Abstract

In the present paper, we show that the Ringel–Hall algebra of a finitary algebra over a finite field satisfies fundamental relations in a more general setting. By twisting the multiplication, we obtain the quantum Serre relations. Finally, certain relations between Ringel–Hall algebras of an algebra and its factor algebras are discussed.

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1. Introduction

Ringel–Hall algebras of finitary rings were introduced by Ringel [12,14] in order to deal with possible filtrations of modules with fixed factors. It turns out that Ringel–Hall algebra approach provides a nice framework for the realization of quantized enveloping algebras and Kac–Moody algebras, see, e.g., [5,8,9,11–14]. One of the key features in this approach is that Ringel–Hall algebras satisfy the so-called fundamental relations, which are similar to the quantum Serre relations—the defining relations for quantized enveloping algebras. Later, it was shown in [16] that by twisting the multiplication in Ringel–Hall algebras of hereditary algebras, fundamental relations become the quantum Serre relations themselves.
Let $A$ be a finitary algebra over a finite field $\mathbb{F}_q$. The integral Ringel–Hall algebra $\mathcal{H}(A)$ of $A$ is by definition the free abelian group with basis $u_{[M]}$, indexed by isoclasses $[M]$ of finite dimensional $A$-modules $M$. The multiplication is given by
\[
 u_{[M]}u_{[N]} = \sum_{[L]} F^L_{M,N} u_{[L]},
\]
where $F^L_{M,N}$ is the number of submodules $X$ of $L$ such that $X \cong N$ and $L/X \cong M$. Let $S_i$ and $S_j$ be two finite dimensional simple $A$-modules. Let $\text{Ext}_A^1(S_i, S_j) = 0$. Ringel showed in [12] that under the assumption $\text{Ext}_A^1(S_i, S_j) = 0$, we have in $\mathcal{H}(A)$, 
\[
 \sum_{r=0}^{n} (-1)^{r} q_i^{\binom{r}{2}} \binom{n}{r}_{q_i} u_{[S_i]}^r u_{[S_j]}^{n-r} = 0,
\]
where $q_i = |\text{End}_A(S_i)|$ and $n = 1 + \dim_{\text{End}_A(S_i)} \text{Ext}_A^1(S_j, S_i)$, and under the assumption $\text{Ext}_A^1(S_j, S_i) = 0$, we have
\[
 \sum_{r=0}^{m} (-1)^{r} q_i^{\binom{r}{2}} \binom{m}{r}_{q_i} u_{[S_i]}^m u_{[S_j]}^{m-r} = 0,
\]
where $m = 1 + \dim \text{Ext}_A^1(S_i, S_j)_{\text{End}_A(S_i)}$. These are called the fundamental relations.

In the present paper we show that to obtain the fundamental relations, the assumption $\text{Ext}_A^1(S_i, S_j) = 0$ or $\text{Ext}_A^1(S_j, S_i) = 0$ is indeed not necessary. Then, by twisting the multiplication of the Ringel–Hall algebra of an arbitrary finitary algebra, the quantum Serre relations are also obtained. We further show that Ringel–Hall algebras satisfy the higher order fundamental relations which give rise to the higher order quantum Serre relations studied in [9]. Finally, some relations between the Ringel–Hall algebra of a finitary algebra and Ringel–Hall algebras of its factor algebras are studied. More precisely, if $B$ is a factor algebra of $A$, then the Ringel–Hall algebra $\mathcal{H}(B)$ of $B$ is a factor algebra of $\mathcal{H}(A)$. Also, the Lie algebra associated with $B$ is a factor algebra of the Lie algebra associated with $A$.

2. Ringel–Hall algebras and fundamental relations

In this section we recall from [12] the definition of the Ringel–Hall algebra $\mathcal{H}(A)$ of a finitary algebra $A$ and then show that $\mathcal{H}(A)$ satisfies fundamental relations. This is a generalization of [12, Proposition]. Throughout, $\mathbb{F}_q$ denotes a finite field of $q$ elements.

We first give some useful facts involving Gaussian polynomials. Let $\mathbb{Z}[q]$ be a polynomial ring in an indeterminate $q$. For each $d \geq 1$, define
\[
 [d]_r = [1][2] \cdots [d] \quad \text{with} \quad [r] = \frac{q^r - 1}{q - 1},
\]
and set $[0]_r = 0$ by convention. For $0 \leq r \leq d$, set
\[
 \binom{d}{r} = \frac{[d]_r}{[r]! [d-r]_r}.
\]
A direct calculation shows that for \( d \geq 0 \) and \( 0 \leq r \leq d \),

\[
\begin{bmatrix} d + 1 \cr r \end{bmatrix} = \begin{bmatrix} d \cr r \end{bmatrix} + q^{d + 1 - r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} = q^r \begin{bmatrix} d \cr r \end{bmatrix} + \begin{bmatrix} d \cr r - 1 \end{bmatrix}.
\]

The following lemma is well known (see [10, p. 26]).

**Lemma 2.1.** For any positive integer \( d \), we have in \( \mathbb{Z}[q] \),

\[
\sum_{r=0}^{d} (-1)^r q^{\frac{(r-1)(r-2)}{2}} \begin{bmatrix} d \cr r \end{bmatrix} = 0.
\]

The following result is a generalization of the above lemma.

**Lemma 2.2.** For any integers \( d \geq 1 \) and \( 0 \leq c \leq d - 1 \), we have

\[
\sum_{r=0}^{d} (-1)^r q^{\frac{(r-c)(r-c-1)}{2}} \begin{bmatrix} d \cr r \end{bmatrix} = 0. \tag{2.2.1}
\]

**Proof.** We prove (2.2.1) by induction on \( d \). For simplicity, we write

\[ f_{d,c} = \sum_{r=0}^{d} (-1)^r q^{\frac{(r-c)(r-c-1)}{2}} \begin{bmatrix} d \cr r \end{bmatrix}. \]

If \( d = 1 \), then \( c = 0 \) and \( f_{1,0} = 1 - 1 = 0 \), as required. Now assume the formula (2.2.1) holds for \( d \geq 1 \), that is, \( f_{d,c} = 0 \) for \( 0 \leq c \leq d - 1 \). We need to prove \( f_{d+1,c} = 0 \) whenever \( 0 \leq c \leq (d+1) - 1 = d \). If \( 0 \leq c \leq d - 1 \), we have

\[
f_{d+1,c} = \sum_{r=0}^{d+1} (-1)^r q^{\frac{(r-c)(r-c-1)}{2}} \begin{bmatrix} d + 1 \cr r \end{bmatrix}
\]

\[
= q^{\frac{c(c+1)}{2}} + \sum_{r=1}^{d} (-1)^r q^{\frac{(r-c)(r-c-1)}{2}} \begin{bmatrix} d + 1 \cr r \end{bmatrix} + \sum_{r=1}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} + q^{\frac{(d+1-c)(d-c)}{2}}
\]

\[
= q^{\frac{c(c+1)}{2}} + \sum_{r=1}^{d} (-1)^r q^{\frac{(r-c)(r-c-1)}{2}} \begin{bmatrix} d \cr r \end{bmatrix} + q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} + \sum_{r=1}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} + \sum_{r=1}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix}
\]

\[
= f_{d,c} + \sum_{r=0}^{d+1} (-1)^{r+1} q^{\frac{(r-c)(r-c-1)}{2} + (d-c)} \begin{bmatrix} d \cr r \end{bmatrix} + \sum_{r=0}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} + \sum_{r=1}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix}
\]

\[
= f_{d,c} + \sum_{r=0}^{d+1} (-1)^{r+1} q^{\frac{(r-c)(r-c-1)}{2} + (d-c)} \begin{bmatrix} d \cr r \end{bmatrix} + \sum_{r=0}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix} + \sum_{r=1}^{d} (-1)q^{d+1-r} \begin{bmatrix} d \cr r - 1 \end{bmatrix}.
\]
\[= f_{d,c} - q^{d-c} f_{d,c} = (1 - q^{d-c}) f_{d,c} = 0 \quad \text{since } c \leq d - 1.\]

