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Partial pluricomplex energy and integrability exponents of plurisubharmonic functions ☆

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Abstract

We first prove a quantitative estimate of the volume of the sublevel sets of a plurisubharmonic function in a hyperconvex domain with boundary values 0 (in a quite general sense) in terms of its Monge–Ampère mass in the domain. Then we deduce a sharp sufficient condition on the Monge–Ampère mass of such a plurisubharmonic function φ for $\exp(-2\varphi)$ to be globally integrable as well as locally integrable.

Keywords: Plurisubharmonic functions; Pluricomplex Monge–Ampère energy; Volume of the sublevel sets; Exponents of integrability; Log canonical thresholds

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1. Introduction

It is well known that estimates on volumes and capacities of sub-level sets of plurisubharmonic functions from various classes as well as integrability theorems for such classes play an important role in many areas of complex analysis (see [23,25,29,30] and references therein).

A classical result in this direction is Skoda's theorem [28] which asserts that if φ is a plurisubharmonic function defined near some point $a \in \mathbb{C}^n$, then $\exp(-2\varphi)$ is locally integrable in a neighborhood of a if its Lelong number satisfies $\nu_a(\varphi) < 1$.

For $n = 1$ the condition $\nu_a(\varphi) < 1$ turns out to be equivalent to the local integrability of $\exp(-2\varphi)$ in a neighborhood of a. It is possible in this case to derive a global integrability result using classical potential theory. Namely, if φ is a subharmonic function defined on the unit disc $\mathbb{D} \subset \mathbb{C}$ with smallest harmonic majorant identically zero and $2\pi\mu := \int_{\mathbb{D}} \Delta\varphi < +\infty$. Then for any $s > 0$

$$V_2\big(\{\varphi \leqslant -s\}\big) \leqslant 4\pi \exp(-2s/\mu), \tag{1.1}$$

where V_2 is the 2-dimensional Lebesgue measure on $\mathbb{C}$.

From this inequality it is easy to derive a uniform bound on $\int_{\mathbb{D}} e^{-2\varphi}\, dV_2$ when $\int_{\mathbb{D}} \Delta\varphi \leqslant 2\pi\mu < 1$ (see Section 4).

When $n \geqslant 2$, the situation is much more delicate (see [5] for a partial result).

In [18], Demailly provided a sharp condition for the local integrability of $e^{-2\varphi}$, if φ is plurisubharmonic, in terms of the mass of its Monge–Ampère measure $(dd^c\varphi)^n$. It says that if $\Omega \Subset \mathbb{C}^n$, $\varphi \in PSH(\Omega)$ satisfies $-A \leqslant \varphi \leqslant 0$ on $\Omega \setminus K$, where $K \Subset \Omega$ and

$$\int_{\Omega} \big(dd^c\varphi\big)^n \leqslant \mu^n < n^n \tag{1.2}$$

then

$$\int_K e^{-2\varphi}\, dV_{2n} \leqslant C(\Omega, K, A, \mu)$$

where dV_{2n} denotes the $2n$-dimensional Lebesgue measure and $dd^c = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}$. In an appendix to [18], the last-named author observed that from this estimate one can actually deduce a global estimate on the whole of Ω.

This result can be viewed as a non-linear version of Skoda's integrability theorem [28] where the assumption that φ is bounded near $\partial\Omega$ gives a much stronger statement. Without any extra hypothesis the estimate is no longer true as functions depending on one variable only show.

Actually Demailly proved, using an approximation theorem [17] and the semicontinuity theorem for complex singularity exponents of plurisubharmonic functions [19] (both rather difficult) that his criterion is equivalent to a local algebra inequality due to Corti [15] for $n = 2$, and de Fernex, Ein, Mustaţă [21] in the general case. The inequality

$$lc(\mathcal{I}) \geqslant n e(\mathcal{I})^{-1/n}, \tag{1.3}$$

relates $e(\mathcal{I})$ — the Hilbert–Samuel multiplicity of the ideal of germs of holomorphic functions with isolated singularity at the origin in $\mathbb{C}^n$ and $lc(\mathcal{I})$ — the log canonical threshold of $\mathcal{I}$ at the origin.

Introducing his result Demailly called for an "analytic proof" of it for the following reasons:

- the criterion involves only plurisubharmonic functions,
- in his proof the constant $C(\Omega, K, A, \mu)$ is not explicitly given in terms of Ω, K, A, μ,
- an alternative proof combined with Demailly's argument would provide an analytic way of proving (1.3).

We refer to [15,18,20,21] for the discussion of the interesting consequences (1.3) has in the study of birational rigidity of varieties.

Our first aim is to generalize (1.1) into the several complex variables setting. In particular this gives a positive answer to the question of Demailly.

In order to state our main results let us introduce some notations. Throughout this paper, $\Omega \Subset \mathbb{C}^n$ denotes a bounded hyperconvex domain (see Section 2 for the definition). The normalized operator $d^c = \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial)$ is used, so that the complex Monge–Ampère measure given by $\log|z|$ is exactly the Dirac measure at the origin, i.e. $(dd^c \log|z|)^n = \delta_0$.

We shall consider the class $\mathcal{E}(\Omega)$ introduced in [8]. It is the largest set of non-positive plurisubharmonic functions defined on the hyperconvex domain Ω for which the complex Monge–Ampère operator is well defined (Theorem 4.5 in [8]). Let $\mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$ contain those functions with smallest maximal plurisubharmonic majorant identically zero and also with finite total Monge–Ampère mass. Note that if $n = 1$, then the condition of belonging to $\mathcal{F}(\Omega)$ coincides with the above conditions on φ that its smallest harmonic majorant is identically zero and bounded Laplace mass.

