Boundedness of solutions for reversible system via Moser’s twist theorem

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Abstract

In this paper we consider the problem of the boundedness of all solutions for the reversible system

\[ x'' + \sum_{j=0}^{l} b_j(t)x^{2j+1} + x^{2n+1} + \sum_{i=0}^{n-1} a_i(t)x^{2i+1} = 0. \]

It is shown that all the solutions are bounded provided that the \( a_i(t) \) (\( 0 \leq i \leq [(n-1)/2] \)) are of bounded variation in \([0, 1]\) and the derivatives of \( b_j(t) \) and \( a_i(t) \) (\( [(n-1)/2] + 1 \leq i \leq n-1, \ 0 \leq j \leq l \)) are Lipschitzian. It is also shown that there exist \( a_i \)'s being discontinuous everywhere such that all solutions of the equation are bounded. This implies that the continuity of \( a_i \)'s is not necessary for the boundedness of solutions of the equation.

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1. Introduction

In this paper, we are concerned with the boundedness of solutions for the nonlinear scalar differential equation of the second order

\[ x'' + f(x, t)x' + g(x, t) = p(t), \]  \( (1) \)

where \( f, g \) and \( p \) are periodic in \( t \). This problem has been widely investigated by many authors since 1940s. In order to show the boundedness of solutions of Eq. (1), one can construct an absorbing compact domain in the phase space such that all solutions of Eq. (1) always enter this domain whenever \( t \geq t_0 \) (see [1–3]). When \( f \equiv 0 \), Eq. (1) is a conservative system

\[ x'' + g(x, t) = p(t). \]  \( (2) \)
Then one cannot construct the above-mentioned absorbing compact domain in the phase space in order to show the boundedness of the solutions. In this case, one could try to apply KAM theory in order to find conditions which guarantee the boundedness of all the solutions. If \( g(x, t) = g(x) \), then Eq. (2) takes the form\(^6\)

\[
x'' + g(x) = p(t).
\]

(3)

It was Littlewood who asked whether or not the solutions of Eq. (3) are bounded for all time. The first boundedness result of Eq. (3) is due to Morris [4], who proved that all solutions of Eq. (3) are bounded for a special case: \( g(x) = 2x^3 \) with \( p(t) \in C^0 \). In 1987, Dieckerhoff and Zehnder [5] considered the following conservative system:

\[
x'' + x^{2n+1} + \sum_{i=0}^{k} a_i(t)x^i = 0, \quad 0 \leq k \leq 2n,
\]

(4)

with sufficient smooth \( a_i(t) \) and proved that all solutions of Eq. (4) are bounded via Moser’s twist theorem if \( a_i(t) \in C^\infty \). Laederich and Levi [6] improved such results and showed that Eq. (4) possesses quasiperiodic solutions and every solution of Eq. (4) is bounded when \( a_i(t) \in C^{(5+\epsilon)}(S^1) \). In 1995, Yuan [7,8] relaxed the smoothness to \( C^2 \). In 1998, Yuan [9] proved that every solution of Eq. (4) is bounded if \( a_i(t) (0 \leq i \leq n) \) are of bounded variation in \([0, 1]\) and the derivatives of \( a_i(t) (n+1 \leq i \leq 2n) \) are Lipschitzian. The above mentioned results deal with concrete nonlinearity and more recent papers [10,11] have enlarged the case of admissible nonlinearity.

In [12,13], the following equation:

\[
x'' + \sum_{i=0}^{l} b_i(t)x^{2i+1}x' + x^{2n+1} + \sum_{i=0}^{k} a_i(t)x^{2i+1} = p(t)
\]

(5)

is considered, where \( n \geq 2(l + 1), n \geq k + 1, a_i(t) \) and \( b_i(t) \) are even, and \( p(t) \) is odd. By using a modified version of Dieckerhoff and Zehnder’s technique, Yuan [13] proved that every solution of Eq. (5) is bounded via Moser’s twist theorem. It is required in [13] that \( a_i(t) \) and \( b_i(t) \) are sufficient smoothness. In [14], Liu proved that every solution is bounded of Eq. (5) if \( b_j(t) \in C^2, a_i(t) \in C^2 (0 \leq j \leq l, \ (n+1)/2 \leq i \leq k), \) and \( a_i(t) \in C^1 (0 \leq i \leq ((n+1)/2) - 1) \). Yuan [15] relaxed the smoothness to \( b_j(t) (0 \leq j \leq l) \) and \( a_i(t) ((n-1)/2) + 1 \leq i \leq k) \) are of \( C^1 \) and their derivatives are Lipschitzian, and \( a_i(t) (0 \leq i \leq ((n-1)/2)) \) is Lipschitzian. To this end, it is nature to ask whether or not the boundedness phenomenon of Eq. (5) is related to the smoothness in the \( t \)-variable, which is similar to that posed by Yuan in [9]. We will try to answer this question.

