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Monotonic Solutions of a Class of Quadratic Integral Equations of Volterra Type

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Abstract—Using the technique associated with measures of noncompactness we prove the existence of monotonic solutions of a class of quadratic integral equation of Volterra type in the Banach space of real functions defined and continuous on a bounded and closed interval.

Keywords—Measure of noncompactness, Fixed-point theorem, Monotonic solutions, Quadratic integral equation.

1. INTRODUCTION

The theory of integral operators and integral equations is an important part of nonlinear analysis. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and mathematical physics, engineering, economics, biology as well in describing problems connected with real world (cf. [1–6]).

In the present paper, we are going to study the solvability of a class of quadratic integral equations of Volterra type. We will look for solutions of those equations in the Banach space of real functions being defined and continuous on a bounded and closed interval. The main tool used in our investigations is the technique of measures of noncompactness which is frequently used in several branches of nonlinear analysis [3,4,7,8]. We will apply the measure of noncompactness defined in [9] in proving the solvability of the considered equations in the class of monotonic functions.

The results of this paper generalize and complete the results obtained earlier in the paper [8].

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2. NOTATION AND AUXILIARY FACTS

Now, we are going to recall the basic results which are needed further on.

Assume that E is a real Banach space with the norm $\|\cdot\|$ and the zero element 0. Denote by $B(x,r)$ the closed ball centered at x and with radius r and by B_r the ball $B(0,r)$. For X being a nonempty subset of E we denote by $\bar{X}, \operatorname{Conv} X$ the closure and the convex closure of X, respectively. With the symbols λX and $X+Y$ we denote the algebraic operations on sets. Finally, let us denote by $\mathbf{M}_E$ the family of nonempty and bounded subsets of E and by $\mathbf{N}_E$ its subfamily consisting of all relatively compact sets.

DEFINITION 1. *(See [7].) A function $\mu : \mathbf{M}_E \to [0,\infty)$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions*

(1) *the family* $\ker \mu = \{X \in \mathbf{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subset \mathbf{N}_E$;
(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$;
(3) $\mu(\bar{X}) = \mu(\operatorname{Conv} X) = \mu(X)$;
(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$;
(5) *if* $\{X_n\}_n$ *is a sequence of closed sets from* $\mathbf{M}_E$ *such that* $X_{n+1} \subset X_n$ *for* $n = 1,2,\dots$ *and if* $\lim_{n\to\infty} \mu(X_n) = 0$ *then, the set* $X_\infty = \bigcap_{n=1}^{\infty} X_n$ *is nonempty.*

The family $\ker\mu$ described above is called *the kernel of the measure of noncompactness* μ. Further facts concerning measures of noncompactness and their properties may be found in [7]. For our purposes we will only need the following fixed-point theorem [7].

THEOREM 1. *Let N be a nonempty, bounded, closed and convex subset of the Banach space E and let $F : N \to N$ be a continuous transformation such that $\mu(FX) \le K\mu(X)$ for any nonempty subset X of N, where $K \in [0,1)$ is a constant. Then, F has a fixed point in the set N.*

REMARK 1. Under the assumptions of the above theorem it can be shown that, the set Fix$F$ of fixed points of F belonging to N is a member of $\ker\mu$. This observation allows us to characterize solutions of considered equations.

In what follows, we will work in the classical Banach space $C[0,M]$ consisting of all real functions defined and continuous on the interval $[0,M]$. For convenience, we write $I = [0,M]$ and $C(I) = C[0,M]$. The space $C(I)$ is furnished by the standard norm

$$\|x\| = \max\{|x(t)| : t \in I\}.$$

Now, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in the sequel. That measure was introduced and studied in the paper [9].