Let us state our main results.

Theorem A. *There exists a uniform constant $c_n > 0$, depending only on n such that for any $\varphi \in \mathcal{F}(\Omega)$, and any $s > 0$, we have that*

$$V_{2n}\big(\{\varphi \leqslant -s\}\big) \leqslant c_n \delta_\Omega^{2n} \big(1 + s\mu^{-1}\big)^{n-1} \exp\big(-2ns\mu^{-1}\big), \tag{1.4}$$

where V_{2n} is the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$, $\mu \geqslant 0$ is defined through $\mu^n = \int_\Omega (dd^c\varphi)^n$ and δ_Ω is the diameter of Ω.

Theorem B. *There exist a uniform constant $a_n > 0$, depending only on n, such that for any positive number $0 \leqslant \mu < n$ and any $\varphi \in \mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c\varphi)^n \leqslant \mu^n$, we have that*

$$\int_\Omega e^{-2\varphi}\, dV_{2n} \leqslant \left(\pi^n + a_n \frac{\mu}{(n-\mu)^n}\right)\delta_\Omega^{2n}, \tag{1.5}$$

where V_{2n} is the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$ and δ_Ω is the diameter of Ω.

Theorem C. *Let $\Omega \Subset \mathbb{C}^n$ and $D \Subset \mathbb{C}$ be hyperconvex domains and $\varphi \in \mathcal{F}(\Omega \times D)$. Then for almost all $\zeta \in D$, the energy (up to a sign) of the slice function $\varphi(\cdot, \zeta)$ is well defined by*

$$\varphi_{n+1}(\zeta) := \int_{\Omega_z} \varphi(z,\zeta)\big(dd^c_z \varphi(z,\zeta)\big)^n,$$

and is equal to

$$\varphi_{n+1}(\zeta) = \int_{\Omega_z \times D_\eta} g(\zeta,\eta)\big(dd^c \varphi(z,\eta)\big)^{n+1},$$

where $g = g_D$ is the Green function of D with logarithmic pole. Moreover, $\varphi_{n+1} \in \mathcal{F}(D)$ and its Laplace mass is given by

$$\int_D dd^c \varphi_{n+1} = \int_{\Omega \times D} \big(dd^c \varphi\big)^{n+1}.$$

Theorem C is proved in Section 3 (see Theorem 3.1). Theorem B follows from Theorem A and gives a precise estimate on the global integrability of $\exp(-2\varphi)$ in terms of its Monge–Ampère mass. This will give in particular a precise and global quantitative version of Demailly's theorem (see Section 5).

The proof of Theorem A goes by induction on the dimension $n \geqslant 1$ starting from (1.1) and uses in a crucial way the result of Theorem C. The first step is to reduce to the case where $\Omega = \mathbb{D}^n$ is the unit polydisc using a subextension theorem [14]. Then the key ingredient in the induction process relies on special properties of the pluricomplex energy of the slices of a function $\varphi \in \mathcal{F}(\Omega \times D)$ defined on a product domain (see Section 3). Namely we are able to compute the $(n+1)$-dimensional complex Monge–Ampère mass of a function $\varphi \in \mathcal{F}(\Omega \times D)$ in terms of the Laplacian mass of its partial n-dimensional Monge–Ampère energy function which turns out to be well defined and subharmonic on D.

In Section 6 we give different applications of our results. We prove a useful inequality between the volume and the Monge–Ampère capacity of a Borel set (Proposition 6.1) and an integral estimate of Monge–Ampère capacities of slices of a Borel set in a product domain (Proposition 6.2). Finally in Section 6.3 we deduce a general local transcendental inequality on complex singularity exponents of plurisubharmonic functions (Proposition 6.3) which implies, following an argument of Demailly in [18], the local algebra inequality (1.3).

2. Preliminaries

Let us recall some definitions. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, $\mathbb{D}^n$ the unit polydisc in $\mathbb{C}^n$ and let V_{2n} denote the Lebesgue measure in $\mathbb{C}^n$. Consider also the usual differential operators d and $d^c = \frac{\sqrt{-1}}{2\pi}(\bar\partial - \partial)$ acting on plurisubharmonic functions on domains in $\mathbb{C}^n$ so that $dd^c = (\sqrt{-1}/\pi)\partial\bar\partial$.

For an open set $\Omega \subset \mathbb{C}^n$, we denote by $PSH(\Omega) \subset L^1_{loc}(\Omega)$ the set of plurisubharmonic functions in Ω.

An open set $\Omega \Subset \mathbb{C}^n$ is said to be hyperconvex if it admits a negative plurisubharmonic exhaustion function i.e. there exists a plurisubharmonic function $\rho : \Omega \mapsto [-1, 0[$ such that for any $c < 0$, $\Omega_c := \{z \in \Omega;\ \rho(z) < c\} \Subset \Omega$.

It is well known that a domain $D \Subset \mathbb{C}$ is hyperconvex if and only if it is regular with respect to the Dirichlet problem for the Laplace operator [27]. Therefore any product of regular planar domains (e.g. a polydisc) is a hyperconvex domain. More generally any bounded pseudoconvex domain with Lipschitz boundary is hyperconvex (see [16] and references therein).

Now we recall some notations from [7,8]. We write $\mathcal{E}_0(\Omega)$ for the set of plurisubharmonic test functions i.e. functions $\varphi \in PSH(\Omega) \cap L^\infty(\Omega)$ which tend to zero at the boundary and satisfy $\int_\Omega (dd^c\varphi)^n < +\infty$.

Denote by $\mathcal{F}(\Omega)$ the set of all $\varphi \in PSH(\Omega)$ such that there exists a sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ and $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$.