It remains to show that \(f_{d,1,d} = 0\) for \(d \geq 1\). We again proceed induction on \(d\). In case \(d = 1\), we have

\[f_{2,1} = \sum_{r=0}^{2} (-1)^r q^{\frac{(r-1)(r-2)}{2}} \left[\begin{array}{c} 2 \\ r \end{array}\right] = q - (q + 1) + 1 = 0.\]

Let now \(d \geq 2\). Then we deduce that

\[f_{d+1,1,d} = \sum_{r=0}^{d+1} (-1)^r q^{\frac{(r-d)(d-d-1)}{2}} \left[\begin{array}{c} d + 1 \\ r \end{array}\right] \]

\[= q^{\frac{d(d+1)}{2}} + \sum_{r=1}^{d} (-1)^r q^{\frac{(r-d)(d-d-1)}{2}} \left[\begin{array}{c} d + 1 \\ r \end{array}\right] + (-1)^{d+1}\]

\[= q^{\frac{d(d+1)}{2}} + \sum_{r=1}^{d} (-1)^r q^{\frac{(r-d)(d-d-1)}{2}} \left( q^r \left[\begin{array}{c} d \\ r \end{array}\right] + \left[\begin{array}{c} d \\ r - 1 \end{array}\right] \right) + (-1)^{d+1}\]

\[= \left( q^{\frac{d(d+1)}{2}} + \sum_{r=1}^{d} (-1)^r q^{\frac{(r-d)(d-d-1)}{2}} \left[\begin{array}{c} d \\ r \end{array}\right] \right)\]

\[+ \left( \sum_{r=1}^{d} (-1)^r q^{\frac{(r-d)(d-d-1)}{2}} \left[\begin{array}{c} d \\ r - 1 \end{array}\right] + (-1)^{d+1} \right)\]

\[= q^d f_{d,d-1} - f_{d,d-1} = (q^d - 1) f_{d,d-1}.\]

Thus, by the induction hypothesis, we get that \(f_{d+1,d} = (q^d - 1) f_{d,d-1} = 0\). This finishes the proof. \(\square\)

Let \(A\) be an \(\mathbb{F}_q\)-algebra. By \(A\)-\textbf{mod} we denote the category of finite dimensional left \(A\)-modules. For \(M, N_1, \ldots, N_t \in A\)-\textbf{mod}, let \(F_{N_1, \ldots, N_t}^M\) be the number of the filtrations

\[M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0,\]

such that \(M_i/M_{i-1} \cong N_i\) for all \(1 \leq i \leq t\).

Now let \(A\) be a finitary \(\mathbb{F}_q\)-algebra, i.e., for any \(M, N \in A\)-\textbf{mod}, \(\text{Ext}_A^1(M, N)\) is finite dimensional; see [14]. For each \(M \in A\)-\textbf{mod}, we denote by \([M]\) the isoclass of \(M\). Following [12], the integral Ringel–Hall algebra \(\mathcal{H}(A)\) of \(A\) is the free abelian group with basis \([u_{[M]}] : M \in A\)-\textbf{mod}\) indexed by the set of isoclasses of \(A\)-modules, with multiplication given by

\[u_{[M]} u_{[N]} = \sum_{[L]} F_{M,N}^L u_{[L]} \quad \text{for all } M, N \in A\)-\textbf{mod}.\]

It is easy to see that \(\mathcal{H}(A)\) is an associative algebra with identity \(1 = u_0\), where 0 denotes the isoclass of the trivial \(A\)-module 0.
Let $S_i, i \in I$, be a complete set of simple modules in $A\text{-mod}$. Thus, for each $i \in I$, the endomorphism algebra $D_i := \text{End}_A(S_i)$ of $S_i$ is a finite field extension of $\mathbb{F}_q$. For $i, j \in I$, we consider the $D_i$-$D_j$-bimodule $\text{Ext}_A^1(S_j, S_i)$ and $D_j$-$D_i$-bimodule $\text{Ext}_A^1(S_i, S_j)$ and define

$$c_{i,j}' = -\dim D_i \text{Ext}_A^1(S_j, S_i), \quad c_{i,j}'' = -\dim \text{Ext}_A^1(S_i, S_j)D_i.$$ 

We finally define a matrix $C_A = (c_{i,j})_{i,j \in I}$ by setting $c_{i,j} = 2\delta_{i,j} + (c_{i,j}' + c_{i,j}'')$, where $\delta_{i,j}$ is the Kronecker symbol.

**Remark 2.3.** If $I$ is finite and $c_{i,i} = 2$ for all $i \in I$, then $C_A$ is a symmetrizable generalized Cartan matrix. In general, $C_A$ is a Borcherds–Cartan matrix in the sense of [1], which defines the so-called generalized Kac–Moody algebra.

Given a polynomial $f(q)$ in $\mathbb{Z}[q]$ and a complex number $a \in \mathbb{C}$, we write $f(q)_a$ for $f(a)$.

For each $i \in I$, we let $q_i = |D_i|$ and write $u_i = u_{[S_i]}$ in $\mathcal{H}(A)$. The subalgebra $C(A)$ of $\mathcal{H}(A)$ generated by $u_i, i \in I$, is called the composition algebra of $A$.

**Theorem 2.4.** Let $i, j \in I$ with $i \neq j$ and suppose $c_{i,i} = 2$, i.e., $\text{Ext}_A^1(S_i, S_i) = 0$. Then we have

$$\sum_{r=0}^{n} (-1)^r q_i^{(r+c_{i,j}')(r+c_{i,j}'-1)} \frac{[n]}{[r]} \frac{[n-r]}{[q_i]} u_i^r u_j u_i^{n-r} = 0,$$

where $n = 1 - c_{i,j}$.