In this paper, we continue to study the boundedness of all the solutions of Eq. (5) when \( k = n - 1, p(t) = 0, \) by using smoothing technique together with the methods developed in [5,14,15], we can prove the following:

Theorem 1. Suppose \( n \geq 2(l + 1), \) then all solutions (in the sense of Carathéodory) of Eq. (5) are bounded under the following conditions:

(C1) the functions \( a_i(t) \) with \( 0 \leq i \leq n - 1 \) and \( b_j(t) \) with \( 0 \leq j \leq l \) are 1-periodic and even functions and they are Lebesgue measurable;

(C2) for any \( \epsilon > 0 \) and every \( a_i (0 \leq i \leq ((n-1)/2)) \), there exists a function \( a_i,\epsilon \) of period 1, of bounded variation in \([0, 1]\) such that the variation, \( BV(a_i,\epsilon; [0, 1]) \), is bounded by some constant \( C \) independent of \( \epsilon \) and

\[
\int_0^1 |a_i(t) - a_i,\epsilon(t)| \, dt < \epsilon, \quad 0 \leq i \leq [(n-1)/2];
\]

(C3) for any \( \epsilon > 0 \) and every \( a_i \) \( ((n-1)/2) + 1 \leq i \leq n - 1 \), \( b_j(t) (0 \leq j \leq l) \), there exists a function \( a_i,\epsilon \) \( b_j,\epsilon \) of period 1, such that \( da_i,\epsilon/dt, db_j,\epsilon/dt \) are Lipschitzian with Lipschitz constant \( C \) independent of \( \epsilon \) and

\[
\int_0^1 |a_i(t) - a_i,\epsilon(t)| \, dt < \epsilon, \quad [(n-1)/2] + 1 \leq i \leq n - 1;
\]

\[
\int_0^1 |b_j(t) - b_j,\epsilon(t)| \, dt < \epsilon, \quad 0 \leq j \leq l.
\]
Then there is $\omega_0 > 0$ such that for every irrational number $\omega > \omega_0$ satisfying
\[
|\omega - \frac{p}{q}| \geq \frac{1}{2}|q|^{2-\delta}, \quad \delta > 0.
\]
The time 1 maps $P : (x, x')|_{t=0} \rightarrow (x, x')|_{t=1}$ of Eq. (5) possesses an invariant circle with rational number $\omega$. In particular, the solutions starting from the circle are quasiperiodic with frequencies $\omega$ and 1 and every solution is bounded, i.e., it exists for all $t \in \mathbb{R}$ and
\[
\sup_{t \in \mathbb{R}} (|x(t)| + |x'(t)|) < \infty.
\]

**Theorem 2.** Suppose $n \geq 2(l + 1)$, then all solutions (in the sense of Carathéodory) of Eq. (5) are bounded under the following conditions:

(C1) the functions $a_i(t)$ with $0 \leq i \leq k$ and $b_j(t)$ with $0 \leq j \leq l$ are 1-periodic and even functions;

(C2) the functions $a_i(t)$ with $0 \leq i \leq \lfloor (n - 1)/2 \rfloor$ are of bounded variation in $[0, 1]$;

(C3) the derivatives of functions $a_i(t), b_j(t)$ with $\lfloor ((n - 1)/2) \rfloor + 1 \leq i \leq n - 1, 0 \leq j \leq l$, are Lipschitzian.

**Proof.** The functions $a_i, \epsilon$ and $b_j, \epsilon$ required in the assumptions can be picked by $a_i, \epsilon = a_i$ and $b_j, \epsilon = b_j$ for any $\epsilon$. Hence, Theorem 2 is a corollary of Theorem 1. □

**Remark 1.** The conditions (C1)–(C3) in Theorems 1 and 2 are relaxed to the smoothness as in [9], that means the coefficients need not to be Lipschitz continuous as in [14].

2. Approximation technique

The following lemma is of importance in relaxing the smoothness requirement.

**Lemma 1.** Assume a function $f$ satisfies the assumption (C2) in Theorem 1, then for any $\epsilon > 0$ there is a differential function $f_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ of period 1 such that
\[
|f_\epsilon(t)| \leq \max_{t \in [0,1]} |f(t)|, \quad \int_0^1 |f'_\epsilon(t)| \, dt \leq C, \quad \int_0^1 |f(t) - f_\epsilon(t)| \, dt < \epsilon,
\]
where $C$ is a constant independent of $\epsilon$. Moreover if $f(t)$ is even (odd), then $f_\epsilon(t)$ is even (odd).

**Proof.** Define $f_\epsilon(t)$ as in [9],
\[
f_\epsilon(t) = \frac{1}{2\delta^*} \int_{t-\delta^*}^{t+\delta^*} f_\delta(s) \, ds
\]
and
\[
f_\delta(t) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} f(s) \, ds,
\]
where $\delta, \delta^* > 0$ are small enough such that $\int_0^1 |f_\delta(t) - f(t)| \, dt < \epsilon$ and $\int_0^1 |f_\epsilon(t) - f_\delta(t)| \, dt < \epsilon$, so the estimates are satisfied as in [9], and one can verify easily that if $f(t)$ is odd (even), then $f_\epsilon(t)$ is odd (even). □

**Lemma 2.** Assume a function $f$ satisfies the assumption (C3) in Theorem 1, then for any $\epsilon > 0$ there is a differential function $f_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ of period 1 such that
\[ |\hat{f}_\epsilon(t)|, |\hat{f}_\epsilon'(t)|, |\hat{f}_\epsilon''(t)| \leq C, \quad |f(t) - \hat{f}_\epsilon(t)| dt < \epsilon, \]

where \(C\) is a constant independent of \(\epsilon\). Moreover if \(f(t)\) is even (odd), then \(f_\epsilon(t)\) is even (odd).

**Proof.** For any \(\epsilon > 0\), define a kernel
\[
\Delta_\epsilon(t) = \begin{cases} \text{constant determined by } A_\epsilon, & \text{for } t \leq \epsilon, \\ 0, & \text{for } |t| > \epsilon, \end{cases}
\]

where \(A_\epsilon\) is determined by
\[
\int_{-\infty}^{+\infty} \Delta_\epsilon(t) dt = 1.
\]

Let \(f_\epsilon(t)\) be the function defined in (C3) of Theorem 1, define
\[
\hat{f}_\epsilon(t) = \int_{-\infty}^{+\infty} \Delta_\epsilon(t - s) f_\epsilon(s) ds,
\]

one can verify easily that the estimates are satisfied, and if \(f(t)\) is odd (even), then \(f_\epsilon(t)\) is odd (even).