To do this let us fix a nonempty and bounded subset X of $C(I)$. For $\varepsilon > 0$ and $x \in X$ denote by $w(x,\varepsilon)$ the modulus of continuity of x defined by

$$w(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in I, |t-s| \le \varepsilon\}.$$

Further, let us put

$$w(X,\varepsilon) = \sup\{w(x,\varepsilon) : x \in X\},$$
$$w_0(X) = \lim_{\varepsilon\to 0} w(X,\varepsilon).$$

Next, let us define the following quantities

$$i(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t,s \in I,\ t \le s\},$$
$$i(X) = \sup\{i(x) : x \in X\}.$$

Observe that, $i(X) = 0$ if and only if all functions belonging to X are nondecreasing on I. Finally, let us put

$$\mu(X) = w_0(X) + i(X).$$

It can be shown [9] that, the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $\ker\mu$ consists of all sets X belonging to $\mathbf{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

3. MAIN RESULT

In this section, we apply the measure of noncompactness μ above defined to the study of monotonic solutions of our integral equation.

We consider the following nonlinear integral equation of Volterra type

$$x(t) = a(t) + (Tx)(t) \int_0^t v(t, \tau, x(\tau))\, d\tau, \qquad t \in I. \tag{1}$$

The functions $a(t)$ and $v(t, \tau, x)$ appearing in this equation are given while $x = x(t)$ is an unknown function.

This equation will be examined under the following assumptions.

(i) $a \in C(I)$ and the function a is nondecreasing and nonnegative on the interval I.

(ii) $v : I \times I \times \mathbf{R} \to \mathbf{R}$ is a continuous function such that $v : I \times I \times \mathbf{R}_+ \to \mathbf{R}_+$ and for arbitrarily fixed $\tau \in I$ and $x \in \mathbf{R}_+$ the function $t \to v(t, \tau, x)$ is nondecreasing on I.

(iii) There exists a nondecreasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that, the inequality

$$|v(t, \tau, x)| \le f(|x|),$$

holds for all $t, \tau \in I$ and $x \in \mathbf{R}$.

(iv) The operator $T : C(I) \to C(I)$ is continuous and satisfies the conditions of Theorem 1 for the measure of noncompactness μ with a constant Q and, moreover, T is a positive operator, i.e., $Tx \ge 0$ if $x \ge 0$.

(v) There exist nonnegative constants c and d such that

$$|(Tx)(t)| \le c + d\|x\|,$$

for each $x \in C(I)$ and $t \in I$.

(vi) The inequality $\|a\| + (c + dr)Mf(r) \le r$ has a positive solution r_0 such that $Mf(r_0)Q < 1$.

Then, we have the following theorem.

THEOREM 2. *Under the Assumptions (i)–(vi) the equation (1) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I.*

PROOF. Let us consider the operator V defined on the space $C(I)$ in the following way

$$(Vx)(t) = a(t) + (Tx)(t) \int_0^t v(t, \tau, x(\tau))\, d\tau.$$

In view of the Assumptions (i), (ii) and (iv) it follows that, the function Vx is continuous on I for any function $x \in C(I)$, i.e., V transforms the space $C(I)$ into itself. Moreover, keeping in mind the Assumptions (iii) and (v) we get

$$\begin{aligned} |(Vx)(t)| &\le |a(t)| + |(Tx)(t)| \left| \int_0^t v(t, \tau, x(\tau))\, d\tau \right| \\ &\le \|a\| + (c + d\|x\|) \int_0^t f(|x(\tau)|)\, d\tau \\ &\le \|a\| + (c + d\|x\|) \int_0^t f(\|x\|)\, d\tau \le \|a\| + (c + d\|x\|)Mf(\|x\|). \end{aligned}$$

Hence,

$$\|Vx\| \le \|a\| + (c + d\|x\|)Mf(\|x\|).$$

Thus, taking into account the Assumption (vi) we infer that there exists $r_0 > 0$ with $Mf(r_0)Q < 1$ and such that, the operator V transforms the ball B_{r_0} into itself.

In what follows, we will consider the operator V on the subset $B^+_{r_0}$ of the ball B_{r_0} defined by

$$B^+_{r_0} = \{x \in B_{r_0} : x(t) \geq 0 \text{ for } t \in I\}.$$

Obviously, the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. In view of these facts and Assumptions (i), (ii), and (iv) we deduce easily that V transforms the set $B^+_{r_0}$ into itself.