**Proof.** For simplicity, we write $c_1 = -c_{i,j}'$ and $c_2 = -c_{i,j}''$. Then $n = 1 + c_1 + c_2$. By the multiplication in $\mathcal{H}(A)$, for each $0 \leq r \leq n$, we have

$$u_i^r u_j u_i^{n-r} = \left[ r \right]_{q_i} \left[ n-r \right]_{q_i} u_{[rS_i]} u_{[S_j]} u_{[(n-r)S_i]} = \left[ r \right]_{q_i} \left[ n-r \right]_{q_i} \sum_{[M]} f_r(M) u_{[M]},$$

where $f_r(M) = F_r^{M}_{rS_i, S_j, (n-r)S_i}$ is the number of the filtrations of $M$

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0$$

such that $M/M_1 \cong rS_i$, $M_1/M_2 \cong S_j$, and $M_2 \cong (n-r)S_i$. Thus, to show the formula (2.4.1), it suffices to show that for each $A$-module $M$,

$$f(M) := \sum_{r=0}^{n} (-1)^r q_i^{(r+c_2)(r+c_2-1)} \frac{[n]}{[r]} \frac{[n-r]}{q_i} f_r(M)$$

$$= \left[ n \right]_{q_i} \sum_{r=0}^{n} (-1)^r q_i^{(r+c_2)(r+c_2-1)} f_r(M) = 0.$$
Let us fix an \( A \)-module \( M \). If \( fr(M) \neq 0 \) for some \( 0 \leq r \leq n \), that is, \( M \) has a filtration of the form (2.4.2), then the Loewy length of \( M \) is at most 3. Since \( M \) has a unique composition factor \( S_j \) and \( \text{Ext}^1_A(S_i, S_i) = 0 \), \( M \) can be decomposed into

\[
M = N \oplus dS_i,
\]

where \( d \geq 0 \), and \( N \) is indecomposable with a composition factor \( S_j \). Hence, \( N \) admits a filtration

\[
N = N_0 \supseteq N_1 \supseteq N_2 \supseteq N_3 = 0
\]

such that \( N/N_1 \cong d_2S_i, N_1/N_2 \cong S_j, \) and \( N_2 \cong d_1S_i \), where \( d_1 \geq 0 \) and \( d_2 \geq 0 \) satisfy \( d_1 + d_2 + d = n \). Moreover, \( \text{top}N_1 = S_j \) and \( \text{soc}(N/N_2) = S_j \). Thus, there are two exact sequences

\[
\begin{align*}
0 & \longrightarrow d_1S_i \longrightarrow N_1 \longrightarrow S_j \longrightarrow 0, \quad (2.4.3) \\
0 & \longrightarrow S_j \longrightarrow N/N_2 \longrightarrow d_2S_i \longrightarrow 0. \quad (2.4.4)
\end{align*}
\]

Applying \( \text{Hom}_A(-, S_i) \) to the sequence (2.4.3) yields the long exact sequence

\[
\begin{array}{c}
\cdots \longrightarrow \text{Hom}_A(N_1, S_i) = 0 \longrightarrow \text{Hom}_A(d_1S_i, S_i) \longrightarrow \text{Ext}^1_A(S_j, S_i) \\
\end{array}
\]

Comparing dimensions over \( D_i = \text{End}_A(S_i) \) shows that

\[
c_1 - d_1 = \dim_{D_i} \text{Ext}^1_A(S_j, S_i) - \dim_{D_i} \text{Hom}_A(d_1S_i, S_i)
= \dim_{D_i} \text{Ext}^1_A(N_1, S_i) \geq 0,
\]

thus \( d_1 \leq c_1 \). On the other hand, by applying \( \text{Hom}_A(S_i, -) \) to (2.4.4), we get the following long exact sequence

\[
\begin{array}{c}
\cdots \longrightarrow \text{Hom}_A(S_i, N/N_2) = 0 \longrightarrow \text{Hom}_A(S_i, d_2S_i) \longrightarrow \text{Ext}^1_A(S_i, S_j) \\
\end{array}
\]

Again, comparing dimensions over \( D_i = \text{End}_A(S_i) \) gives that

\[
c_2 - d_2 = \dim \text{Ext}^1_A(S_i, S_j)_{D_i} - \dim \text{Hom}_A(S_i, d_2S_i)_{D_i}
= \dim \text{Ext}^1_A(S_i, N/N_2)_{D_i} \geq 0.
\]

Hence, \( d_2 \leq c_2 \).

Clearly, if \( M \) has a filtration of the form (2.4.2), then it is determined by a submodule \( X \) of \( dS_i \) isomorphic to \( (n - r - d_1)S_i \) with \( M_2 = N_2 \oplus X \) and \( M_1 = N_1 \oplus X \). Thus, a necessary condition for \( fr(M) \neq 0 \) is that \( 0 \leq n - r - d_1 \leq d \), i.e.,

\[
d_2 = n - d_1 - d \leq r \leq n - d_1 = d_2 + d.
\]
Moreover, in each of these cases, \( f_r(M) \) is the number of submodules \( X \) of \( dS_i \) satisfying \( X \cong (n - r - d_1)S_i \). Thus, \( f_r(M) = \left[ \frac{d}{n - r - d_1} \right] q_i \). Consequently,

\[
f(M) = [n]^{d + d}_{q_i} \sum_{r = d_2}^{d + d} (-1)^r q_i^{\frac{(r - c_2)(r - c_2 - 1)}{2}} \left[ \frac{d}{n - r - d_1} \right] q_i
\]

\[
= (-1)^{d_2} [n]^{d}_{q_i} \sum_{t = 0}^{d} (-1)^t q_i^{\frac{(t - c_2)(t - c_2 - d_2 - 1)}{2}} \left[ \frac{d}{d - t} \right] q_i \quad \text{(setting } t = r - d_2) .
\]

Since \( n = 1 + c_1 + c_2 = d + d_1 + d_2, d_1 \leq c_1 \) and \( d_2 \leq c_2 \), we get

\[
0 \leq c_2 - d_2 = d + d_1 - c_1 - 1 \leq d - 1.
\]

It follows from Lemma 2.2 that \( f(M) = 0 \). This finishes the proof. □

**Remarks 2.5.**

1. The fundamental relations obtained in [12, Proposition] are exactly the formula (2.4.1) for the cases \( c'_{i,j} = 0 \) and \( c''_{i,j} = 0 \). In some sense, the theorem means that the fundamental relations are “universal.”

2. The theorem can be formulated for the Hall algebra of a finitary and skeletally small exact \( F_q \)-category (see the definition in [7] and [17]).

By \( n(A) \) we denote the subgroup of \( \mathcal{H}(A) \) generated by \( u_{[M]} \) for \( M \) indecomposable. Clearly, \( n(A) \) is not closed under multiplication. However, we have the following lemma (see [7,11,15]).

**Lemma 2.6.** Over the ring \( \mathbb{Z}/(q - 1) \), \( n(A) \) is a Lie subalgebra of \( \mathcal{H}(A) \), i.e., \( n(A)/(q - 1)n(A) \) is a Lie subalgebra of \( \mathcal{H}(A)/(q - 1)\mathcal{H}(A) \), where the Lie bracket is the commutator \( [u_{[M]}, u_{[N]}] = u_{[M]}u_{[N]} - u_{[N]}u_{[M]} \).