### 3. Action-angle variables

Let us consider Eq. (1), which is equivalent to the plane system
\[
x' = y, \quad y' = -g(x, t) - f(x, t)y.
\]

First of all, we consider a special Hamiltonian system
\[
x' = y, \quad y' = -x^{2n+1}
\]

with Hamiltonian
\[
h(x, y) = \frac{y^2}{2} + \frac{x^{2(n+1)}}{2(n+1)}.
\]

Suppose that \((x_0(t), y_0(t))\) is the solution of Eq. (7), with the initial conditions \((x_0(0), y_0(0)) = (0, 1)\). And let \(T_0\) be its minimal positive period. It follows from (7) that \(x_0(t)\) and \(y_0(t)\) possess the following qualities:

(a) \(x_0(t + T_0) = x_0(t)\) and \(y_0(t + T_0) = y_0(t)\);
(b) \(x_0'(t) = y_0(t)\) and \(y_0'(t) = -(x_0(t))^{2n+1}\);
(c) \((n+1)(y_0(t))^2 + (x_0(t))^{2n+1} = 1\);
(d) \(x_0(-t) = x_0(t)\) and \(y_0(-t) = -y_0(t)\).

The action-angle variables are now defined by the mapping \(\psi : R^+ \times S^1 \to R^2/\{0\}\), where \((x, y) = \psi(\lambda, \theta)\) with \(\lambda > 0\) and \(\theta \text{ (mod 1)}\) being given by the formula
\[
\psi : \quad x = c^\alpha \rho^\alpha x_0(\theta T_0), \quad y = c^\beta \rho^\beta y_0(\theta T_0),
\]

where \(\alpha = \frac{1}{n+2}, \beta = 1 - \alpha\) and \(c = \frac{1}{\beta T_0}\).

By using the transformation \(\psi\), Eq. (6) is transformed into
\[
\begin{cases}
\rho' = h_1(\rho, \theta, t) + h_2(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t) \equiv f_1(\rho, \theta, t), \\
\theta' = c_0 \rho^{2\beta - 1} + h_3(\rho, \theta, t) + h_4(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t) \equiv f_2(\rho, \theta, t),
\end{cases}
\]

where
\[ h_1(\rho, \theta, t) = -\sum_{i=[(n-1)/2]+1}^{n-1} \hat{a}_{i,\epsilon} T_0 c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(i+1)}(\theta T_0) \]
\[ - \sum_{j=0}^{l} \hat{b}_{j,\epsilon} T_0 c^{(2j+1)\alpha} \rho^{(2j+1)\alpha} y_0(\theta T_0) x_0^{2(j+1)}(\theta T_0), \]
\[ h_2(\rho, \theta, t) = -\sum_{i=0}^{[(n-1)/2]} \hat{a}_{i,\epsilon} T_0 c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(j+1)}(\theta T_0) \]
\[ g_1(\rho, \theta, t) = -\frac{1}{\epsilon} \sum_{i=0}^{n-1} \hat{a}_{i,\epsilon} T_0 c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(i+1)}(\theta T_0) \]
\[ - \frac{1}{\epsilon} \sum_{j=0}^{l} \hat{b}_{j,\epsilon} T_0 c^{(2j+1)\alpha} \rho^{(2j+1)\alpha} y_0(\theta T_0) x_0^{2(j+1)}(\theta T_0), \]
\[ h_3(\rho, \theta, t) = \sum_{i=[(n-1)/2]+1}^{n-1} \hat{a}_{i,\epsilon} \alpha c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(i+1)}(\theta T_0) \]
\[ + \sum_{j=0}^{l} \hat{b}_{j,\epsilon} \alpha c^{(2j+1)\alpha} \rho^{(2j+1)\alpha} y_0(\theta T_0) x_0^{2(j+1)}(\theta T_0), \]
\[ h_4(\rho, \theta, t) = \sum_{i=0}^{[(n-1)/2]} \hat{a}_{i,\epsilon} \alpha c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(i+1)}(\theta T_0), \]
\[ g_2(\rho, \theta, t) = \frac{1}{\epsilon} \sum_{i=0}^{n-1} \hat{a}_{i,\epsilon} \alpha c^{2(i+1)\alpha} \rho^{2(i+1)\alpha} y_0(\theta T_0) x_0^{2(i+1)}(\theta T_0) \]
\[ + \frac{1}{\epsilon} \sum_{j=0}^{l} \hat{b}_{j,\epsilon} \alpha c^{(2j+1)\alpha} \rho^{(2j+1)\alpha} y_0(\theta T_0) x_0^{2(j+1)}(\theta T_0), \]

where \( \hat{a}_{i,\epsilon} = a_i - \hat{a}_{i,\epsilon}, \hat{b}_{j,\epsilon} = b_j - \hat{b}_{j,\epsilon} \) and \( i = 0, 1, 2, \ldots, k, j = 0, 1, 2, \ldots, l. \)

**Definition 1.** \( f \in C(R^n \times S^1, R^n). \) We call the system

\[ x' = f(x, t) \]

a reversible system if there is an involution \( G : R^n \rightarrow R^n \) such that

\[ DG \circ f(Gx, -t) = -f(x, t). \]

Such system is also called a reversible system with respect to \( G. \)

Clearly, \( f_1(\rho, -\theta, -t) = -f_1(\rho, \theta, t), f_2(\rho, -\theta, -t) = f_2(\rho, \theta, t). \) So Eq. (10) is a reversible system with respect to \( G : (\rho, \theta) \rightarrow (\rho, -\theta). \)

**4. Main propositions**

First of all, we introduce a space of function \( P(r) \) which is much similar to that in [15]. Let \( f(\lambda, \theta, t) : (l, +\infty) \times R^2 \rightarrow R \) be a function which is \( 1 \)-periodic in \( t \) as well as in \( \theta \) and possesses continuous partial derivatives \( D^j_\lambda D^l_\theta \) for each pair of nonnegative integers \( (j, l) \).

