K + (\ln M)\,1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j))\right)\right) \\ &\quad\leqslant \exp(r \ln m)1_K + \exp(r \ln M)1_K - \sum_{j=1}^{k} \Phi_j(\exp(r \ln(A_j))) \\ &\quad= m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r) \end{aligned} \tag{8}$$

or

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^r \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^r.$$

Raising both sides to the power $-\frac{1}{r}$ $(0 < \frac{1}{r} \leqslant 1)$, it follows from Theorem C that

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^{-1}.$$

Hence, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_0(\mathbf{A}, \mathbf{\Phi}).$$

(ii) *Step 1:* Suppose that $0 < r < s < 1$.

In the same way as in (i) *Step 1* we obtain inequality (5). Observe that, since $m^s 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant M^s 1_K$, it follows that $m^s 1_K \leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant M^s 1_K$. Raising both sides of (5) to the power $\frac{1}{s}$ $(\frac{1}{s} > 1)$, it follows from Theorem D (i) that

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant K\left(m^s, M^s, \frac{1}{s}\right) \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 2: Suppose that $0 = r < s < 1$.

In the same way as in (i) *Step 3* we obtain inequality (6). With the same observation as in (ii) *Step 1* and raising both sides of (6) to the power $\frac{1}{s}$ $(\frac{1}{s} > 1)$, it follows from Theorem D (i) that

$$\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi}) \leqslant K\left(m^s, M^s, \frac{1}{s}\right) \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 3: Suppose that $-1 < r < s < 0$.

Applying reversed inequality (1) to the concave function $f(t) = t^{\frac{s}{r}}$ (note that $0 < \frac{s}{r} < 1$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we obtain reversed inequality (5). Observe that, since $M^s 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant m^s 1_K$, it follows that $M^s 1_K \leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant m^s 1_K$. Raising both sides of reversed (5) to the power $\frac{1}{s}(\frac{1}{s} < -1)$, it follows from Theorem D (ii) that

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant K\left(M^s, m^s, \frac{1}{s}\right)\widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

Since $K(M,m,p) = K(m,M,p)$ (see [5, p. 77]), we have

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant K\left(m^s, M^s, \frac{1}{s}\right)\widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

Step 4: Suppose that $-1 < r < s = 0$.

Applying the inequality (1) to the convex function $f(t) = \frac{1}{r}\ln t$ and replacing A_j, m and M with A_j^r, M^r and m^r, respectively, we obtain

$$\frac{1}{r}\ln\left(m^r 1_K + M^r 1_K - \sum_{j=1}^{k}\Phi_j(A_j^r)\right) \leqslant (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^{k}\Phi_j(\ln(A_j)).$$

Observing that both sides have spectra in $[\ln m, \ln M]$, it follows from Theorem E that

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant \Delta(m,M,0)\widetilde{M}_0(\mathbf{A},\mathbf{\Phi}).$$

Step 5: Suppose that $-1 < r < 0 < s < 1$.

In the same way as in (i) *Step 2* we obtain inequality (5). With the same observation as in (ii) *Step 1* and raising both sides of (5) to the power $\frac{1}{s}$ ($\frac{1}{s} > 1$), it follows from Theorem D (i) that

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant K\left(m^s, M^s, \frac{1}{s}\right)\widetilde{M}_s(\mathbf{A},\mathbf{\Phi}). \qquad \square$$

If we use inequalities (2) instead of the inequality (1), then we have the following results:

Theorem 3. *Let $r, s \in \mathbf{R}$, $r < s$.*

(i) *If $s \geqslant 1$, then*

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant S(r,s,\mathbf{A},\mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

If $r \leqslant -1$, then

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant S(s,r,\mathbf{A},\mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

(ii) *If $-1 < r$ and $s < 1$, then*

$$\frac{1}{\Delta(m,M,s)}\cdot\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant S(r,s,\mathbf{A},\mathbf{\Phi}) \leqslant \Delta(m,M,s)\cdot\widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

Proof. (i) *Step 1:* Suppose that $0 < r < s$ and $s \geqslant 1$.