Now, we show that V is continuous on the set $B^+_{r_0}$. To do this let us fix $\varepsilon > 0$ and take arbitrarily $x, y \in B^+_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, for $t \in I$ we get the following inequalities

$$\begin{aligned}
|(Vx)(t) - (Vy)(t)| &= \left|(Tx)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau - (Ty)(t) \int_0^t v(t,\tau,y(\tau))\,d\tau\right| \\
&\leq \left|(Tx)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau - (Ty)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau\right| \\
&\quad + \left|(Ty)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau - (Ty)(t) \int_0^t v(t,\tau,y(\tau))\,d\tau\right| \\
&\leq |(Tx)(t) - (Ty)(t)| \int_0^t v(t,\tau,x(\tau))\,d\tau \\
&\quad + |Ty(t)| \int_0^t |v(t,\tau,x(\tau)) - v(t,\tau,y(\tau))|\,d\tau \\
&\leq \|Tx - Ty\| \int_0^t f(r_0)\,d\tau + (c + dr_0) \int_0^t \beta_{r_0}(\varepsilon)\,d\tau \\
&\leq \|Tx - Ty\| M f(r_0) + (c + dr_0)\beta_{r_0}(\varepsilon) M,
\end{aligned}$$

where $\beta_{r_0}(\varepsilon)$ is defined as

$$\beta_{r_0}(\varepsilon) = \sup\{|v(t,\tau,x) - v(t,\tau,y)| : t,\tau \in I,\ x,y \in [0,r_0],\ |x - y| \leq \varepsilon\}.$$

From the above estimate we derive the following inequality

$$\|Vx - Vy\| \leq \|Tx - Ty\| M f(r_0) + (c + dr_0) M \beta_{r_0}(\varepsilon).$$

From the uniform continuity of the function v on the set $I \times I \times [0, r_0]$ and the continuity of T, the last inequality implies the continuity of the operator V on the set $B^+_{r_0}$.

In what follows let us take a nonempty set $X \subset B^+_{r_0}$. Further, fix arbitrarily a number $\varepsilon > 0$ and choose $x \in X$ and $t, s \in [0, M]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we may assume that $t \leq s$. Then, in view of our assumptions we obtain

$$\begin{aligned}
|(Vx)(s) - (Vx)(t)| &\leq |a(s) - a(t)| + \left|(Tx)(s) \int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau\right| \\
&\leq w(a,\varepsilon) + \left|(Tx)(s) \int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t) \int_0^s v(s,\tau,x(\tau))\,d\tau\right| \\
&\quad + \left|(Tx)(t) \int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t) \int_0^s v(t,\tau,x(\tau))\,d\tau\right| \\
&\quad + \left|(Tx)(t) \int_0^s v(t,\tau,x(\tau))\,d\tau - (Tx)(t) \int_0^t v(t,\tau,x(\tau))\,d\tau\right| \\
&\leq w(a,\varepsilon) + |(Tx)(s) - (Tx)(t)| \int_0^s v(s,\tau,x(\tau))\,d\tau \\
&\quad + (Tx)(t) \int_0^s |v(s,\tau,x(\tau)) - v(t,\tau,x(\tau))|\,d\tau + (Tx)(t) \int_t^s v(t,\tau,x(\tau))\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le w(a,\varepsilon) + w(Tx,\varepsilon)\int_0^s f(r_0)\,d\tau + (c+dr_0)\int_0^s \gamma_{r_0}(\varepsilon)\,d\tau \\
&\quad + (c+dr_0)f(r_0)(s-t) \\
&\le w(a,\varepsilon) + w(Tx,\varepsilon)Mf(r_0) + (c+dr_0)M\gamma_{r_0}(\varepsilon) + (c+dr_0)f(r_0)\varepsilon,
\end{aligned}
$$

where $\gamma_{r_0}(\varepsilon)$ is defined as

$$\gamma_{r_0}(\varepsilon) = \sup\{|v(s,\tau,x) - v(t,\tau,x)| : t,s\in I,\ |s-t|\le\varepsilon,\ x\in[0,r_0]\}.$$