**Definition 2.** For a given constant \( r \in R, \) we say that \( f(\lambda, \theta, t) \in P(r) \), if for all nonnegative integers \( j \) and \( l, \) the inequalities

\[ \sum_{j=0}^{l} \sum_{i=[(n-1)/2]+1}^{n-1} \sum_{j=0}^{l} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{l} \sum_{i=[(n-1)/2]+1}^{n-1} \sum_{j=0}^{l} \sum_{i=0}^{[(n-1)/2]} \text{...} \]
Define the following domain

$$A_{\lambda_0} = \{ (\lambda, \theta, t) \mid \lambda \geq \lambda_0, (\theta, t) \in S^1 \times S^1 \}.$$ 

Let $d_1 = \max \{ 1 + (2l + 1)\alpha, 2n\alpha \}$, $d_2 = 2[(n + 1)/2]\alpha$, $d_3 = d_1 - 1$, $d_4 = d_2 - 1$. By using Lemma 3, we have

$$h_1(\lambda, -\theta, t) = -h_1(\lambda, \theta, t) = -h_1(\lambda, \theta, -t) \in P(d_1),$$

$$h_2(\lambda, -\theta, t) = -h_2(\lambda, \theta, t) \in P(d_2),$$

$$h_3(\lambda, -\theta, t) = h_3(\lambda, \theta, t) = h_3(\lambda, \theta, t) \in P(d_3),$$

$$h_4(\lambda, -\theta, t) = h_4(\lambda, \theta, t) \in P(d_4),$$

$$g_1(\lambda, -\theta, t) = -g_1(\lambda, \theta, t) \in P(d_1),$$

$$g_2(\lambda, -\theta, t) = g_2(\lambda, \theta, t) \in P(d_3).$$

(11)

**Proposition 1.** There exists a diffeomorphism depending periodically on $t$,

$$\Psi^1: \rho = \mu + U_1(\mu, \phi, t), \quad \theta = \phi,$$

such that $A_{\mu_+} \subset \Psi^1(\mathcal{A}_{\mu_0}) \subset A_{\mu_-}$ for some $\mu_+ < \mu_0 < \mu_-$, and Eq. (10) is transformed into

$$\begin{cases}
\mu' = \hat{h}_1(\mu, \phi, t) + \hat{h}_2(\mu, \phi, t) + \epsilon \hat{g}_1(\mu, \phi, t), \\
\phi' = c_0 \mu^{2\beta - 1} + \hat{h}_3(\mu, \phi, t) + \hat{h}_4(\mu, \phi, t) + \epsilon \hat{g}_2(\mu, \phi, t),
\end{cases}$$

(12)

where $\hat{h}_1 \in P(d_1), i = 1, 2, 3, 4$, $\hat{g}_1 \in P(d_1)$ and $\hat{g}_2 \in P(d_3)$ with

$$\hat{d}_1 = d_1 - \delta_1, \quad \hat{d}_i = d_i, \quad i = 2, 3, 4, \quad \delta_1 = \min\{2\beta - d_1, 2\beta - 1 - d_3\} \geq \alpha > 0.$$

Moreover

$$\hat{h}_1(\mu, -\phi, t) = -\hat{h}_1(\mu, \phi, t) = -\hat{h}_1(\mu, \phi, -t),$$

$$\hat{h}_3(\mu, -\phi, t) = \hat{h}_3(\mu, \phi, -t) = \hat{h}_3(\mu, \phi, t).$$

The system (12) is reversible with respect to $G : (\mu, \phi) \to (\mu, -\phi)$.
Proof. Set

$$\Phi^1: \mu = \rho + V_1(\rho, \theta, t), \quad \phi = \theta.$$ 

Under the transformation $\Phi^1$, we have

$$\frac{d\mu}{dt} = \frac{d\rho}{dt} + \frac{\partial V_1}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial V_1}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial V_1}{\partial t} \frac{dt}{dt} = h_1 + h_2 + g_1 + \frac{\partial V_1}{\partial \rho} (h_1 + h_2 + g_1) + \frac{\partial V_1}{\partial \rho} (c_0 \rho^{2\beta-1} + h_3 + h_4 + g_2) + \frac{\partial V_1}{\partial t}.$$ 

Since $h_1(\rho, -\theta, t) = -h_1(\rho, \theta, t), [h_1] = 0$. Thus, we can define $V_1$ by setting

$$V_1(\rho, \theta, t) = -\int_{0}^{\theta} \frac{h_1(\rho, s, t)}{c_0 \rho^{2\beta-1}} ds.$$ 