Applying inequalities (2) to the convex function $f(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} \geqslant 1$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we have

$$\left[m^r 1_K + M^r 1_K - \sum_{j=1}^{k}\Phi_j(A_j^r)\right]^{\frac{s}{r}} \leqslant \frac{M^r 1_K - S_r}{M^r - m^r}\cdot M^s + \frac{S_r - m^r 1_K}{M^r - m^r}\cdot m^s$$
$$\leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^{k}\Phi_j(A_j^s). \tag{9}$$

Raising these inequalities to the power $\frac{1}{s}(0 < \frac{1}{s} \leqslant 1)$, it follows from Theorem C that

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(r, s, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 2: Suppose that $r < 0$ and $s \geqslant 1$.

Applying inequalities (2) to the convex function $f(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} < 0$ here) and proceeding in the same way as in *Step 1*, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(r, s, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 3: Suppose that $r = 0$ and $s \geqslant 1$.

Applying inequalities (2) to the convex function $f(t) = \exp(s \cdot t)$ and replacing A_j, m and M with $\ln(A_j)$, $\ln m$ and $\ln M$, respectively, we have

$$\begin{aligned}
&\exp\left(s\left((\ln m)1_K + (\ln M)1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j))\right)\right)\\
&\quad\leqslant \frac{(\ln M)1_K - S_0}{\ln M - \ln m} \cdot \exp(s \ln M) + \frac{S_0 - (\ln m)1_K}{\ln M - \ln m} \cdot \exp(s \ln m)\\
&\quad\leqslant \exp(s \ln m)1_K + \exp(s \ln M)1_K - \sum_{j=1}^{k} \Phi_j(\exp(s \ln(A_j)))\\
&\quad= m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s)
\end{aligned} \tag{10}$$

or

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^s \leqslant [S(0, s, \mathbf{A}, \mathbf{\Phi})]^s \leqslant [\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})]^s.$$

Raising these inequalities to the power $\frac{1}{s}(0 < \frac{1}{s} \leqslant 1)$, it follows from Theorem C that

$$\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi}) \leqslant S(0, s, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 4: Suppose that $r < s < 0$ and $r \leqslant -1$.

Applying inequalities (2) to the convex function $f(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} \geqslant 1$ here) and replacing A_j, m and M with A_j^s, m^s and M^s, respectively, we have

$$\begin{aligned}
\left[m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s)\right]^{\frac{r}{s}} &\leqslant \frac{M^s 1_K - S_r}{M^s - m^s} \cdot M^r + \frac{S_r - m^s 1_K}{M^s - m^s} \cdot m^r\\
&\leqslant m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r).
\end{aligned}$$

Raising these inequalities to the power $-\frac{1}{r}$ $(0 < -\frac{1}{r} \leqslant 1)$, it follows from Theorem C that

$$[\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [S(s, r, \mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^{-1}.$$

Hence, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(s, r, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 5: Suppose that $s > 0$ and $r \leqslant -1$.

Applying inequalities (2) to the convex function $f(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} < 0$ here) and proceeding in the same way as in *Step 4*, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(s, r, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}).$$

Step 6: Suppose that $s = 0$ and $r \leqslant -1$.

Applying inequalities (2) to the convex function $f(t) = \exp(r \cdot t)$ and replacing A_j, m and M with $\ln(A_j)$, $\ln m$ and $\ln M$, respectively, we have

$$\begin{aligned}
&\exp\left(r\left((\ln m)1_K + (\ln M)1_K - \sum_{j=1}^{k} \Phi_j(\ln(A_j))\right)\right) \\
&\quad\leqslant \frac{(\ln M)1_K - S_0}{\ln M - \ln m} \cdot \exp(r \ln M) + \frac{S_0 - (\ln m)1_K}{\ln M - \ln m} \cdot \exp(r \ln m) \\
&\quad\leqslant \exp(r \ln m)1_K + \exp(r \ln M)1_K - \sum_{j=1}^{k} \Phi_j(\exp(r \ln(A_j))) \\
&\quad= m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)
\end{aligned}$$

or

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^r \leqslant [S(0, r, \mathbf{A}, \mathbf{\Phi})]^r \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^r.$$

Raising these inequalities to the power $-\frac{1}{r}$ $(0 < \frac{1}{r} \leqslant 1)$, it follows from Theorem C that

$$[\widetilde{M}_0(\mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [S(0, r, \mathbf{A}, \mathbf{\Phi})]^{-1} \leqslant [\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})]^{-1}.$$

Hence, we have

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(0, r, \mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_0(\mathbf{A}, \mathbf{\Phi}).$$

(ii) *Step 1:* Suppose that $0 < r < s < 1$.