Notice, that in view of the uniform continuity of the function v on the set $I\times I\times[0,r_0]$ we have that $\gamma_{r_0}(\varepsilon)\to 0$ as $\varepsilon\to 0$. This fact in conjunction with the above obtained estimate allows us to derive the following inequality

$$w_0(VX) \le Mf(r_0)w(TX). \tag{2}$$

On the other hand, fix arbitrarily $x\in X$ and $t,s\in I$ such that $t\le s$. Then, we have the following estimate

$$
\begin{aligned}
&|(Vx)(s) - (Vx)(t)| - [(Vx)(s) - (Vx)(t)] \\
&= \left|a(s) + (Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau - a(t) - (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right| \\
&\quad - \left[a(s) + (Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau - a(t) - (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right] \\
&\le [|a(s)-a(t)| - [a(s)-a(t)]] + \left[\left|(Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau\right.\right. \\
&\quad \left.- (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right| - \left.\left[(Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right]\right] \\
&\le \left|(Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t)\int_0^s v(s,\tau,x(\tau))\,d\tau\right| \\
&\quad + \left|(Tx)(t)\int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right| \\
&\quad - \left[(Tx)(s)\int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t)\int_0^s v(s,\tau,x(\tau))\,d\tau\right] \\
&\quad - \left[(Tx)(t)\int_0^s v(s,\tau,x(\tau))\,d\tau - (Tx)(t)\int_0^t v(t,\tau,x(\tau))\,d\tau\right] \\
&\le |(Tx)(s) - (Tx)(t)|\int_0^s v(s,\tau,x(\tau))\,d\tau - [(Tx)(s) - (Tx)(t)]\int_0^s v(s,\tau,x(\tau))\,d\tau \\
&\quad + |(Tx)(t)|\left[\left|\int_0^t (v(s,\tau,x(\tau)) - v(t,\tau,x(\tau)))\,d\tau\right|\right. \\
&\quad + \left.\left|\int_t^s v(s,\tau,x(\tau))\,d\tau\right|\right] - (Tx)(t)\left[\int_0^s v(s,\tau,x(\tau))\,d\tau - \int_0^t v(t,\tau,x(\tau))\,d\tau\right] \\
&\le [|(Tx)(s) - (Tx)(t)| - [(Tx)(s) - (Tx)(t)]]\int_0^s v(s,\tau,x(\tau))\,d\tau \\
&\quad + (Tx)(t)\left[\int_0^t (v(s,\tau,x(\tau)) - v(t,\tau,x(\tau)))\,d\tau + \int_t^s v(s,\tau,x(\tau))\,d\tau\right] \\
&\quad - (Tx)(t)\left[\int_0^s v(s,\tau,x(\tau))\,d\tau - \int_0^t v(t,\tau,x(\tau))\,d\tau\right] \\
&\le [|(Tx)(s) - (Tx)(t)| - [(Tx)(s) - (Tx)(t)]]Mf(r_0) \le Mf(r_0)i(Tx).
\end{aligned}
$$

Hence, we get

$$i(Vx) \le Mf(r_0)i(Tx),$$

and consequently,

$$i(VX) \le Mf(r_0)i(TX). \tag{3}$$

Finally, linking (2), (3) and keeping in mind the definition of the measure of noncompactness μ (cf. Section 2) and, moreover, the Assumption (iv) we obtain

$$\mu(VX) \le Mf(r_0)\mu(TX) \le Mf(r_0)Q\mu(X).$$

As $Mf(r_0)Q < 1$ (Assumption (vi)) applying Theorem 1 we complete the proof. ∎

REMARK 2. Taking into account Remark 1 and the description of the kernel of the measure of noncompactness μ given in Section 2, we deduce easily from the proof of Theorem 2 that solutions of our integral equation belonging to the set $B_{r_0}^+$ are nondecreasing and continuous on the interval $I = [0, M]$. Moreover, those solutions are also positive provided $a(t) > 0$ for $t \in I$.

4. SOME REMARKS

First, we are going to make an observation related with the Hypothesis (v) of Theorem 2. The Hypothesis (v) of the Theorem 2 can be modified in the following sense. We can consider that, the operator T verifies that there exist nonnegative constants c and d such that,

$$|(Tx)(t)| \le c + d|x(t)|, \qquad \text{for each } x \in C(I) \text{ and } t \in I. \tag{4}$$

In this case, the proof of Theorem 2 with this assumption can be done in the same way as the proof given in this paper.