The class $\mathcal{E}(\Omega)$ will be the set of all $\varphi \in PSH(\Omega)$ such that for any open subset $\omega \Subset \Omega$ there is a function $\psi \in \mathcal{F}(\Omega)$ such that $\psi = \varphi$ on ω.

The complex Monge–Ampère operator is well defined and continuous under decreasing limits in the class $\mathcal{E}(\Omega)$. Moreover in the class $\mathcal{F}(\Omega)$, we have the following strong convergence theorem, namely if (φ_j) is a decreasing sequence of functions in $\mathcal{F}(\Omega)$ which converges to $\varphi \in \mathcal{F}(\Omega)$, then for any $h \in PSH(\Omega)$ such that $h \leqslant 0$, we have (see [8,10])

$$\lim_j \int_\Omega h\big(dd^c\varphi_j\big)^n = \int_\Omega h\big(dd^c\varphi\big)^n.$$

Define $\mathcal{E}_1(\Omega)$ to be the class of plurisubharmonic functions $\varphi \in PSH(\Omega)$ with finite energy i.e. there exists a sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ and $\sup_j \int_\Omega (-\varphi_j)(dd^c\varphi_j)^n < +\infty$. It can be proved that $\mathcal{E}_1(\Omega) \subset \mathcal{E}(\Omega)$ (see [8]).

We will need the following lemma.

Lemma 2.1. *Let $v \in PSH(\Omega) \cap L^\infty(\Omega)$ be such that $\lim_{z\to\zeta} v(\zeta) = 0$ for any $\zeta \in \partial\Omega$. Assume that $\int_\Omega (-v)(dd^cv)^n < +\infty$. Then $v \in \mathcal{E}_1(\Omega)$.*

Proof. Let (Ω_j) be an exhaustion of Ω by bounded domains. It follows from [24] that for $j \in \mathbb{N}$, there exists $v_j \in \mathcal{E}_0(\Omega)$ such that

$$\big(dd^cv_j\big)^n = \mathbf{1}_{\Omega_j}\big(dd^cv\big)^n$$

in Ω. By the comparison principle $(v_j)_j$ is a decreasing sequence from $\mathcal{E}_0(\Omega)$ converging to v. Integration by parts gives that $\int_\Omega (-v_j)(dd^cv_j)^n \leqslant \int_\Omega (-v)(dd^cv)^n < +\infty$, so $v \in \mathcal{E}_1(\Omega)$ by definition. □

Now we introduce the notion of capacity due to Bedford and Taylor [3]. For a given Borel subset $E \subset \Omega$ we define the Monge–Ampère capacity of the condenser (E, Ω) by

$$Cap(E,\Omega) = Cap_\Omega(E) := \sup\left\{\int_E \big(dd^cv\big)^n;\ v \in PSH(\Omega),\ -1 \leqslant v \leqslant 0\right\}.$$

Then by [3] if $E \Subset \Omega$ we have the formula

$$Cap(E,\Omega) = \int_\Omega \big(dd^ch^*_{E,\Omega}\big)^n,$$

where $h_{E,\Omega}$ is the extremal function of (E,Ω) defined by

$$h_{E,\Omega} := \sup\big\{v \in PSH(\Omega);\ v \leqslant 0,\ v|E \leqslant -1\big\}.$$

We will also need the following estimates on the capacity of the sub-level sets of functions in $\mathcal{E}_1(\Omega)$ (see [12]).

Lemma 2.2. *Let $v \in \mathcal{E}_1(\Omega)$. Then for any $s > 0$, we have that*

$$s^{n+1} Cap_\Omega\big(\{v \leqslant -s\}\big) \leqslant \int\limits_\Omega (-v)\big(dd^c v\big)^n.$$

Proof. By homogeneity, it is enough to prove the estimate for $s = 1$. Then take an arbitrary compact subset $K \subset \{v \leqslant -1\}$. If h_K is the extremal function of (K,Ω) the function $h := h_K^* \in \mathcal{E}_0(\Omega)$ and satisfies $v \leqslant h$. Thus using repeatedly integration by parts we obtain that

$$\begin{aligned} Cap(K,\Omega) &= \int\limits_\Omega (-h)\big(dd^c h\big)^n \leqslant \int\limits_\Omega (-v)\big(dd^c h\big)^n \\ &\leqslant \int\limits_\Omega (-h) dd^c v \wedge \big(dd^c h\big)^{n-1} \leqslant \int\limits_\Omega (-v) dd^c v \wedge \big(dd^c h\big)^{n-1} \\ &\leqslant \cdots \leqslant \int\limits_\Omega (-v)\big(dd^c v\big)^n. \qquad \square \end{aligned} \tag{2.1}$$

3. Partial pluricomplex energies

In this section we present the main technical tool which will be used in the proofs of many results to follow. Namely we show that given a function $\varphi \in \mathcal{F}(\Omega \times D)$, its partial n-dimensional Monge–Ampère energy function (up to a sign) is well defined and subharmonic on D and moreover its Laplacian mass is equal to the $(n+1)$-dimensional Monge–Ampère mass of φ. The precise statement is the following (this is Theorem C in the introduction).