In the same way as in (i) *Step 1* we obtain inequalities (9). Observe that, since $m^r 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^r) \leqslant M^r 1_K$ and $m^s 1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant M^s 1_K$, it follows that $m^s 1_K \leqslant [m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)]^{\frac{s}{r}} \leqslant M^s 1_K$ and $m^s 1_K \leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s) \leqslant M^s 1_K$. Raising inequalities (9) to the power $\frac{1}{s}$ $(\frac{1}{s} > 1)$, it follows from Theorem D (i) that

$$\begin{aligned}
&K\left(m^s, M^s, \frac{1}{s}\right)^{-1}\left[m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j(A_j^r)\right]^{\frac{1}{r}} \\
&\quad\leqslant \left[\frac{M^r 1_K - S_r}{M^r - m^r} \cdot M^s + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}} \\
&\quad\leqslant K\left(m^s, M^s, \frac{1}{s}\right)\left[m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j(A_j^s)\right]^{\frac{1}{s}},
\end{aligned}$$

or

$$\Delta(m, M, s)^{-1}\widetilde{M}_r(\mathbf{A}, \boldsymbol{\Phi}) \leqslant S(r, s, \mathbf{A}, \boldsymbol{\Phi}) \leqslant \Delta(m, M, s)\widetilde{M}_s(\mathbf{A}, \boldsymbol{\Phi}).$$

Step 2: Suppose that $0 = r < s < 1$.

In the same way as in (i) *Step 3* we obtain inequalities (10). Observe that, since $(\ln m)1_K \leqslant (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \leqslant (\ln M)1_K$ and $m^s 1_K \leqslant \sum_{j=1}^k \Phi_j(A_j^s) \leqslant M^s 1_K$, it follows that

$$m^s 1_K \leqslant \exp\left(s\left((\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j))\right)\right) \leqslant M^s 1_K$$

and $m^s 1_K \leqslant m^s 1_K + M^s 1_K - \sum_{j=1}^k \Phi_j(A_j^s) \leqslant M^s 1_K$. Raising inequalities (10) to the power $\frac{1}{s}$ ($\frac{1}{s} > 1$), it follows from Theorem D (i) that

$$\Delta(m, M, s)^{-1}\widetilde{M}_0(\mathbf{A}, \boldsymbol{\Phi}) \leqslant S(0, s, \mathbf{A}, \boldsymbol{\Phi}) \leqslant \Delta(m, M, s)\widetilde{M}_s(\mathbf{A}, \boldsymbol{\Phi}).$$

Step 3: Suppose that $-1 < r < s < 0$.

Applying reversed inequalities (2) to the concave function $f(t) = t^{\frac{s}{r}}$ (note that $0 < \frac{s}{r} < 1$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we obtain reversed (9). With the same observation as in *Step 1* and raising reversed (9) to the power $\frac{1}{s}$ ($\frac{1}{s} < -1$), it follows from Theorem D (ii) that

$$\Delta(m, M, s)^{-1}\widetilde{M}_r(\mathbf{A}, \boldsymbol{\Phi}) \leqslant S(r, s, \mathbf{A}, \boldsymbol{\Phi}) \leqslant \Delta(m, M, s)\widetilde{M}_s(\mathbf{A}, \boldsymbol{\Phi}).$$

Step 4: Suppose that $-1 < r < s = 0$.