Clearly, $V_1(\rho, -\theta, t) = V_1(\rho, \theta, t) = V_1(\rho, \theta, -t)$ and $V_1 \in P(d_1 - 2\beta + 1)$. It follows from $V_1 \in P(d_1 - 2\beta + 1)$ that there exists the inverse of $\Phi^1$. Setting

$$\Psi^1 = (\Phi^1)^{-1}: \rho = \mu + U_1(\mu, \phi, t), \quad \theta = \phi,$n

then $U_1(\mu, -\phi, t) = U_1(\mu, \phi, -t) = U_1(\mu, \phi, t) \in P(d_1 - 2\beta + 1)$. It follows from the same arguments as those of Proposition 1 in [13] that

$$\hat{h}_1(\mu, \phi, t) = \frac{\partial V_1}{\partial \theta} h_3(\rho, \theta, t) + \frac{\partial V_1}{\partial \rho} h_1(\rho, \theta, t),$$

$$\hat{h}_2(\mu, \phi, t) = h_2(\rho, \theta, t) + \frac{\partial V_1}{\partial \theta} h_4(\rho, \theta, t) + \frac{\partial V_1}{\partial \rho} h_2(\rho, \theta, t) + \frac{\partial V_1}{\partial t},$$

$$\hat{h}_3(\mu, \phi, t) = c_0 \rho^{2\beta-1} - c_0 \mu^{2\beta-1} + h_3(\rho, \theta, t),$$

$$\hat{h}_4(\mu, \phi, t) = h_4(\rho, \phi, t),$$

$$\hat{g}_1(\mu, \phi, t) = g_1(\rho, \theta, t) + \frac{\partial V_1}{\partial \rho} g_1(\rho, \theta, t) + \frac{\partial V_1}{\partial \theta} g_2(\rho, \theta, t),$$

$$\hat{g}_2(\mu, \phi, t) = g_2(\rho, \phi, t).$$

We note that the transformed system (12) satisfies the assumptions of Proposition 1. Hence, there is an integer $i_0$ such that after $i_0$ successive applications of this proposition we find that the corresponding term $\hat{h}_1$ belongs to $P(d_2)$. We write the transformed system in the form

$$\begin{cases} \rho' = h_2(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t), \\ \theta' = c_0 \rho^{2\beta-1} + h_3(\rho, \theta, t) + h_4(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t), \end{cases} \quad (13)$$

where $h_i \in P(d_i)$ ($i = 2, 3, 4$), $g_1 \in P(d_1)$ and $g_2 \in P(d_3)$, $h_2(\rho, -\theta, -t) = -h_2(\rho, \theta, t)$, $h_3(\rho, -\theta, -t) = h_3(\rho, \theta, t)$, $h_4(\rho, -\theta, -t) = h_4(\rho, \theta, t)$, $g_1(\lambda, -\theta, -t) = -g_1(\lambda, \theta, t)$, $g_2(\lambda, -\theta, -t) = g_2(\lambda, \theta, t)$. Hence system (13) is reversible with respect to $G: (\mu, \phi) \rightarrow (\mu, -\phi)$.

Let us consider the following system:

$$\begin{cases} \rho' = h_1(\rho, t) + h_2(\rho, \theta, t) + h_5(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t), \\ \theta' = c_0 \rho^{2\beta-1} + h_3(\rho, \theta, t) + h_4(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t), \end{cases} \quad (14)$$

where $h_i \in P(a_i)$, $i = 1, 2, 3, 4, 5$, $g_1 \in P(d_1)$ and $g_2 \in P(d_3)$ with

$$a_1 = d_2, \quad a_2 = d_2, \quad a_3 = d_3, \quad a_4 = d_4, \quad a_5 < 0,$$

and $h_1(\rho, -t) = -h_1(\rho, t)$, $h_2(\rho, -\theta, -t) = -h_2(\rho, \theta, t)$, $h_3(\rho, -\theta, -t) = -h_3(\rho, \theta, t)$, $h_4(\rho, -\theta, -t) = h_4(\rho, \theta, t)$, $h_5(\rho, -\theta, -t) = -h_5(\rho, \theta, t)$, $g_1(\lambda, -\theta, -t) = -g_1(\lambda, \theta, t)$, $g_2(\lambda, -\theta, -t) = g_2(\lambda, \theta, t)$.

Clearly, system (14) is also reversible with respect to $G: (\mu, \phi) \rightarrow (\mu, -\phi)$.
Proposition 2. There exists a diffeomorphism

\[ \Psi^2 : \rho = \mu + U_2(\mu, \phi, t), \quad \theta = \phi, \]

such that \( A_{\mu_+} \subset \Psi^1(A_{\mu_0}) \subset A_{\mu_-} \) for some \( \mu_+ < \mu_0 < \mu_- \), and Eq. (14) is transformed into

\[
\begin{aligned}
\mu' &= \hat{h}_1(\mu, t) + \hat{h}_2(\mu, \psi, t) + \hat{h}_3(\mu, \psi, t), \\
\phi' &= c_0\mu^{2\beta - 1} + \hat{h}_4(\mu, \psi, t) + \hat{g}_2(\mu, \psi, t),
\end{aligned}
\]

(15)

where \( \hat{h}_i \in P(\hat{a}_i), i = 1, 2, 3, 4, g_1 \in P(d_1) \) and \( g_2 \in P(d_3) \) with \( \hat{a}_1 = a_1, \hat{a}_2 = a_2 - \delta_2, \hat{a}_3 = a_3, \hat{a}_4 < 0, \hat{a}_5 < 0 \) and \( \delta_2 = \min\{2\beta - a_1, 2\beta - 1 - a_3\} > 0 \).

The system (15) is reversible with respect to \( G : (\mu, \phi) \rightarrow (\mu, -\phi) \).