Theorem 3.1. *Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}$ be two bounded hyperconvex domains and $g = g_D$ the Green function of D with logarithmic pole. If $\varphi \in \mathcal{F}(\Omega \times D)$, then the slice function $\Omega \ni z \to \varphi(z,\zeta)$ belongs to $\mathcal{E}_1(\Omega)$ for all $\zeta \in D$ with*

$$\int\limits_{\Omega_z \times D_\eta} g(\zeta,\eta)\big(dd^c \varphi(z,\eta)\big)^{n+1} > -\infty.$$

Furthermore, if we define

$$\varphi_{n+1}(\zeta) := \int\limits_{\Omega_z} \varphi(z,\zeta)\big(dd_z^c \varphi(z,\zeta)\big)^n$$

if the integral is well defined and $\varphi_{n+1}(\zeta) = -\infty$ *otherwise, then for any* $\zeta \in D$,

$$\varphi_{n+1}(\zeta) = \int_{\Omega_z \times D_\eta} g(\zeta, \eta)\big(dd^c \varphi(z, \eta)\big)^{n+1}.$$

In particular we have that $\varphi_{n+1} \in \mathcal{F}(D)$ *and it satisfies*

$$\int_D dd^c \varphi_{n+1} = \int_{\Omega \times D} \big(dd^c \varphi\big)^{n+1}.$$

Proof. Assume first that $\varphi \in \mathcal{E}_0(\Omega \times D) \cap C^\infty(\Omega \times D)$, and let $K \Subset \Omega$, $L \Subset D$. Then $0 \geqslant \varphi(z, \zeta)(dd^c_z \varphi(z, \zeta))^n \in C^\infty(\Omega \times D)$ and thus

$$\varphi^K(\zeta) := \int_K \varphi(z, \zeta)\big(dd^c_z \varphi(z, \zeta)\big)^n \in C^\infty(D).$$

For $h \in \mathcal{E}_0(D) \cap C(D)$ we have

$$\begin{aligned}\int_L \varphi^K(\zeta) dd^c h(\zeta) &= \int_L \int_K \varphi(z, \zeta)\big(dd^c_z \varphi(z, \zeta)\big)^n \wedge dd^c h(\zeta) \\ &= \int_K \int_L \varphi(z, \zeta)\big(dd^c \varphi(z, \zeta)\big)^n \wedge dd^c h(\zeta).\end{aligned}$$

Then it follows that

$$\int_K \int_L \varphi(z, \zeta)\big(dd^c \varphi(z, \zeta)\big)^n \wedge dd^c h(\zeta) \geqslant \int_\Omega \int_D \varphi(z, \zeta)\big(dd^c \varphi(z, \zeta)\big)^n \wedge dd^c h(\zeta).$$

By a generalized Jensen–Lelong–Demailly type formula in $\mathcal{F}(\Omega \times D)$ [11, Remark 1], we have

$$\int_{\Omega \times D} \varphi(z, \zeta)\big(dd^c \varphi(z, \zeta)\big)^n \wedge dd^c h(\zeta) = \int_{\Omega \times D} h(\zeta)\big(dd^c \varphi(z, \zeta)\big)^{n+1} > -\infty,$$

since h as a function of $(z, \zeta) \in \Omega \times D$, only depends on $\zeta \in D$ and vanishes on the distinguished boundary of $\Omega \times D$.

Then by letting L increase to D, it follows that φ^K is a decreasing family of continuous functions on Ω which are uniformly integrable on Ω as K increases to Ω. This implies that φ_{n+1} is upper semicontinuous and integrable on Ω and satisfies

$$\int_D \varphi_{n+1}\, dd^c h = \int_{D_\zeta} \int_{\Omega_z} h(\zeta)\big(dd^c \varphi(z, \zeta)\big)^{n+1}, \tag{3.1}$$

for any test function $h \in \mathcal{E}_0(D) \cap C(D)$. Since $C_0^\infty(D) \subset \mathcal{E}_0(D) \cap C(\bar{D}) - \mathcal{E}_0(D) \cap C(\bar{D})$ (see [8]), we get from (3.1) that $dd^c \varphi_{n+1} \geqslant 0$ in the weak sense on Ω, which proves that φ_{n+1} is subharmonic on D.

Now fix $\zeta \in D$ and apply (3.1) to the function $h = \sup\{g(\zeta, \cdot), -j\}$. Then by classical potential theory in $\mathbb{C}$, we deduce that

$$\varphi_{n+1}(\zeta) = \int\limits_{\Omega_z \times D_\eta} g(\zeta, \eta)\big(dd^c \varphi(z, \eta)\big)^{n+1}, \tag{3.2}$$

since $dd^c g(\zeta, \cdot)$ is the Dirac mass at the point ζ. This also proves that $\varphi_{n+1} \in \mathcal{F}(D)$.

If $\varphi_{n+1}(\zeta) > -\infty$, then $v := \varphi(\cdot, \zeta)$ has boundary values 0 and

$$\int\limits_{\Omega} (-v)\big(dd^c v\big)^n =: -\varphi_{n+1}(\zeta) < +\infty,$$

which implies by Lemma 2.1 that $v = \varphi(\cdot, \zeta) \in \mathcal{E}_1(\Omega)$.

For the general case, assume that $\varphi \in \mathcal{F}(\Omega \times D)$. By [9] we can choose a sequence $\varphi^j \in \mathcal{E}_0 \cap C^\infty(\Omega \times D)$ such that $\varphi^j \searrow \varphi$, $j \to +\infty$. It follows from (3.2) that $(\varphi^j_{n+1})_j$ is a decreasing sequence of functions in $\mathcal{F}(D)$ such that

$$\lim_{j \to +\infty} \varphi^j_{n+1}(\zeta) = \int\limits_{\Omega \times D} g(\zeta, \eta)\big(dd^c \varphi\big)^{n+1}.$$

It follows now from the previous case that $(\varphi^j(\cdot, \zeta))_j$ is a decreasing sequence of functions in $\mathcal{E}_1(\Omega)$ with uniformly bounded energies which converges to $\varphi(\cdot, \zeta)$ if $\varphi_{n+1}(\zeta) > -\infty$. Then from Theorem 3.8 in [7], we deduce that $\varphi(\cdot, \zeta) \in \mathcal{E}_1(\Omega)$ if $\varphi_{n+1}(\zeta) > -\infty$. □

Theorem 3.1 says that almost all the slices of a function in $\mathcal{F}(\mathbb{D}^2)$ belong to the space $\mathcal{E}_1(\mathbb{D}^2)$. However these slices do not always belong to $\mathcal{F}(\mathbb{D})$ as the following example shows.