Applying inequalities (2) to the convex function $f(t) = \frac{1}{r}\ln t$ (note that $\frac{1}{r} < 0$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we obtain

$$\begin{aligned}
&\frac{1}{r}\ln\left(m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r)\right)\\
&\quad\leqslant \frac{M^r 1_K - S_r}{M^r - m^r}\cdot\ln M + \frac{S_r - m^r}{M^r - m^r}\cdot\ln m\\
&\quad\leqslant (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)).
\end{aligned}$$

Observe that, since $r < 0$, $M^r 1_K \leqslant m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r) \leqslant m^r 1_K$ and $(\ln m)1_K \leqslant \sum_{j=1}^k \Phi_j(\ln(A_j)) \leqslant (\ln M)1_K$, it follows that

$$\ln m \leqslant \frac{1}{r}\ln\left(m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j(A_j^r)\right) \leqslant \ln M$$

and $(\ln m)1_K \leqslant (\ln m)1_K + (\ln M)1_K - \sum_{j=1}^k \Phi_j(\ln(A_j)) \leqslant (\ln M)1_K$. Now, it follows from Theorem E that

$$S(e^{\ln M - \ln m})^{-1}\widetilde{M}_r(\mathbf{A}, \boldsymbol{\Phi}) \leqslant S(r, 0, \mathbf{A}, \boldsymbol{\Phi}) \leqslant S(e^{\ln M - \ln m})\widetilde{M}_0(\mathbf{A}, \boldsymbol{\Phi}).$$

Step 5: Suppose that $-1 < r < 0 < s < 1$.

Applying inequalities (2) to the convex function $f(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} < 0$ here) and replacing A_j, m and M with A_j^r, m^r and M^r, respectively, we obtain inequalities (9). Proceeding in the same way as in *Step 1*, we have

$$\Delta(m, M, s)^{-1}\, \widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \leqslant S(r, s, \mathbf{A}, \mathbf{\Phi}) \leqslant \Delta(m, M, s)\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}). \qquad \square$$

Remark 1. Some considerations in Theorems 2 and 3 can be shortened using obvious properties $\widetilde{M}_{-s}(\mathbf{A}^{-1}, \mathbf{\Phi}) = \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})^{-1}$ and $S(-s, -r, \mathbf{A}^{-1}, \mathbf{\Phi}) = S(s, r, \mathbf{A}, \mathbf{\Phi})^{-1}$, where $\mathbf{A}^{-1} = (A_1^{-1}, \ldots, A_k^{-1})$.

Remark 2. Since obviously $S(r, r, \mathbf{A}, \mathbf{\Phi}) = \widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})$, inequalities in Theorem 3 (i) give us

$$S(r, r, \mathbf{A}, \mathbf{\Phi}) \leqslant S(r, s, \mathbf{A}, \mathbf{\Phi}) \leqslant S(s, s, \mathbf{A}, \mathbf{\Phi}),\ r < s,\ s \geqslant 1$$

and

$$S(r, r, \mathbf{A}, \mathbf{\Phi}) \leqslant S(s, r, \mathbf{A}, \mathbf{\Phi}) \leqslant S(s, s, \mathbf{A}, \mathbf{\Phi}),\ r < s,\ r \leqslant -1.$$

An open problem is to give the list of inequalities comparing "mixed means" $S(r, s, \mathbf{A}, \mathbf{\Phi})$ in remaining cases.

4. Quasi-arithmetic means of Mercer's type

Let $\mathbf{A}$ and $\mathbf{\Phi}$ be as in the previous section. Let $\varphi, \psi \in C([m, M])$ be strictly monotonic functions on an interval $[m, M]$. We define

$$\widetilde{M}_\varphi(\mathbf{A}, \mathbf{\Phi}) = \varphi^{-1}\left(\varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k} \Phi_j(\varphi(A_j))\right).$$

Observe that, since $m1_H \leqslant A_j \leqslant M1_H$, it follows that

- $\varphi(m)1_H \leqslant \varphi(A_j) \leqslant \varphi(M)1_H$ if φ is increasing,
- $\varphi(M)1_H \leqslant \varphi(A_j) \leqslant \varphi(m)1_H$ if φ is decreasing.