Proof. Define a transformation

\[ \Phi^2 : \mu = \rho + V_2(\rho, \theta, t), \quad \phi = \theta, \]

where

\[ V_2(\rho, \theta, t) = -\int_0^\theta \frac{h_2(\rho, s, t) - [h_2](\rho, t)}{c_0\rho^{2\beta - 1}} \, ds. \]

Clearly, \( V_2(\rho, -\theta, -t) = V_2(\rho, \theta, t) \in P(\alpha_2 - 2\beta + 1) \). It follows from \( V_2(\rho, \theta, t) \in P(\alpha_2 - 2\beta + 1) \) that there exists the inverse of \( \Phi^2 \). Set

\[ \Psi^2 = (\Phi^2)^{-1} : \rho = \mu + U_2(\mu, \phi, t), \quad \theta = \phi, \]

then \( U_2(\mu, -\phi, -t) = U_2(\mu, \phi, t) \in P(\alpha_2 - 2\beta + 1) \). The remaining proof is similar to that of Proposition 1, and the details are omitted. \( \square \)

We can see that system (13) is in the class of systems of the form (14) and therefore it satisfies the assumptions of Proposition 2. Hence, there exists an integer \( j_0 \) such that after \( j_0 \) successive applications of Proposition 2, we transform system (13) into a form such as

\[
\begin{aligned}
\rho' &= h_1(\rho, t) + h_2(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t), \\
\theta' &= c_0\rho^{2\beta - 1} + h_3(\rho, \theta, t) + h_4(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t),
\end{aligned}
\]

(16)

where \( h_i \in P(\alpha_i) \) for \( i = 1, 2, 3, 4, g_1 \in P(d_1) \) and \( g_2 \in P(d_3) \) with

\[ \alpha_1 < 1, \quad \alpha_2 < 0, \quad \alpha_3 < 2\beta - 1, \quad \alpha_4 < 0, \]

and \( h_1(\rho, t) = -h_1(\rho, t), h_2(\rho, -\theta, -t) = -h_2(\rho, \theta, t), h_3(\rho, -\theta, -t) = h_3(\rho, \theta, t), h_4(\rho, -\theta, -t) = h_4(\rho, \theta, t), g_1(\lambda, -\theta, -t) = -g_1(\lambda, \theta, t), g_2(\lambda, -\theta, -t) = g_2(\lambda, \theta, t) \). So system (16) is reversible with respect to \( G : (\mu, \phi) \rightarrow (\mu, -\phi) \).

Proposition 3. There exists a diffeomorphism

\[ \Psi^3 : \rho = \mu + U_3(\mu, \phi, t), \quad \theta = \phi, \]

such that \( A_{\mu_+} \subset \Psi^1(A_{\mu_0}) \subset A_{\mu_-} \) for some \( \mu_+ < \mu_0 < \mu_- \), and Eq. (16) is transformed into

\[
\begin{aligned}
\mu' &= \hat{h}_1(\mu, t) + \hat{h}_2(\mu, \psi, t) + \hat{g}_1(\mu, \psi, t), \\
\phi' &= c_0\mu^{2\beta - 1} + \hat{h}_3(\mu, \psi, t) + \hat{h}_4(\mu, \psi, t) + \hat{g}_2(\mu, \psi, t),
\end{aligned}
\]

(17)

where \( \hat{h}_i \in P(e_i) \) for \( i = 1, 2, 3, 4, g_1 \in P(d_1) \) and \( g_2 \in P(d_3) \) with

\[ e_1 = \alpha_1 - \delta_3, \quad \delta_3 = 1 - \alpha_1 > 0, \quad e_i = \alpha_i, \quad i = 2, 3, 4. \]

And the system (17) is reversible with respect to \( G : (\mu, \phi) \rightarrow (\mu, -\phi) \).
Proof. Define a transformation

\[ \Phi^3: \quad \mu = \rho + V_3(\rho, t), \quad \phi = \theta, \]

where

\[ V_3(\rho, t) = - \frac{1}{t} \int_0^t h_1(\rho, t) \, dt \in \mathcal{P}(\alpha_1). \]

As \( h_1 \) is odd and 1-periodic in \( t \), we can see that \( V_3(\rho, t) \) is 1-periodic and even in \( t \). Let \( \Psi^3 = (\Phi^3)^{-1}: \rho = \mu + U_3(\mu, \phi, t), \theta = \phi \). Then it is easy to see that \( U_3(\mu, -t) = U_3(\mu, t) \in \mathcal{P}(\alpha_1) \). Under the transformation \( \Psi^3 \), system (16) is transformed into Eq. (17), where

\[ \begin{align*}
\hat{h}_1 &= \frac{\partial U_3}{\partial \mu} h_1(\rho, \theta, t), \\
\hat{h}_2 &= h_2(\rho, \theta, t), \\
\hat{h}_3 &= c_0 \rho^{2\beta-1} - c_0 \mu^{2\beta-1} + h(\rho, \theta, t), \\
\hat{h}_4 &= h(\rho, \theta, t), \\
\hat{g}_1 &= \frac{\partial U_3}{\partial \mu} g_1(\rho, \theta, t) + g_1(\rho, \theta, t), \\
\hat{g}_2 &= g_2(\rho, \theta, t).
 \end{align*} \]

The further proof is similar to that of Proposition 1. \( \square \)

Now, it is easy to see that there exists an integer \( k_0 \) such that after \( k_0 \) successive applications of Proposition 3, the corresponding \( \hat{h}_1 \) belongs to \( P(\hat{\alpha}_1) \) with \( \hat{\alpha}_1 < 0 \). After applying the previous transformations, we can transform system (10) into

\[ \begin{align*}
\rho' &= h_1(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t), \\
\theta' &= c_0 \rho^{2\beta-1} + h(\rho, t) + h_3(\rho, \theta, t) + h_4(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t),
\end{align*} \tag{18} \]

where \( h_1 \in \mathcal{P}(\alpha_1), h \in \mathcal{P}(b), h_3 \in \mathcal{P}(\alpha_3), h_4 \in \mathcal{P}(\alpha_4), g_1 \in \mathcal{P}(d_1) \) and \( g_2 \in \mathcal{P}(d_3) \) with \( \alpha_1 < 0, \alpha_3 < b < 2\beta - 1 \) and \( \alpha_4 < 0 \). Hence system (18) is reversible with respect to \( G : (\mu, \phi) \to (\mu, -\phi) \).