Applying Theorem 2.4 gives the following

**Proposition 2.7.** Let \( i, j \in I \) with \( i \neq j \) and suppose \( c_{i,i} = 2 \). Then we have in \( n(A)/(q - 1) \),

\[
(ad u_i)^{1-c_{i,j}} u_j = 0.
\]

Now let \( A \) be a finite dimensional \( F_q \)-algebra. Thus, \( I \) is a finite set. For a simple \( A \)-module \( S_i \), a finite field extension \( K \) of \( F_q \) is said to be conservative for \( S_i \) if \( S_i^K := S_i \otimes_{F_q} K \) is again a simple module over \( A^K := A \otimes_{F_q} K \). Define

\[
\Omega = \{ K \mid F_q \subseteq K \subseteq \overline{F}_q \text{ is a finite field extension and conservative for all } S_i, i \in I \}.
\]

We claim that \( \Omega \) is an infinite set. Indeed, by [2, §7], for each \( i \in I \) and each finite field extension \( K \) of \( F_q, S_i^K \) is a semisimple \( A^K \)-module and

\[
\text{End}_{A^K}(S_i^K) \cong \text{End}_A(S_i) \otimes_{F_q} K.
\]
Thus, $S^K_i$ is a simple $A^K$-module if and only if $\text{End}_A(S_i) \otimes_{\mathbb{F}_q} K$ is again a finite field extension of $\mathbb{F}_q$. Since $\text{End}_A(S_i)$ is a finite field extension of $\mathbb{F}_q$, it is isomorphic to $\mathbb{F}_{q^e}$ for some $e_i \geq 1$. Let $d = \dim_{\mathbb{F}_q} K$, i.e., $K \cong \mathbb{F}_{q^d}$. It is known that the tensor product $\mathbb{F}_{q^e} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^f}$ is a finite field extension of $\mathbb{F}_q$ if and only if $d$ is coprime to $e_i$. In other words, $K$ is conservative for $S_i$ if and only if $d$ is coprime to $e_i$. Therefore, $\Omega$ is an infinite set.

Moreover, for each $K \in \Omega$, $\{S^K_i = S_i \otimes_{\mathbb{F}_q} K \mid i \in I\}$ forms a complete set of simple $A^K$-modules. And the matrix $C_{A^K}$ associated with $A^K$ coincides with $C_A$.

Consider the direct product

$$\Pi = \prod_{K \in \Omega} n(A^K)_{(|K|−1)},$$

which is an abelian group (i.e., a $\mathbb{Z}$-module) with componentwise addition. Also, using componentwise Lie bracket, $\Pi$ becomes a Lie algebra over $\mathbb{Z}$ (or a Lie ring). For each $i \in I$, we write $u_{i,K} = u_{\{S^K_i\}} \in n(A^K)_{(|K|−1)}$ and set $\tilde{u}_i = (u_{i,K})_{K \in \Omega} \in \Pi$. Following [11, 4.4], the Lie subalgebra of $\Pi$ generated by $\tilde{u}_i$, $i \in I$, is called the degenerated composition Lie algebra of $A$. We denote it by $\tilde{n}_c(A)$. Proposition 2.7 implies that for $i \neq j \in I$ with $c_{i,j} = 2$, we have in $\tilde{n}_c(A)$,

$$(\text{ad} \tilde{u}_i)^{1−c_{i,j}} \tilde{u}_j = 0.$$

Consequently, the Lie algebra $\tilde{n}_c(A)$ satisfies the Serre relations.

3. Twisted Ringel–Hall algebras and quantum Serre relations

In this section we define the twisted Ringel–Hall algebras of arbitrary finitary $\mathbb{F}_q$-algebras and show that they satisfy the quantum Serre relations.

As in the previous section, let $A$ be a finitary $\mathbb{F}_q$-algebra and let $\{S_i \mid i \in I\}$ be a complete set of simple modules in $A\text{-mod}$.

The Grothendieck group $K_0(A)$ of $A\text{-mod}$ is by definition the quotient $F/R$, where $F$ is the free abelian group with basis elements the isoclasses of modules in $A\text{-mod}$ and $R$ is the subgroup generated by $[L] − [M] − [N]$ for all short exact sequences $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$ in $A\text{-mod}$. Then it is easily seen that $K_0(A)$ identifies with the free abelian group $\mathbb{Z}[I]$ with basis $I$. Given a module $M$ in $A\text{-mod}$, we denote by $\dim M$ the image of $M$ in $K_0(A)$, called the dimension vector of $M$. From the definition, if $\dim M = \sum_{i \in I} x_i i$, then $x_i$ is the number of composition factors isomorphic to $S_i$ in a composition series of $M$. We define a bilinear form $\langle−,−\rangle: K_0(A) \times K_0(A) \rightarrow \mathbb{Z}$ by setting for $M, N \in A\text{-mod}$,

$$\langle \dim M, \dim N \rangle = \sum_{i \in I} \varepsilon_i x_i y_i - \sum_{i,j \in I} x_i y_j \dim_{\mathbb{F}_q} \text{Ext}_A^1(S_i, S_j),$$

where $\dim M = \sum_{i \in I} x_i i$, $\dim N = \sum_{i \in I} y_i i$, and $\varepsilon_i = \dim_{\mathbb{F}_q} \text{End}_A(S_i)$.

Remark 3.1. If $A$ is hereditary, i.e., $\text{Ext}_A^2(M, N) = 0$ for all $M, N \in A\text{-mod}$, then the form $\langle−,−\rangle$ is just the Euler form associated with $A$. In general, it is not the Euler form of $A$.

Let $\mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate $v$. For $n \geq 0$ and $0 \leq t \leq n$, let
\[ [n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1}, \]
\[ [n]^! = \prod_{r=1}^{n} [r], \quad \text{and} \quad \begin{bmatrix} n \\ t \end{bmatrix} = \frac{[n]^!}{[t]^! [n-t]^!}. \]

For each polynomial \( f(v) \) in \( \mathbb{Z}[v, v^{-1}] \) and a complex number \( a \in \mathbb{C} \), we write \( f(v)_a \) for \( f(a) \).

**Definition 3.2.** The twisted Ringel–Hall algebra \( \mathcal{H}^n(A) \) of \( A \) is the free module over \( \mathbb{Z}[v, v^{-1}] \) with basis \( \{ u_M \mid M \in A\text{-mod} \} \); and the multiplication is given by
\[ u_M \ast u_N = v^{\langle \dim M, \dim N \rangle} \sum_{[L]} \mathcal{F}_{M,N,L} u_L \]
for all \( M, N \in A\text{-mod} \).

The subalgebra of \( \mathcal{H}^n(A) \) generated by \( u_i = u_{[S_i]} \), \( i \in I \), is called the (twisted) composition algebra of \( A \); it is denoted by \( C^*(A) \).