Applying positive linear maps Φ_j and summing, it follows that

- $\varphi(m)1_K \leqslant \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \leqslant \varphi(M)1_K$ if φ is increasing,
- $\varphi(M)1_K \leqslant \sum_{j=1}^{k} \Phi_j(\varphi(A_j)) \leqslant \varphi(m)1_K$ if φ is decreasing,

since $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. Hence, $\widetilde{M}_\varphi(\mathbf{A}, \mathbf{\Phi})$ is well defined.

A function $f \in C([m, M])$ is said to be *operator increasing* if f is operator monotone, i.e., if $A \leqslant B$ implies $f(A) \leqslant f(B)$, for all selfadjoint operators A and B on a Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M]$. A function $f \in C([m, M])$ is said to be *operator decreasing* if $-f$ is operator monotone.

Theorem 4. *Under the above hypotheses, we have*

(i) *if either $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator decreasing, then*

$$\widetilde{M}_{\varphi}(\mathbf{A}, \mathbf{\Phi}) \leqslant \widetilde{M}_{\psi}(\mathbf{A}, \mathbf{\Phi}). \tag{11}$$

In fact, to be more specific, we have the following series of inequalities

$$\begin{aligned}
&\widetilde{M}_{\varphi}(\mathbf{A}, \mathbf{\Phi}) \\
&\quad\leqslant \psi^{-1}\left(\frac{\varphi(M)1_K - \sum_{j=1}^{k}\Phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)} \cdot \psi(M) \right.\\
&\qquad\qquad\left. + \frac{\sum_{j=1}^{k}\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M)-\varphi(m)} \cdot \psi(m)\right) \\
&\quad\leqslant \widetilde{M}_{\psi}(\mathbf{A}, \mathbf{\Phi})
\end{aligned} \tag{12}$$

(ii) *if either $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator increasing, or $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator decreasing, then inequalities (11) and (12) are reversed.*

Proof. Suppose that $\psi \circ \varphi^{-1}$ is convex. If in Theorem 1 we let $f = \psi \circ \varphi^{-1}$ and replace A_j, m and M with $\varphi(A_j)$, $\varphi(m)$ and $\varphi(M)$, respectively, then we obtain

$$\begin{aligned}
&(\psi \circ \varphi^{-1})\left(\varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k}\Phi_j(\varphi(A_j))\right) \\
&\quad\leqslant \frac{\varphi(M)1_K - \sum_{j=1}^{k}\Phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)} \cdot (\psi \circ \varphi^{-1})(\varphi(M)) \\
&\qquad + \frac{\sum_{j=1}^{k}\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M)-\varphi(m)} \cdot (\psi \circ \varphi^{-1})(\varphi(m)) \\
&\quad\leqslant (\psi \circ \varphi^{-1})(\varphi(m))1_K + (\psi \circ \varphi^{-1})(\varphi(M))1_K - \sum_{j=1}^{k}\Phi_j((\psi \circ \varphi^{-1})(\varphi(A_j))).
\end{aligned}$$

or

$$\begin{aligned}
&\psi\left(\varphi^{-1}\left(\varphi(m)1_K + \varphi(M)1_K - \sum_{j=1}^{k}\Phi_j(\varphi(A_j))\right)\right) \\
&\quad\leqslant \frac{\varphi(M)1_K - \sum_{j=1}^{k}\Phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)} \cdot \psi(M) + \frac{\sum_{j=1}^{k}\Phi_j(\varphi(A_j)) - \varphi(m)1_K}{\varphi(M)-\varphi(m)} \cdot \psi(m) \\
&\quad\leqslant \psi(m)1_K + \psi(M)1_K - \sum_{j=1}^{k}\Phi_j(\psi(A_j)).
\end{aligned} \tag{13}$$

If $\psi \circ \varphi^{-1}$ is concave then we obtain the reverse of inequalities (13).

If ψ^{-1} is operator increasing, then (13) implies (12). If ψ^{-1} is operator decreasing, then the reverse of (13) implies (12). Analogously, we get the reverse of (12) in the cases when $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator decreasing, or $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator increasing. □

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