Moreover, notice that if the operator T verifies the above assumption then it satisfies the Hypothesis (v) too, i.e.,

$$|(Tx)(t)| \le c + d|x(t)|, \qquad \text{for each } t \in I \Rightarrow |(Tx)(t)| \le c + d\|x\|, \text{ for each } t \in I.$$

But, there exist operators which verify the Hypothesis (v) and do not satisfy the inequality (4), for example the operator T defined by $(Tx)(t) = x(t^2)$. In this case, we have that there exist no constants c and d such that $|(Tx)(t)| \le c + d|x(t)|$. Suppose that, there exist these constants c and d and we take $t_0 \in (0,1)$. We consider the straight line joining the points $(t_0, 0)$ and $(t_0^2, \max(c,d)+1)$ which has the expression

$$x(t) = \frac{\max(c,d)+1}{t_0^2 - t_0}(t - t_0).$$

Then, it is clear that $x(t)$ belongs to $C(I)$ and Tx does not satisfy the inequality (4). In fact, taking $t = t_0$ we have

$$|(Tx)(t_0)| = |x(t_0^2)| = \max(c,d) + 1,$$

and $c + d|x(t_0)| = c$. Therefore, $|Tx(t_0)| > c + d|x(t_0)|$.

Thus, we can conclude that our Assumption (v) is less restrictive than the inequality (4).

Next, we give some examples which show the relevance of the hypotheses of Theorem 2.

EXAMPLE 1. Take the function $a \in C[0,1]$ given by $a(t) = t^2 - 2t + 1$ which is not nondecreasing and, consequently, this function does not satisfy our Assumption (i).

We put $v(t, \tau, x) = 1$, this function satisfies the Assumptions (ii) and (iii) and, moreover, we can take as f the function $f(r) = 1$ and in this case $M = 1$.

The operator T will be defined as $(Tx)(t) = 1$ and satisfies the hypotheses (iv) and (v) with $c = 1$, $d = 0$ and we can take $Q = 1/2$. As $\|a\| = 1$ the inequality from (vi) has the form

$$1 + 1 \cdot 1 \cdot 1 \leq r,$$

and the number $r_0 = 2$ is a positive solution of the above inequality. Moreover, $Mf(r_0)Q = 1 \cdot 1 \cdot (1/2) \leq 1$. Our integral equation can be expressed by

$$x(t) = (t^2 - 2t + 1) + \int_0^t dt = t^2 - t + 1,$$

and, consequently, the solution is not nondecreasing.

EXAMPLE 2. In this case our interval is $[0, 1]$. Take as $a \in C[0, 1]$ the function zero which satisfies the Assumption (i) of the Theorem 2. We put $v(t, \tau, x) = t^2 - t + (1/4)$ which does not satisfy the Assumption (ii) and satisfies the Hypothesis (iii) with $f = 1/4$. Take $(Tx)(t) = 1$ and this operator satisfies (iv) and (v) with $c = 1$, $d = 0$ and $Q = 1/2$. The inequality from (vi) appears as

$$1 \cdot 1 \cdot \frac{1}{4} \leq r,$$

which admits the number $r_0 = 1/4$ as a positive solution. Moreover, $Mf(r_0)Q = 1 \cdot (1/4) \cdot (1/2) < 1$ and our integral equation has the form

$$x(t) = \int_0^t \left(t^2 - t + \frac{1}{4} \right) d\tau = t^3 - t^2 + \frac{t}{4},$$

and the solution is not nondecreasing.

EXAMPLE 3. Take as $a \in C[0, 1]$ the function zero, and as T the operator $(Tx)(t) = 1$ which satisfies (iv) and (v) with $c = 1$, $d = 0$.

Now, we put $v(t, \tau, x) = x - 1$. Obviously, $v : I \times I \times \mathbf{R} \to \mathbf{R}$ is a continuous function where $I = [0, 1]$ but this function does not satisfy the hypothesis $v : I \times I \times \mathbf{R}_+ \to \mathbf{R}_+$. The Assumption (iii) is satisfied with $f(r) = 1 + r$. The inequality appearing in Assumption (vi) has the form

$$1 \cdot 1 \cdot (1 + r) \leq r,$$

and this inequality has not solution. Our integral equation has the form

$$x(t) = \int_0^t (x(\tau) - 1) \, d\tau.$$

Obviously, $x(t) \equiv 0$ is not a solution of this integral equation. Moreover, as $x(0) = 0$ and if this equation has a solution which is nondecreasing by using the continuity and monotony of $x(t)$, and the form of our integral equation we can find $t_0 \in I$ such that $x(t_0) < 0$ and this is not possible. Thus, our equation has not a nondecreasing solution.