Example 3.2. The function

$$\varphi(z, \zeta) := \sum_{k=1}^{+\infty} \max\big\{\log|z|, k^{-4} \log|\zeta|\big\}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}$$

is an example of a function $\varphi \in \mathcal{F}(\mathbb{D}^2)$ with all slices $\varphi(\cdot, \zeta) \in \mathcal{E}_1(\mathbb{D}) \setminus \mathcal{F}(\mathbb{D})$ if $\zeta \neq 0$ (see Example 5.7 [13]).

The last result can be generalized as follows.

Theorem 3.3. *Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}$ be two bounded hyperconvex domains and $g = g_D$ the Green function of D with logarithmic pole. If $\varphi_j \in \mathcal{F}(\Omega \times D)$, $0 \leqslant j \leqslant n$, then the slice function $\Omega \ni z \to \varphi_0(z, \zeta)$ is integrable on Ω with respect to the measure $dd^c_z \varphi_1(z, \zeta) \wedge \cdots \wedge dd^c_z \varphi_n(z, \zeta)$ for all $\zeta \in D$ with*

$$\int_{\Omega_z \times D_\eta} g(\zeta,\eta)\, dd^c\varphi_0(z,\eta) \wedge dd^c\varphi_1(z,\eta) \wedge \cdots \wedge dd^c\varphi_n(z,\eta) > -\infty.$$

Furthermore, if we define

$$u(\zeta) := \int_{\Omega_z} \varphi_0(z,\zeta)\, dd^c_z\varphi_1(z,\zeta) \wedge \cdots \wedge dd^c_z\varphi_n(z,\zeta), \quad \zeta \in D$$

then for any $\zeta \in D$,

$$u(\zeta) = \int_{\Omega_z \times D_\eta} g(\zeta,\eta)\, dd^c\varphi_0(z,\eta) \wedge dd^c\varphi_1(z,\eta) \wedge \cdots \wedge dd^c\varphi_n(z,\eta)$$

so in particular we have that $u \in \mathcal{F}(D)$ and

$$\int_D dd^c u = \int_{\Omega \times D} dd^c\varphi_0 \wedge dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_n.$$

Proof. Let $h \in \mathcal{E}_0 \cap C(D)$ be a given test function. As in the first part of the proof of Theorem 3.1 we get

$$\int_D u\, dd^c h = \int_{\Omega \times D} \varphi_0(z,\zeta)\, dd^c\varphi_1(z,\zeta) \wedge \cdots \wedge dd^c\varphi_n(z,\zeta) \wedge dd^c h(\zeta).$$

So if we prove that the right-hand side equals

$$\int_{\Omega \times D} h(\zeta)\, dd^c\varphi_0(z,\zeta) \wedge \cdots \wedge dd^c\varphi_n(z,\zeta),$$

the proof can be completed in the same way as in the second part of the proof of the previous theorem.

Indeed for $0 \leqslant k_j \leqslant n$, $0 \leqslant j \leqslant n$ the inequality

$$\int_{\Omega \times D} \varphi_{k_0}(z,\zeta)\, dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta) \wedge dd^c h(\zeta)$$
$$\geqslant \int_{\Omega \times D} h(\zeta)\, dd^c\varphi_{k_0}(z,\zeta) \wedge dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta),$$

can be obtained by approximating h by a decreasing sequence of functions $h_p \in \mathcal{E}_0(\Omega \times D)$, using partial integration in $\mathcal{F}$ and observing that $dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta) \wedge dd^c h_p(\zeta)$ tends weakly to $dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta) \wedge dd^c h(\zeta)$ when p tends to ∞.

Again, since $\mathcal{F}$ is a convex cone, $\psi := \sum_{j=0}^n \varphi_j \in \mathcal{F}(\Omega \times D)$ and then it follows from [11] that

$$\int_{\Omega\times D} \psi(z,\zeta)\big(dd^c\psi(z,\zeta)\big)^n \wedge dd^c h(\zeta) = \int_{\Omega\times D} h(\zeta)\big(dd^c\psi(z,\zeta)\big)^{n+1}.$$

Using the separate linearity of the wedge product, we can expand both sides to obtain sums of terms of the form

$$\int_{\Omega\times D} \varphi_{k_0}(z,\zeta)\, dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta) \wedge dd^c h(\zeta)$$

on the left-hand side, while on the right-hand side we get terms

$$\int_{\Omega\times D} h(\zeta)\, dd^c\varphi_{k_0}(z,\zeta) \wedge dd^c\varphi_{k_1}(z,\zeta) \wedge \cdots \wedge dd^c\varphi_{k_n}(z,\zeta).$$

Since they have the same sum, they have all to be equal which completes the proof of the theorem. □

4. Volume estimates of sub-level sets

Here we prove Theorem A (see Corollary 4.2 below). Let us first prove it for the polydisc.

Theorem 4.1. *There exists a constant $c_n > 0$ such that for any $\mu > 0$, any $\varphi \in \mathcal{F}(\mathbb{D}^n)$ with $\int_{\mathbb{D}^n}(dd^c\varphi)^n \leqslant \mu^n$ and any $s > 0$, we have that*

$$V_{2n}\big(\{\varphi \leqslant -s\}\big) \leqslant c_n(1+s/\mu)^{n-1}\exp(-2ns/\mu). \tag{4.1}$$

Proof. We prove the theorem using induction over the dimension n.