For each \( m \geq 1 \) and each \( i \in I \), we set
\[ u^*_m = u_i \ast u_i \ast \cdots \ast u_i \in \mathcal{H}^n(A). \]

**Theorem 3.3.** Let \( A \) be a finitary \( \mathbb{F}_q \)-algebra and let \( C_A = (c_{i,j})_{i,j \in I} \) be the associated matrix. Then for \( i \neq j \in I \) with \( c_{i,i} = 2 \), we have in \( \mathcal{H}^n(A) \) (or \( C^*(A) \)),
\[ \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} v_i^{r} \ast u_j \ast u_i^{*(n-r)} = 0, \]
where \( n = 1 - c_{i,j} \).

**Proof.** Let
\[ c_1 = \dim_{\text{End}_A(S_j)} \text{Ext}^1_A(S_j, S_i), \quad c_2 = \dim \text{Ext}^1_A(S_i, S_j)_{\text{End}_A(S_i)}. \]
Then
\[ (\dim S_j, \dim S_i) = -\varepsilon_i c_1 \quad \text{and} \quad (\dim S_i, \dim S_j) = -\varepsilon_i c_2. \]

From the definition of the twisted multiplication, we have for \( 0 \leq r \leq n \),
\[ u_i^{r} \ast u_j \ast u_i^{*(n-r)} = v_i^{\frac{n^2-n}{2} - (n-r)c_1 - rc_2} u_i^{r} u_j u_i^{n-r}. \]
Since \( \begin{bmatrix} n \\ r \end{bmatrix}_{vi} = v_i^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_{v_i} \), we obtain by Theorem 2.4,
\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} u_i^{n-r} u_j u_i^{n-r} u_i^{n-r}
\]

\[
= \sum_{r=0}^{n} (-1)^r v_i^{r^2 - rn + \frac{n^2 - n}{2} - (n-r)c_1 - r c_2} \binom{n}{r} q_i u_i^{n-r} u_i^{n-r}
\]

\[
= v_i^{n^2 - c_2^2 - c_2 - nc_1} \sum_{r=0}^{n} (-1)^r q_i^{(r-c_1)(r-c_2-1)} \binom{n}{r} q_i u_i^{n-r} u_i^{n-r}
\]

\[
= 0.
\]

This completes the proof. \[\square\]

**Remark 3.4.** The relations (3.3.1) are exactly the quantum Serre relations—the defining relations of quantized enveloping algebras. In other words, the twisted Ringel–Hall algebra satisfies the quantum Serre relations.

### 4. Higher order fundamental relations

We keep all the notations in Sections 2 and 3. Let \( A \) be a finitary \( \mathbb{F}_q \)-algebra and let \( \mathcal{H}(A) \) be the Ringel–Hall algebra of \( A \).

The following result shows that \( \mathcal{H}(A) \) satisfies higher order fundamental relations. Its proof is analogous to that of Theorem 2.4. However, we provide the proof for completeness.

**Theorem 4.1.** Let \( i, j \in I \) with \( i \neq j \). Suppose \( c_{i,i} = c_{j,j} = 2 \), i.e., \( \text{Ext}_A^1(S_i, S_i) = 0 = \text{Ext}_A^1(S_j, S_j) \). Then for \( n \geq 1 \) and \( m \geq 1 - nc_{i,j} \),

\[
\sum_{r=0}^{m} (-1)^r q_i^{(r-m-c'_i,j)(r-m-n c'_i,j+1)} \binom{m}{r} q_i u_i^{n-r} u_i^{m-r} = 0.
\]  \( (4.1.1) \)

**Proof.** We simply write \( c_1 = -c'_{i,j} \) and \( c_2 = -c''_{i,j} \). Then \( m \geq 1 + nc_1 + nc_2 \). By definition, for each \( 0 \leq r \leq m \), we have

\[
u_i^n u_i^{m-r} = [r]_{q_i} [n]_{q_i} [m-r]_{q_i} u_i^{[r]} u_i^{[n]} u_i^{[m-r]} = \sum_{[M]} f_r(M) u_i^{[M]} ,
\]

where \( f_r(M) = F_{r S_i, n S_j, (m-r) S_i} \) is the number of the filtrations of \( M \)

\[
M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0
\]  \( (4.1.2) \)

such that \( M/M_1 \cong r S_i \), \( M_1/M_2 \cong n S_j \), and \( M_2 \cong (m-r) S_i \). Thus, to show the formula (4.1.1), it suffices to show that for each \( A \)-module \( M \)
\[ f(M) := \sum_{r=0}^{m} (-1)^r q_i^{\frac{(r-m+nc_1)(r-m+nc_1+1)}{2}} [m]_{q_i} [n]_{q_i} [m-r]_{q_i} f_r(M) \]

\[ = [m]_{q_i} [n]_{q_i} \sum_{r=0}^{m} (-1)^r q_i^{\frac{(r-m+nc_1)(r-m+nc_1+1)}{2}} f_r(M) = 0. \]

Let us fix an \( A \)-module \( M \). Suppose \( f_r(M) \neq 0 \). Then the Loewy length of \( M \) is at most 3. Since \( \text{Ext}^1_A(S_i, S_i) = 0 \) and \( \text{Ext}^1_A(S_j, S_j) = 0 \), \( M \) can be decomposed into

\[ M = N_1 \oplus N_2 \oplus N_3 \oplus dS_i \oplus yS_j, \quad (4.1.3) \]

where \( d, y \geq 0 \), each indecomposable summand of \( N_1 \) has Loewy length 3, each indecomposable summand of \( N_2 \) has Loewy length 2 with top \( N_2 \) isomorphic to various copies of \( S_i \), and each indecomposable summand of \( N_3 \) has Loewy length 2 with soc \( N_3 \) isomorphic to various copies of \( S_i \). Note that some of \( N_i \) could be zero. Hence, \( N_1 \) has a filtration

\[ N_1 \supseteq N_1' \supseteq N_1'' \supseteq 0 \]

such that \( N_1/N_1' \cong a_1 S_i \), \( N_1'/N_1'' \cong a_2 S_j \), and \( N_1'' \cong a_3 S_i \) for some \( a_1, a_2, a_3 \geq 0 \). Moreover, \( \text{soc}(N_1/N_1'') \cong \text{top} N_1' \cong a_2 S_j \). Similarly, we have

\[ \text{top} N_2 \cong b_1 S_i, \quad \text{soc} N_2 \cong b_2 S_j, \]

\[ \text{top} N_3 \cong x_1 S_j, \quad \text{soc} N_3 \cong x_2 S_i, \]

for some nonnegative integers \( b_1, b_2, x_1, x_2 \). Comparing dimension vectors gives

\[ \begin{cases} a_2 + b_2 + x_1 + y = n, \\ a_1 + a_3 + b_1 + x_2 + d = m. \end{cases} \quad (4.1.4) \]

Using an argument similar to that in the proof of Theorem 2.4, we can show that

\[ a_1 \leq a_2 c_2, \quad a_3 \leq a_2, \quad c_1 b_1 \leq b_2 c_2, \quad \text{and} \quad x_2 \leq x_1 c_1. \]

By the decomposition (4.1.3) of \( M \), if \( M \) has a filtration of the form (4.1.2), then it is determined by a submodule \( X \) of \( dS_i \) isomorphic to \( (m-r-a_3-x_2) S_i \) with

\[ M_2 = N_1'' \oplus \text{soc} N_3 \oplus X \quad \text{and} \quad M_1 = N_1' \oplus \text{soc} N_2 \oplus N_3 \oplus X \oplus yS_j. \]