5. EXAMPLES

Now, we give some examples concerning the Assumptions (iii) and (iv).

EXAMPLE 4. Let us assume that, the function $f(r)$ appearing in the Assumption (iii) has the form $f(r) = \lambda r^3$, where λ is a positive constant. Take a function $a \in C[0, 1]$ such that $\|a\| < 1/2$ and such that $\lambda < 8 - 16\|a\|$. In this case $M = 1$ and we consider as T the operator $(Tx)(t) = x(t)$. Then, the inequality from (vi) has the form

$$\|a\| + \lambda r^4 \leq r.$$

Applying Bolzano's theorem to the function

$$g(r) = \|a\| + \lambda r^4 - r,$$

in the interval $[0, 1/2]$ we can find a positive solution r_0 to the above inequality with $0 < r_0 < 1/2$. Moreover, we have

$$Mf(r_0)Q = f(r_0) = \lambda r_0^3 < \frac{\lambda}{8} < 1 - 2\|a\| < 1.$$

EXAMPLE 5. Take a function $a \in C[0, 1]$ such that $\|a\| < 1/2$. Assume that, the function f from the Assumption (iii) has the form $f(r) = \lambda \ln(r + 1)$ where λ is a positive constant such that

$$\lambda < \frac{1/2 - \|a\|}{\ln(\sqrt{3}/2)},$$

and $(Tx)(t) = 1$. Then, (for $M = 1$) the inequality from (vi) has the form

$$\|a\| + \lambda r \ln(r + 1) \leq r.$$

Taking into account our conditions and applying Bolzano's theorem to the function

$$g(r) = \|a\| + \lambda \ln(r + 1) - r,$$

in the interval $[0, 1/2]$ we can find a positive solution r_0 to the above inequality with $0 < r_0 < 1/2$. Moreover, we have

$$Mf(r_0)Q = f(r_0) = \lambda \ln(r_0 + 1) < \lambda \ln\left(\frac{3}{2}\right) < 1 - 2\|a\| < 1.$$

In what follows let us observe that, the assumptions of our existence result contained in Theorem 2 are rather easy to verify. We illustrate this assertion with help of some examples.

EXAMPLE 6. Consider the following nonlinear quadratic integral equation

$$x(t) = t^3 + \left(\frac{1}{4}x(t) + \frac{1}{4}\right) \int_0^t \left(t + \cos\left(\frac{x^2(\tau)}{1 + x^2(\tau)}\right)\right) d\tau.$$

We investigate the solvability of this equation in the space $C[0, 1]$. Observe that, in our situation we have that $a(t) = t^3$, $\|a\| = 1$ and

$$v(t, \tau, x) = t + \cos\left(\frac{x^2}{1 + x^2}\right).$$

Further, we get $|v(t, \tau, x)| \leq 2$ for all $t, \tau \in [0, 1]$ and $x \in \mathbf{R}$. Thus, the function $f(r)$ has the form $f(r) = 2$. Moreover, the function $t \to v(t, \tau, x)$ is nondecreasing on $[0, 1]$ for fixed $\tau \in [0, 1]$ and $x \in \mathbf{R}$. The operator T is defined by $(Tx)(t) = (1/4)x(t) + 1/4$ and verifies the Assumptions (iv) and (v) with $Q = 1/4$, $c = d = 1/4$. Moreover, the inequality

$$\|a\| + (c + dr)Mf(r) \leq r,$$

has the form

$$1 + \left(\frac{1}{4}r + \frac{1}{4}\right) 2 \leq r,$$

or equivalently,

$$\frac{3}{2} + \frac{1}{2}r \leq r,$$

which has the positive solution $r_0 = 3$ and we have that

$$Mf(r_0)Q = 1 \cdot 2 \cdot \frac{1}{4} = \frac{1}{2} < 1.$$

This allows us to infer that in the space $C[0,1]$ there exists a solution $x = x(t)$ of our equation which is nondecreasing on the interval $[0,1]$.