For $n = 1$, the estimate was proved in [5]. Let us recall its proof here for the convenience of the reader. We use the classical Pólya's inequality which we recall. Let $K \subset \mathbb{C}$ be a compact subset in the complex plane with area $A(K)$ and logarithmic capacity $c(K)$. In [26], Pólya proved what we today could write as

$$A(K) \leqslant \pi c(K)^2 \tag{4.2}$$

(for an elegant proof see e.g. Theorem 5.3.5 in [27]). We can assume that K is not polar. Then from the Riesz representation formula for the Green function V_K of K with pole at infinity, we obtain that

$$c(K) \leqslant 2\exp\Big(-\sup_{|z|=1} V_K(z)\Big).$$

Now denote by $M_K := \sup_{|z|=1} V_K(z)$. Then $M_K^{-1} V_K \leqslant h_{K,\mathbb{D}}$ on $\mathbb{D}$. Since by our normalization $\int_{\mathbb{D}} dd^c V_K = \int_{\mathbb{C}} dd^c V_K = 1$, it follows from the comparison principle for the Laplace operator that

$$M_K^{-1} = M_K^{-1} \int_{\mathbb{D}} dd^c V_K \leqslant \int_{\mathbb{D}} dd^c h_K^* = Cap(K, \mathbb{D}).$$

This inequality is due to Alexander and Taylor (see [2]). Then putting all together we obtain the inequality

$$V_2(K) \leqslant 4\pi \exp\bigl(-2/Cap(K, \mathbb{D})\bigr). \tag{4.3}$$

It is clear that this inequality is still true for Borel subsets $K \subset \mathbb{D}$. Now if $\varphi \in \mathcal{F}(\mathbb{D})$, we know that $Cap(\{\varphi \leqslant -s\}, \mathbb{D}) \leqslant \int_{\mathbb{D}} dd^c\varphi/s$. Therefore we have that

$$V_2\bigl(\{\varphi \leqslant -s\}\bigr) \leqslant 4\pi \exp(-2s/\mu), \tag{4.4}$$

where $\mu := \int_{\mathbb{D}} dd^c\varphi$ (see [5]).

Now assume that the estimate (4.1) is true in dimension n and let us prove it in dimension $n+1$.

Fix $\varphi \in \mathcal{F}(\mathbb{D}^{n+1})$ such that

$$\int_{\mathbb{D}^{n+1}} \bigl(dd^c\varphi\bigr)^{n+1} \leqslant \mu^{n+1}.$$

By homogeneity, it is enough to prove the estimate for $s = 1$. Then we want to estimate the volume $V_{2n+2}(\{\varphi \leqslant -1\})$ by applying Fubini's theorem. So fix $\zeta \in \mathbb{D}$ and estimate the volume $V_{2n}(\{z \in \mathbb{D}^n\colon \varphi(z,\zeta) \leqslant -1\})$. Indeed, define $E_\zeta := \{z \in \mathbb{D}^n\colon \varphi(z,\zeta) \leqslant -1\}$, consider its relative extremal function $h_\zeta := h^*_{E_\zeta}$ and observe that $V_{2n}(E_\zeta) = V_{2n}(\{h_\zeta \leqslant -1\})$, since the two sets coincide up to a pluripolar set. We want to apply the induction hypothesis to the function h_ζ.

Fix $\zeta \in \mathbb{D}$ such that $\varphi_{n+1}(\zeta) > -\infty$ and observe that $h_\zeta \geqslant v := \varphi(\cdot,\zeta)$. By Theorem 3.1, the function $v = \varphi(\cdot,\zeta) \in \mathcal{E}_1(\mathbb{D}^n)$ and then $h_\zeta \in \mathcal{E}_1(\mathbb{D}^n)$. On the other hand, by [3], we know that

$$\int_{\mathbb{D}^n} \bigl(dd^c h_\zeta\bigr)^n = Cap\bigl(E_\zeta, \mathbb{D}^n\bigr).$$

Then since $v = \varphi(\cdot,\zeta) \in \mathcal{E}_1(\mathbb{D}^n)$, it follows from Lemma 2.2 that

$$Cap\bigl(E_\zeta, \mathbb{D}^n\bigr) = Cap\bigl(\{v \leqslant -1\}, \mathbb{D}^n\bigr) \leqslant \int_{\mathbb{D}^n} (-v)\bigl(dd^c v\bigr)^n = -\varphi_{n+1}(\zeta) < +\infty.$$

This implies that for any $\zeta \in \mathbb{D}$ such that $\varphi_{n+1}(\zeta) > -\infty$,

$$\int_{\mathbb{D}^n} \bigl(dd^c h_\zeta\bigr)^n \leqslant -\varphi_{n+1}(\zeta) < +\infty$$

and then $h_\zeta \in \mathcal{F}(\mathbb{D}^n)$.