Hence, a necessary condition for \( f_r(M) \neq 0 \) is that \( 0 \leq m-r-a_3-x_2 \leq d \), that is,

\[ a_1 + b_1 = m - (a_3 + x_2 + d) \leq r \leq m - (a_3 + x_2) = a_1 + b_1 + d. \]

In all these cases, we have \( f_r(M) = [m-r-a_3-x_2]_{q_i} = [d+(a_1+b_1)-r]_{q_i} \). Hence,
\[ f(M) = [m]_{q_i}^1 [n]_{k_i}^1 \sum_{r=a_1+b_1}^{a_1+b_1+d} (-1)^r q_i \left( \frac{(r-m+nc_1)(r-m+nc_1+1)}{2} \begin{bmatrix} d + (a_1 + b_1) - r \\ d \end{bmatrix}_{q_i} \right) \]

\[ = (-1)^{a_1+b_1} [m]_{q_i}^1 [n]_{k_i}^1 \sum_{t=0}^{d} (-1)^t q_i \left( \frac{(t+a_1+b_1-m+nc_1)(t+a_1+b_1-m+nc_1+1)}{2} \begin{bmatrix} d - t \\ d \end{bmatrix}_{q_i} \right) \]

\[ (\text{setting } t = r - (a_1 + b_1)) \]

\[ = (-1)^{a_1+b_1} [m]_{q_i}^1 [n]_{k_i}^1 \sum_{t=0}^{d} (-1)^t q_i \left( \frac{(t-(m-nc_1-a_1-b_1-1)(t-(m-nc_1-a_1-b_1-1)+1))}{2} \begin{bmatrix} d - t \\ t \end{bmatrix}_{q_i} \right). \]

Since \( a_1 \leq a_2 c_2 \) and \( b_1 \leq b_2 c_2 \), we get

\[ m - nc_1 - a_1 - b_1 - 1 \geq nc_2 - (a_2 + b_2) c_2 = (x_1 + y) c_2 \geq 0. \]

Finally, from \( m = a_1 + a_3 + b_1 + x_2 + d \) it follows that

\[ m - nc_1 - a_1 - b_1 - 1 = d - 1 + (a_3 + x_2) - nc_1 \]

\[ \leq d - 1 + (a_2 + x_1) c_1 - nc_1 \leq d - 1, \]

that is, \( 0 \leq m - nc_1 - a_1 - b_1 - 1 \leq d - 1 \). We conclude by Lemma 2.2 that \( f(M) = 0. \)

Let \( \mathcal{H}^a(A) \) be the twisted Ringel–Hall algebra of \( A \). The following corollary is the twisted version of Theorem 4.1.

**Corollary 4.2.** Let \( i, j \in I \) with \( i \neq j \). Suppose \( c_{i,i} = c_{j,j} = 2 \). Then for \( n \geq 1 \) and \( m \geq 1 - nc_{i,j} \), we have in \( \mathcal{H}^n(A) \),

\[ \sum_{r=0}^{m} (-1)^r v_i^{r(m-nc_{i,j}+1)} \begin{bmatrix} m \\ r \end{bmatrix} v_i^{*(m-r)} u_i^r u_j^{*n} u_i^{*(m-r)} = 0. \]  

(4.2.1)

**Remark 4.3.** The relations (4.2.1) are the so-called higher order quantum Serre relations studied in [9, Chapter 7].

5. Quotients of Ringel–Hall algebras

In this section we study relations between the Ringel–Hall algebra of a finitary \( \mathbb{F}_q \)-algebra and Ringel–Hall algebras of its factor algebras.

**Proposition 5.1.** Let \( A \) be a finitary \( \mathbb{F}_q \)-algebra and let \( B \) be a factor algebra of \( A \). Then there are epimorphisms \( \mathcal{H}(A) \to \mathcal{H}(B) \) and \( C(A) \to C(B) \) of \( \mathbb{Z} \)-algebras.

**Proof.** Suppose \( B = A/J \) for some ideal \( J \) of \( A \). Then \( B \text{-mod} \) can be viewed as a full subcategory of \( A \text{-mod} \). Since \( B \)-modules are exactly those \( A \)-modules which are annihilated by \( J \), \( B \text{-mod} \) as a subcategory of \( A \text{-mod} \) is closed under submodules and factor modules.
Let $\mathcal{P}$ denote the set of isoclasses of $A$-modules and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ consisting of the isoclasses of $B$-modules. Let $\mathcal{I}$ denote the free $\mathbb{Z}$-submodule of $\mathcal{H}(A)$ with basis $u_{[M]}, [M] \in \mathcal{P} \setminus \mathcal{P}_0$. We claim that $\mathcal{I}$ is an ideal of $\mathcal{H}(A)$ and $\mathcal{H}(A)/\mathcal{I} \cong \mathcal{H}(B)$. Indeed, for each $u_{[M]} \in \mathcal{I}$ and each $u_{[N]} \in \mathcal{H}(A)$, we have by definition

$$u_{[M]} u_{[N]} = \sum_{[L]} F^L_{M,N} u_{[L]}.$$  

If $F^L_{M,N} \neq 0$, then there is an exact sequence

$$0 \to N \to L \to M \to 0.$$  

The assumption that $[M] \in \mathcal{P} \setminus \mathcal{P}_0$ implies that $[L] \in \mathcal{P} \setminus \mathcal{P}_0$, that is, $u_{[L]} \in \mathcal{I}$, so $u_{[M]} u_{[N]} \in \mathcal{I}$. Similarly, we have $u_{[N]} u_{[M]} \in \mathcal{I}$. Consequently, $\mathcal{I}$ is an ideal of $\mathcal{H}(A)$. Moreover, the $\mathbb{Z}$-linear map

$$\mathcal{H}(B) \to \mathcal{H}(A)/\mathcal{I}, \quad u_{[M]} \mapsto u_{[M]} + \mathcal{I}$$

is an isomorphism of algebras. This gives an epimorphism $\mathcal{H}(A) \to \mathcal{H}(B)$.

Since each simple $B$-module is also a simple $A$-module, the epimorphism $\mathcal{H}(A) \to \mathcal{H}(B)$ induces an epimorphism $\mathcal{C}(A) \to \mathcal{C}(B)$. \qed

**Remark 5.2.** (1) The proposition also follows from a general result of Schiffmann [17] which deals with functorial properties of Hall algebras of finitary $\mathbb{F}_q$-categories.

(2) If $B$ is a factor algebra of $A$, there are also epimorphisms $\mathcal{H}^*(A) \to \mathcal{H}^*(B)$ and $\mathcal{C}^*(A) \to \mathcal{C}^*(B)$ of $\mathbb{Z}[v, v^{-1}]$-algebras.