Proposition 4. There exists a diffeomorphism

\[ \Psi^4: \quad \rho = \mu, \quad \theta = \phi + U_4(\mu, \phi, t), \]

where \( U_4 \in \mathcal{P}(\alpha_3 - 2\beta - 1) \). Under this transformation, Eq. (18) is transformed into

\[ \begin{align*}
\mu' &= \hat{h}_1(\mu, t) + \epsilon \hat{g}_1(\mu, \psi, t), \\
\phi' &= c_0 \mu^{2\beta-1} + \hat{h}(\mu, t) + \hat{h}_3(\mu, \psi, t) + \hat{h}_4(\mu, \psi, t) + \epsilon \hat{g}_2(\mu, \psi, t),
\end{align*} \tag{19} \]

where \( h \in \mathcal{P}(b), \hat{h}_3 \in \mathcal{P}(\hat{\alpha}_3), \hat{h}_4 \in \mathcal{P}(\hat{\alpha}_4), g_1 \in \mathcal{P}(d_1) \) and \( g_2 \in \mathcal{P}(d_3) \) with

\[ \hat{\alpha}_3 = \alpha_3 - \delta_4, \quad \delta_4 = 2\beta - 1 - \alpha_3 > 0, \quad \hat{\alpha}_4 = \max\{\alpha_4, 2\beta - \alpha_3 - 1\}. \]

Proof. Define a transformation

\[ \Phi^4: \quad \mu = \rho, \quad \phi = \theta + V_4(\rho, \theta, t), \]

where

\[ V_4(\rho, \theta, t) = - \frac{1}{\theta} \int_0^\theta \frac{h_3(\rho, s, t) - [h_3](\rho, t)}{c_0 \rho^{2\beta-1} + h(\rho, t)} \, ds \in \mathcal{P}(\alpha_3 - 2\beta - 1). \]

Let \( \Psi^4 = (\Phi^4)^{-1}: \rho = \mu, \theta = \phi + U_4(\mu, \phi, t) \). As in the proof in Proposition 1, we have \( U_4(\rho, -\theta, -t) = -U_4(\rho, \theta, t) \in \mathcal{P}(\alpha_3 - 2\beta - 1) \). Under this mapping, Eq. (18) can be transformed into (19), where
Moreover for every pair \((r, s)\)
\[\text{Proof.} \]

**Lemma 5.**
\[\hat{h}_1 = h_1(\rho, \theta, t), \quad \hat{h} = h(\rho, t) + \int_0^1 h_3(\rho, \theta, t) d\theta, \quad \hat{h}_3 = \frac{\partial V_4}{\partial \theta} h_3(\rho, \theta, t),\]
\[\hat{h}_4 = \frac{\partial V_4}{\partial \rho} h_1(\rho, \theta, t) + \frac{\partial V_4}{\partial \theta} h_4(\rho, \theta, t) + \frac{\partial V_4}{\partial t}, \quad \hat{g}_1 = g_1(\rho, \theta, t),\]
\[\hat{g}_2 = g_2(\rho, \theta, t) + \frac{\partial V_4}{\partial \rho} g_1(\rho, \theta, t) + \frac{\partial V_4}{\partial \theta} g_2(\rho, \theta, t).\]

This completes the proof of Proposition 4. \(\square\)

As a consequence of all the above discussion, there is a positive integer \(l_0\) such that after \(l_0\) successive applications of Proposition 4, the corresponding term \(h_3\) belongs to \(P(\hat{a}_3)\) with \(\hat{a}_3 < 0\).

5. The proof of Theorem 2

It follows from Section 4 that by using a series of diffeomorphisms, Eq. (10) is transformed into
\[
\begin{aligned}
\dot{\lambda}' &= h_1(\rho, \theta, t) + \epsilon g_1(\rho, \theta, t), \\
\dot{\theta}' &= c_0 \rho^{2\beta - 1} + h(\rho, t) + h_2(\rho, \theta, t) + \epsilon g_2(\rho, \theta, t),
\end{aligned}
\] (20)
where \(h_1, h_2 \in P(-\epsilon_0)\) with \(\epsilon_0 > 0\) and \(h \in P(\alpha_0)\) with \(0 < \alpha_0 < 2\beta - 1\), \(g_1 \in P(d_1)\) and \(g_2 \in P(d_3)\). Moreover, \(h_1(\rho, -\theta, -t) = -h_1(\rho, \theta, t), h(\rho, -t) = h(\rho, t), h_2(\rho, -\theta, -t) = h_2(\rho, \theta, t), g_1(\rho, -\theta, -t) = -g_1(\rho, \theta, t), g_2(\rho, -\theta, -t) = g_2(\rho, \theta, t)\). Thus system (20) is reversible with respect to \(G : (\mu, \phi) \to (\mu, -\phi)\).