EXAMPLE 7. Let us consider the following integral equation

$$x(t) = t + \left(\frac{1}{e^3}\int_0^t |x(\tau)|\,d\tau\right)\int_0^t \tau e^{t+(|x(\tau)|/(1+|x(\tau)|))}\,d\tau.$$

We will study if this equation has solution in the space $C[0,1]$. In this case, we have that $M = 1$, $a(t) = t$ and $\|a\| = 1$. Moreover, the function $v(t,\tau,x)$ is defined by

$$v(t,\tau,x) = \tau e^{t+(|x|/(1+|x|))}.$$

We can obtain the following estimate

$$|v(t,\tau,x)| = v(t,\tau,x) = \tau e^{t+(|x|/(1+|x|))} \le e^{1+(|x|/(1+|x|))} \le e^2.$$

Thus, we can take as f the function $f(r) = e^2$. Now, the operator T is $(Tx)(t) = (1/e^3)\int_0^t |x(\tau)|\,d\tau$ and satisfies the Darbo condition with $Q = 0$ and satisfies the Hypothesis (v) with $c = 0$ and $d = 1/e^3$. Now, the inequality of the Assumption (vi) has the following form

$$\|a\| + (c + dr)Mf(r) = 1 + \frac{1}{e^3}r \cdot e^2 \le r,$$

or equivalently

$$1 + \frac{r}{e} \le r.$$

Taking $r_0 = e/(e-1)$ we have that r_0 is a positive solution of the above inequality. Moreover, we deduce that

$$Mf(r_0)Q = 1 \cdot e^2 \cdot 0 = 0 < 1.$$

All the above established facts show that, the assumptions of Theorem 2 are satisfied. So, we conclude that our integral equation has a nondecreasing solution in the space $C[0,1]$.

EXAMPLE 8. Now, let us consider the following integral equation

$$x(t) = \frac{1}{e}t + \frac{1}{4}x(t)\int_0^t \tau(t + \ln(1 + |x(\tau)|))\,d\tau.$$

We will look for solutions of this equation in the space $C[0,1]$. Here, we have that $M = 1$, $a(t) = (1/e)t$, $\|a\| = 1/e$. Moreover, the function $v(t,\tau,x)$ has the form

$$v(t,\tau,x) = \tau(t + \ln(1 + |x|)).$$

We can obtain the following estimate

$$|v(t,\tau,x)| = v(t,\tau,x) = \tau(t + \ln(1 + |x|)) \le 1 + \ln(1 + |x|).$$

As $\ln(1 + |x|) \le |x|$ we have that

$$|v(t,\tau,x)| = v(t,\tau,x) \le 1 + |x|.$$

Thus, we can take as f the function $f(r) = 1+r$. Moreover, the operator T is $(Tx)(t) = (1/4)x(t)$ and satisfies the Darbo condition with $Q = 1/4$ and $c = 0$ and $d = 1/4$. Now, the inequality

$$\|a\| + (c + dr)Mf(r) \leq r,$$

has the expression

$$\frac{1}{e} + \frac{1}{4}r(1+r) \leq r,$$

or equivalently

$$\frac{1}{4}r^4 - \frac{3}{4}r + \frac{1}{e} \leq 0.$$

It is easy to check that, the number

$$r_0 = \frac{3/4 - \sqrt{9/16 - 1/e}}{1/2},$$

is a positive solution of the above inequality. Moreover, we obtain

$$Mf(r_0)Q = 1 \cdot \left(1 + 2\left(\frac{3}{4} - \sqrt{\frac{9e-16}{16e}}\right)\right)\frac{1}{4} < 3 \cdot \frac{1}{4} = \frac{3}{4} < 1.$$

All the above established facts show that, the assumptions of Theorem 2 are satisfied. In view of that theorem we deduce the existence of a solution of our integral equation which belongs to the space $C[0,1]$ and is nondecreasing on the interval $[0,1]$.

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