Now applying the induction hypothesis to the function $h_\zeta \in \mathcal{F}(\mathbb{D}^n)$, we deduce that for almost all $\zeta \in \mathbb{D}$,

$$V_{2n}\big(\{\varphi(\cdot,\zeta)\leqslant -1\}\big)\leqslant c_n\big(1+\big(-\varphi_{n+1}(\zeta)\big)^{-1/n}\big)^{n-1}\exp\big(-2n\big(-\varphi_{n+1}(\zeta)\big)^{-1/n}\big).$$

Then integrating in $\zeta\in\mathbb{D}$, we get

$$V_{2n+2}\big(\{\varphi\leqslant -1\}\big)\leqslant c_n\int\limits_{\mathbb{D}}\chi\big(-\varphi_{n+1}(\zeta)\big)\,dV_2(\zeta),\tag{4.5}$$

where

$$\chi(t):=\big(1+t^{-1/n}\big)^{n-1}\exp\big(-2nt^{-1/n}\big),\quad t\geqslant 0.$$

It is easy to check that the function χ is increasing with $\chi(0)=0$ and $\chi(+\infty)=1$. Therefore from (4.5), it follows that

$$V_{2n+2}\big(\{\varphi\leqslant -1\}\big)\leqslant c_n\int\limits_0^{+\infty}\chi'(t)V_2\big(\{\varphi_{n+1}\leqslant -t\}\big)\,dt.\tag{4.6}$$

Since $\int_{\mathbb{D}}dd^c\varphi_{n+1}=\int_{\mathbb{D}^{n+1}}(dd^c\varphi)^{n+1}\leqslant\mu^{n+1}$ by Theorem 3.1, it follows from (4.6) that

$$V_{2n+2}\big(\{\varphi\leqslant -1\}\big)\leqslant c_n\int\limits_0^{+\infty}\chi'(t)\exp\big(-2t\mu^{-n-1}\big)\,dt.$$

Now using the change of variable $x=t^{-1/n}$ and observing that

$$\chi'(t)\,dt=-(2nx+n+1)(1+x)^{n-2}e^{-2nx}\,dx,$$

we get the following estimate

$$V_{2n+2}\big(\{\varphi\leqslant -1\}\big)\leqslant 8n\pi c_n\int\limits_0^{+\infty}(x+1)^{n-1}\exp\big(-2\big(nx+x^{-n}\mu^{-n-1}\big)\big)\,dx.\tag{4.7}$$

Now observe that the function $\mathbb{R}^+\ni x\mapsto 2(nx+x^{-n}\mu^{-n-1})$ reaches its minimum at the point $x=1/\mu$ and this minimum is precisely equal to $2(n+1)\mu^{-1}$. Then splitting the integral in (4.7) into two parts, integrating first from 0 to $3/\mu$ and then from $3/\mu$ to $+\infty$, we easily get

$$\begin{aligned}V_{2n+2}\big(\{\varphi\leqslant -1\}\big)\leqslant{}&8n\pi c_n(3/\mu)(1+3/\mu)^{n-1}\exp\big(-2(n+1)\mu^{-1}\big)\\&+8n\pi c_n\int\limits_{3/\mu}^{+\infty}(1+x)^{n-1}\exp(-2nx)\,dx.\end{aligned}\tag{4.8}$$

It is easy to see that the last terms is much smaller that the first one and can be easily estimated from above by $8\pi c_n\exp(-2(n+1)/\mu)$ so that we finally get

$$V_{2n+2}\big(\{\varphi \leqslant -1\}\big) \leqslant 8\pi\big(n3^n+1\big)c_n(1+1/\mu)^n \exp\big(-2(n+1)\mu^{-1}\big).$$

This implies that the inequality (4.1) holds with

$$c_n := 2^{3n-1}\pi^n \prod_{0\leqslant k\leqslant n-1}\big(k3^k+1\big), \quad n \geqslant 1. \qquad \square \tag{4.9}$$

The volume estimate actually holds in the following general setting which will prove Theorem A.

Corollary 4.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain. Then for any $\varphi \in \mathcal{F}(\Omega)$ and any $s > 0$, we have*

$$V_{2n}\big(\{\varphi \leqslant -s\}\big) \leqslant c_n \delta_\Omega^{2n}\big(1+s\mu^{-1}\big)^{n-1} \exp\big(-2ns\mu^{-1}\big), \tag{4.10}$$

where $\mu^n := \int_\Omega (dd^c\varphi)^n$, δ_Ω is the diameter of Ω and c_n is the constant defined by (4.9).

Proof. Observe that the inequality is invariant under holomorphic linear change of variables so that we can always assume that $\Omega \Subset \mathbb{D}^n$. Then by the subextension theorem [14], there exists a function $\psi \in \mathcal{F}(\mathbb{D}^n)$ such that $\psi \leqslant \varphi$ and $\int_{\mathbb{D}^n}(dd^c\psi)^n \leqslant \int_{\mathbb{D}^n}(dd^c\varphi)^n = \mu^n$. Applying the estimate of Theorem 4.1 to ψ, we obtain the required estimate. $\square$

Remark 1. An analogous estimate was obtained by Demailly and Kollár for plurisubharmonic functions of the type $\varphi = \log(\sum_{j=1}^N |g_j|^2)$, where $g_1, \ldots, g_N$ are holomorphic functions (see [19, Proposition 1.7, p. 531]).

Remark 2. It is possible to improve slightly the estimate (4.1) at least asymptotically by replacing in the right-hand side the factor $(1+s/\mu)^{n-1}$ by $(1+\sqrt{s/\mu})^{n-1}$. This can be easily done by studying the asymptotics of the integral on the right-hand side of the estimate (4.7) using the classical method of Laplace. We do not know if this estimate is sharp.

5. Integrability theorems in terms of Monge–Ampère masses

In this section we prove Theorem B (see Corollary 5.2) stated in the introduction, which will give a pluripotential proof of a theorem due to Demailly [18]. We also prove a theorem on local integrability.

5.1. Global integrability

Theorem 5.1. *Let $\varphi \in \mathcal{F}(\mathbb{D}^n)$ such that $\int_{\mathbb{D}^n}(dd^c\varphi)^n \leqslant \mu^n$ with $\mu < n$. Then*

$$\int_{\mathbb{D}^n} e^{-2\varphi}\, dV_{2n} \leqslant \pi^n + a_n \frac{\mu}{(n-\mu)^n},$$

where $a_n > 0$ is a dimensional constant.