Let \(\rho = c_0 \rho^{2\beta - 1} + \int_0^1 h(\rho, t) d\tau\), then we have
\[
\begin{aligned}
\dot{\lambda}' &= \hat{h}_1(\lambda, \theta, t) + \epsilon \hat{g}_1(\lambda, \theta, t), \\
\dot{\theta}' &= \lambda + \hat{h}(\lambda, t) + \hat{h}_2(\lambda, \theta, t) + \epsilon \hat{g}_2(\lambda, \theta, t),
\end{aligned}
\] (21)
where \(h_1, h_2 \in P(-\epsilon_1)\) with \(\epsilon_1 > 0\) and \(h \in P(\beta_0)\) with \(\beta_0 < 0\). Moreover, system (21) is reversible with respect to \(G : (\mu, \phi) \to (\mu, -\phi)\). Setting \(\rho_m = 2m, \epsilon = \rho_m^{-2}\), we have the following lemma.

**Lemma 4.** The transformed Eq. (21) can be written in the form
\[
\begin{aligned}
\dot{\lambda}' &= \hat{f}(\lambda, \theta, t), \\
\dot{\theta}' &= \lambda + \hat{h}_0(\lambda, t) + \hat{g}(\lambda, \theta, t),
\end{aligned}
\] (22)
where
\[
|D^r_x D^s_\theta \hat{f}(\lambda, \theta, t)|, |D^r_x D^s_\theta \hat{g}(\lambda, \theta, t)| < \rho_m^{-1/n},
\] (23)
for \((\lambda, \theta, t) \in A_{\rho_m} A_{\rho_m+1}\) and for all nonnegative integers \(r, s\) and sufficiently large \(m\).

Since (23), one can easily check that the solutions of (22) exist for all \(t \in [0, 1]\) if the initial value \(\lambda(0) = \lambda \in [\rho_m, \rho_{m+1}]\) is sufficiently large.

**Lemma 5.** The Poincaré map of Eq. (22) takes the form
\[
J^1: \quad \lambda_1 = \lambda + f(\lambda, \theta), \quad \theta_1 = \theta + g(\lambda, \theta).
\] (24)
Moreover for every pair \((r, s)\),
\[
|D^r_x D^s_\theta f(\lambda, \theta)|, |D^r_x D^s_\theta g(\lambda, \theta)| < \rho_m^{-1/n}
\] (25)
and \(G \circ J^1 \circ G = J^{-1}\).

**Proof.** Set
\[
r(\lambda, t) = t \lambda + \int_0^t \hat{h}_0(\lambda, \tau) d\tau.
\]
and note that the flow $(\lambda(t), \theta(t))$ with $(\lambda(t), \theta(t)) = (\lambda, \theta)$ takes the form

$$
\lambda(t) = \lambda + A(\lambda, \theta, t), \quad \theta(t) = \theta + r(\lambda, t) + B(\lambda, \theta, t).
$$

Thus we have

$$
A(\lambda, \theta, t) = \int_0^t \hat{f}(\lambda + A, \theta + r + B, s) \, ds,
$$

$$
B(\lambda, \theta, t) = \int_0^t \int_0^s \hat{f}(\lambda + A, \theta + r + B, \tau) \, d\tau + \int_0^t \int_0^1 \frac{\hat{h}_0}{\partial \lambda} \left( \lambda + \tau A, s \right) A \, d\tau \, ds + \int_0^t \hat{g}(\lambda + A, \theta + r + B, s) \, ds.
$$

One verifies easily that for $\lambda \geq \lambda_0$ these equations have a unique solution in the space $|A|, |B| \leq 1$ by using the contraction principle, moreover $A$ and $B$ are smooth functions. The required estimates in (25) can be inductively verified from the equations for $A$ and $B$. Moreover, one can verify easily that it is reversible with $G : (\lambda, \theta) \to (\lambda, -\theta)$. The proof is completed.

Now we give the proof of Theorem 2.

**Proof of Theorem 2.** It follows from Lemma 5 that the Poincaré map $P$ satisfies the conditions of Theorem 1.1 in [16]. It follows that for any sufficiently large $\omega$ satisfying

$$
|\omega - \frac{p}{q}| \geq \frac{1}{2} |q|^{-2-\delta}, \quad \delta > 0.
$$

There exists an embedding $\Psi : S^1 \to A_{\lambda_0}$ of a circle, which is $C^1$ close to the injection map $\sigma$ of the circle $\{\omega\} \times S^1 \to A_{\lambda_0}$, and which is invariant under the map. Moreover, on this invariant curve, the map $J$ is conjugated to a rotation with rotation number $\omega$,

$$
J \circ \Psi(s) = \Psi(s+\omega), \quad \forall s \in S^1.
$$

The solutions of (22) starting at time $t = 0$ on the invariant curve determine a 1-periodic cylinder in the space $(\lambda, \theta, t) \in A_{\lambda_0} \times R^1$. Since the vectorfield $X(\lambda, \theta, t)$ defined by the right-hand side of Eq. (22) is 1-periodic in time, its associated extended phase space is $A_{\lambda_0} \times R^1$. Let $F_1^t$ with $F_0^t = \text{id}$, be the flow of the time-independent vectorfield $(X, 1)$ on $A_{\lambda_0} \times R^1$, and define the embedded torus $\Psi^* : T^2 \to A_{\lambda_0} \times R^1$ by

$$
\Psi^*(s, \tau) = F_1^t (\tilde{\Psi}(s - \tau \omega), 0) = (F_1^t \circ \tilde{\Psi}(s - \tau \omega), \tau).
$$

So, we have that $\Psi^*(s+1, \tau) = \Psi^*(s, \tau+1) = \Psi^*(s, \tau)$ and $F_1^t \circ \Psi^*(s, \tau) = \Psi^*(s+\omega t, \tau+t)$. Hence, the flow on the torus $\Psi^*(T^2)$ is quasiperiodic having frequencies $(\omega, 1)$. This completes the proof.

**Remark 3.** In fact, one could prove that the statements of Theorems 1–2 are also valid for Eq. (5).

**References**