**Corollary 5.3.** Let $A$ be a finitary $\mathbb{F}_q$-algebra and let $B$ be a factor algebra of $A$. Then there is an epimorphism $n(A)_{(q-1)} \to n(B)_{(q-1)}$ of Lie algebras over $\mathbb{Z}/(q - 1)$. If, moreover, $A$ is finite dimensional, then there is an epimorphism $n_e(A) \to n_e(B)$ of Lie algebras over $\mathbb{Z}$.

Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0$ (resp., $Q_1$) denotes the set of vertices (resp., arrows) of $Q$. For each arrow $\rho$ in $Q_1$, we denote by $h\rho$ and $t\rho$ the head and the tail of $\rho$, respectively. Let $\sigma$ be an automorphism of $Q$, that is, $\sigma$ is a permutation on the vertices of $Q$ and on the arrows of $Q$ such that $\sigma(h\rho) = h\sigma(\rho)$ and $\sigma(t\rho) = t\sigma(\rho)$ for any $\rho \in Q_1$. Let $k$ be the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$ and $kQ$ be the path algebra of $Q$. Then $\sigma$ induces a Frobenius morphism

$$F = F_{Q, \sigma; q} : kQ \to kQ,$$

$$\sum_s x_s p_s \mapsto \sum_s x_s^q \sigma(p_s).$$

Here $\sum_s x_s p_s$ is a $k$-linear combination of paths $p_s$, and, for a path $p_s = \rho_1 \cdots \rho_l$ with $\rho_1, \ldots, \rho_l \in Q_1$, define $\sigma(p_s) := \sigma(\rho_l) \cdots \sigma(\rho_1)$. Then the set of fixed points

$$\mathfrak{A}(Q, \sigma; q) := (kQ)^F = \{ a \in kQ \mid F(a) = a \}$$

becomes an $\mathbb{F}_q$-algebra. It is known from [3,4] that $\mathfrak{A}(Q, \sigma; q)$ is hereditary. Moreover, each finite dimensional $\mathbb{F}_q$-algebra is Morita equivalent to a factor algebra of some $\mathfrak{A}(Q, \sigma; q)$. Proposition 5.1 and Remark 5.2(2) give the following result.
Corollary 5.4. Let $A$ be a finite dimensional $\mathbb{F}_q$-algebra. Then there exists a quiver $Q$ with automorphism $\sigma$ such that there are epimorphisms $\mathcal{H}(\mathfrak{A}) \to \mathcal{H}(A)$ and $\mathcal{C}(\mathfrak{A}) \to \mathcal{C}(A)$ of $\mathbb{Z}$-algebras, where $\mathfrak{A} = \mathfrak{A}(Q, \sigma; q)$. Moreover, there are epimorphisms $\mathcal{H}^*(\mathfrak{A}) \to \mathcal{H}^*(A)$ and $\mathcal{C}^*(\mathfrak{A}) \to \mathcal{C}^*(A)$ of $\mathbb{Z}[v, v^{-1}]$-algebras.

Now let $A$ be a finitary $\mathbb{F}_q$-algebra and let $J$ be an ideal of $A$. We set $B = A/J$. As in the proof of Proposition 5.1, let $\mathcal{P}$ denote the set of isoclasses of $A$-modules and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ consisting of the isoclasses of $B$-modules, that is, those $A$-modules which are annihilated by $J$. Let $\mathcal{I}$ denote the free $\mathbb{Z}$-submodule of $\mathcal{H}(A)$ with basis $u_{[M]}$, $[M] \in \mathcal{P} \setminus \mathcal{P}_0$. Finally, let $\mathcal{I}$ be the set of isoclasses $[M] \in \mathcal{P} \setminus \mathcal{P}_0$.

Proposition 5.5. The ideal $I_Q$ of $\mathcal{H}(A)_Q$ is generated by $u_{[M]}$, $[M] \in \mathcal{I}$.

To prove the proposition, we need the following lemma. We refer to [6] for its proof.

Lemma 5.6. Let $L$ be a module in $A\text{-}\text{mod}$ with a filtration

$$L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_{t-1} \supseteq L_t = 0.$$

Then

$$\dim_{\mathbb{F}_q} \text{Ext}^1_A(L, L) \leq \dim_{\mathbb{F}_q} \text{Ext}^1_A\left( \bigoplus_{s=1}^{t-1} L_{s-1}/L_s, \bigoplus_{s=1}^{t-1} L_{s-1}/L_s \right).$$

Moreover, the equality holds if and only if $L \cong \bigoplus_{s=1}^{t-1} L_{s-1}/L_s$.

Proof of Proposition 5.5. Let $\overline{\mathcal{I}}$ be the ideal of $\mathcal{H}(A)_Q$ generated by $u_{[M]}$, $[M] \in \mathcal{I}$. It is clear that $\overline{\mathcal{I}} \subseteq I_Q$.

Now take $u_{[M]} \in I_Q$, i.e., $[M] \in \mathcal{P} \setminus \mathcal{P}_0$. We show by induction on $\delta(M) := \dim_{\mathbb{F}_q} \text{Ext}^1_A(M, M)$ that $u_{[M]}$ lies in $\overline{\mathcal{I}}$. Write

$$M = M_1 \oplus \cdots \oplus M_t,$$

where $M_s$, $1 \leq s \leq t$, are indecomposable. Since $[M] \in \mathcal{P} \setminus \mathcal{P}_0$, there is some $1 \leq s \leq t$ such that $[M_s] \in \mathcal{I}$. If $\delta(M) = 0$, then

$$u_{[M]} = F_{[M]} = F_{M_1, \ldots, M_t} u_{[M]}.$$

Thus, we obtain that

$$u_{[M]} = (F_{M_1, \ldots, M_t})^{-1} u_{[M]} u_{[M]} \in \overline{\mathcal{I}}.$$

Suppose now $\delta(M) \geq 1$. Then

$$u_{[M]} = F_{M_1, \ldots, M_t} u_{[M]} + \sum_{[L] \neq [M]} F_{M_1, \ldots, M_t} u_{[L]}.$$

If $F^L_{M_1,\ldots,M_t} \neq 0$, then we have by Lemma 5.6 that $\delta(L) < \delta(M)$. Since $[M_\delta] \in \mathcal{S}$, we have also $u_{[L]} \in \mathcal{I}_Q$ whenever $F^L_{M_1,\ldots,M_t} \neq 0$. By the induction hypothesis, all such $u_{[L]}$ lie in $\mathcal{I}$. We finally get that

$$u_{[M]} = (F^M_{M_1,\ldots,M_t})^{-1} \left( u_{[M_1]} \cdots u_{[M_t]} - \sum_{[L] \neq [M]} F^L_{M_1,\ldots,M_t} u_{[L]} \right) \in \mathcal{I}.$$

This finishes the proof. \qed

**Remark 5.7.** In $\mathcal{H}(A)$, the ideal generated by $u_{[M]}$, $[M] \in \mathcal{S}$, is always contained in $\mathcal{I}$, but in general, they may not coincide. For example, let $\mathbb{F}_q[x]$ denote the polynomial ring in an indeterminate $x$. Define $A = \mathbb{F}_q[x]/(x^n)$, where $n \geq 2$. Let further $J = (x)/(x^n)$ and set $B = A/J$. Then $S = (x^{n-1})/(x^n)$ is the unique simple $A$-module, up to isomorphism. For each $1 \leq i \leq n$, there is a unique indecomposable $A$-module $S(i)$ with dimension $i$. Note that $S(1) = S$. It is easy to see that

$$\mathcal{P}_0 = \{ [mS] \mid m \geq 1 \} \quad \text{and} \quad \mathcal{S} = \{ [S(i)] \mid 2 \leq i \leq n \}.$$

Let $\mathcal{I}_1$ be the ideal of $\mathcal{H}(A)$ generated by $u_{[M]}$, $[M] \in \mathcal{S}$. Take $M = S \oplus S(2)$. Clearly, $u_{[M]} \in \mathcal{I}$. Since

$$u_{[S]} u_{[S(2)]} = u_{[S(2)]} u_{[S]} = u_{[S(3)]} + qu_{[M]},$$

we get $qu_{[M]} \in \mathcal{I}_1$, but $u_{[M]} \notin \mathcal{I}_1$. Hence, $\mathcal{I}_1 \subsetneq \mathcal{I}$.

Now let $A$ be a finite dimensional $\mathbb{F}_q$-algebra. Suppose $C_A$ is a generalized Cartan matrix, i.e., $\text{Ext}^1_A(S_i, S_i) = 0$ for all $i \in I$. Then we have the quantized enveloping algebra $U_v(C_A)$ of the Kac–Moody algebra $\mathfrak{g}(C_A)$ associated with $C_A$. Let further $U^+ = U^+_v(C_A)$ be the positive part of $U_v(C_A)$, which is by definition the $\mathbb{Q}(v)$-algebra generated by $E_i, i \in I$, with the quantum Serre relations, where $\mathbb{Q}(v)$ is the rational function field in the indeterminate $v$. For a prime power $q$ with $v = \sqrt[q]{q}$, we denote by $U^+_v(C_A)$ the specialization of $U^+_v(C_A)$ at $v = v$ via the Lusztig $\mathbb{Z}[v, v^{-1}]$-form of $U$ (see [9]). In particular, $U_v(C_A)$ is a $\mathbb{Q}(v)$-algebra, where $\mathbb{Q}(v)$ denotes the field extension of $\mathbb{Q}$ by $v$.

**Corollary 5.8.** Let $A$ be a finite dimensional $\mathbb{F}_q$-algebra. Suppose $C_A$ is a generalized Cartan matrix. Then there is an epimorphism

$$U^+_v(C_A) \twoheadrightarrow C^*(A) \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v),$$

where $C^*(A)$ is the twisted composition algebra of $A$.

**Proof.** Without loss of generality, we may suppose that $A$ is basic. Then there is a quiver $Q$ with automorphism $\sigma$ such that $Q$ contains no loops and there is an epimorphism $\mathfrak{A} \twoheadrightarrow A$, where $\mathfrak{A} = \mathfrak{A}(Q, \sigma; q)$. Moreover, $C_A = C_{\mathfrak{A}}$. By [5,12], there is a $\mathbb{Q}(v)$-algebra isomorphism

$$U^+_v(C_A) \cong C^*(\mathfrak{A}) \otimes \mathbb{Q}(v), \quad E_i \mapsto u_i, \quad i \in I.$$

The corollary then follows from Corollary 5.4. \qed
Example 5.9. Let $Q$ be the following quiver:

$$Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$ 

Let $A = \mathbb{F}_q Q$ be the path algebra of $Q$ over the finite field $\mathbb{F}_q$. Then

$$C_A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\mathfrak{g}(C_A) \cong \mathfrak{sl}_4,$

where $\mathfrak{sl}_4$ is the complex simple Lie algebra of type $A_3$.

On the other hand, it is known that there are 6 indecomposable $A$-modules, up to isomorphism. For each $i \in \{1, 2, 3\}$, let $P_i$ and $I_i$ be respectively the projective cover and injective hull of the simple $A$-module $S_i$. Then the set $\{P_1 = I_3, P_2, P_3 = S_3, I_1 = S_1, I_2\}$ is a complete list of indecomposable $A$-modules. Let now $B = A/J$ with $J = (\beta \alpha)$ the ideal of $A$ generated by $\beta \alpha$. Then $C_B = C_A$. It is easy to see that $P_1 = I_3$ is the only indecomposable $A$-module which is not annihilated by $J$. By Propositions 5.1 and 5.5, we have $\mathcal{H}(B)_{\mathbb{Q}} \cong \mathcal{H}(A)_{\mathbb{Q}}/\langle u_{\{P_1\}} \rangle$.

Let $\mathcal{H}^*(A)$ and $\mathcal{H}^*(B)$ be the twisted Ringel–Hall algebras of $A$ and $B$, respectively. We remark that as $\mathbb{Z}[v, v^{-1}]$-algebras, $\mathcal{H}^*(B)$ is not isomorphic to $\mathcal{H}^*(A)/\langle u_{\{P_1\}} \rangle$. Since $A$ is hereditary of finite representation type, we have

$$C^*(A) \otimes \mathbb{Q}(v) = \mathcal{H}^*(A) \otimes \mathbb{Q}(v) \xrightarrow{\sim} U_v^+(\mathfrak{sl}_4), \quad u_i \otimes 1 \mapsto E_i.$$ 

Moreover, the twisting version of Proposition 5.5 gives that

$$\mathcal{H}^*(B) \otimes \mathbb{Q}(v) \cong \mathcal{H}^*(A) \otimes \mathbb{Q}(v)/\langle u_{\{P_1\}} \rangle.$$ 

By an easy calculation, we have in $\mathcal{H}^*(A)$,

$$u_1 * u_2 * u_3 = v^{-2}(u_{\{P_1\}} + u_{[S_1 \oplus P_2]} + u_{[I_2 \oplus S_3]} + u_{[S_1 \oplus S_2 \oplus S_3]}),$$

$$u_1 * u_3 * u_2 = v^{-1}(u_{[I_2 \oplus S_3]} + u_{[S_1 \oplus S_2 \oplus S_3]}),$$

$$u_2 * u_3 * u_1 = v^{-1}(u_{[P_2 \oplus S_1]} + u_{[S_1 \oplus S_2 \oplus S_3]}),$$

$$u_3 * u_2 * u_1 = u_{[S_1 \oplus S_2 \oplus S_3]}.$$

Consequently, we get

$$u_{\{P_1\}} = v^2 u_1 * u_2 * u_3 - vu_2 * u_3 * u_1 + u_3 * u_2 * u_1 - vu_1 * u_3 * u_2 - vu_1 * (vu_2 * u_3 - u_3 * u_2) - (vu_2 * u_3 - u_3 * u_2) * u_1.$$ 

For $x, y \in U_v^+(\mathfrak{sl}_4)$, we set $[x, y]_v = vx - yx$. We finally get by Corollary 5.8 that

$$\mathcal{H}^*(B) \otimes \mathbb{Q}(v) \cong U_v^+(\mathfrak{sl}_4)/\left[[E_1, [E_2, E_3]]_v \right].$$

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References