Proof. By Theorem 4.1, we have

$$\int_{\mathbb{D}^n} e^{-2\varphi}\, dV_{2n} = \pi^n + 2\int_0^{+\infty} e^{2s} V_{2n}\big(\{\varphi < -s\}\big)\, ds$$
$$\leqslant \pi^n + 2c_n \int_0^{+\infty} (1+s/\mu)^{n-1} e^{2s-2ns/\mu}\, ds.$$

Now it is easy to see by integration by parts that the integrals

$$I_n := \int_0^{+\infty} (1+s/\mu)^{n-1} e^{2s-2ns/\mu}\, ds$$

satisfy the inequality

$$I_n \leqslant \frac{(n-1)!}{2^n}\frac{\mu}{(n-\mu)^n},$$

for $n \geqslant 1$, and the required estimate follows with the constant

$$a_n := \frac{(n-1)!}{2^{n-1}} c_n. \qquad \square \tag{5.1}$$

We have a more general result.

Corollary 5.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain. Then for any $\varphi \in \mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c\varphi)^n \leqslant \mu^n < n^n$ with $\mu < n$, we have that*

$$\int_\Omega e^{-2\varphi}\, dV_{2n} \leqslant \left(\pi^n + a_n \frac{\mu}{(n-\mu)^n}\right) \delta_\Omega^{2n},$$

where δ_Ω is the diameter of the domain Ω and a_n is the constant defined by (5.1).

Proof. The proof is the same as before using Corollary 4.2. $\square$

Remark. The proof of Corollary 5.2 uses a subextension argument (see the proof of Corollary 4.2). It is interesting to note that it follows from results in [1] that to every $\varphi \in \mathcal{F}(\Omega)$ there are two uniquely determined functions $\varphi_1, \varphi_2 \in \mathcal{F}(\Omega)$, $\varphi \leqslant \varphi_1$, $\varphi \leqslant \varphi_2$ such that

$$\big(dd^c\varphi_1\big)^n = \mathbf{1}_{\{\varphi > -\infty\}}\big(dd^c\varphi\big)^n, \qquad \big(dd^c\varphi_2\big)^n = \mathbf{1}_{\{\varphi = -\infty\}}\big(dd^c\varphi\big)^n.$$

Furthermore, $\varphi_1 + \varphi_2 \leqslant \varphi$, so the integrability exponent of φ is completely determined by its "singular part" φ_2. Indeed since $(dd^c\varphi_1)^n$ puts no mass on pluripolar sets, the Lelong numbers of φ_1 are 0 and then it follows that $e^{-2\varphi_1} \in L^1(\Omega)$ (see [14]).

Also, let $\Omega \Subset \tilde{\Omega}$ be another hyperconvex domain and let $\tilde{\varphi}_2$ be the maximal subextension of φ to $\tilde{\Omega}$. Then it follows that $\tilde{\varphi}_2$ has the same Monge–Ampère measure as φ_2.

As a corollary we get a strengthened version of Demailly's theorem [17].

Corollary 5.3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain and $M > 0$ a fixed constant. Then for any $\varphi \in PSH(\Omega)$ with $0 \geqslant \varphi \geqslant -M$ near the boundary, $\varphi \in \mathcal{E}(\Omega)$. Moreover if $\int_\Omega (dd^c\varphi)^n \leqslant \mu^n$ with $\mu < n$, we have that*

$$\int_\Omega e^{-2\varphi}\,dV_{2n} \leqslant \left(\pi^n + a_n \frac{\mu}{(n-\mu)^n}\right) e^{2M} \delta_\Omega^{2n},$$

where $\delta_\Omega := \operatorname{diam}(\Omega)$ is the diameter of Ω and a_n is the constant defined by (5.1).

Proof. We can assume the domain Ω to be hyperconvex. It follows from Theorem 2.1 in [10] that there exists $\psi \in \mathcal{F}(\Omega)$ with $\int_\Omega (dd^c\psi)^n \leqslant \int_\Omega (dd^c\varphi)^n$ such that $\varphi \geqslant \psi - M$ on Ω. Then the result follows from Corollary 5.2. □

Now we investigate integrability in the critical case when the total Monge–Ampère mass has the maximal value n^n.

Theorem 5.4. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\varphi \in \mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c\varphi)^n = n^n$. Then for any real number $\lambda > n$, we have that*

$$\int_\Omega \frac{e^{-2\varphi}}{(1-\varphi)^\lambda}\,dV_{2n} \leqslant \bigl(1 + (2/\lambda)^\lambda e^{\lambda-2}\bigr) V_{2n}(\Omega) + 2c_n \delta_\Omega^{2n} \frac{1}{\lambda - n},$$

where c_n is the constant defined by (4.9).

Proof. Indeed, set $\chi(t) := (1+t)^{-\lambda} e^{2t}$, for $t \geqslant 0$. Since

$$\chi'(t) = \bigl(-\lambda(1+t)^{-\lambda-1} + 2(1+t)^{-\lambda}\bigr)e^{2t} = (2 - \lambda + 2t)(1+t)^{-\lambda-1}e^{2t},$$

it follows that the function χ is increasing for $t \geqslant t_0 := (\lambda - 2)/2$ and decreasing on $[0, t_0]$.

Therefore we have that

$$\begin{aligned}
\int_\Omega \frac{e^{-2\varphi}}{(1-\varphi)^\lambda}\,dV_{2n} &= \int_{-\varphi < t_0} \frac{e^{-2\varphi}}{(1-\varphi)^\lambda}\,dV_{2n} + \int_{-\varphi \geqslant t_0} \frac{e^{-2\varphi}}{(1-\varphi)^\lambda}\,dV_{2n} \\
&\leqslant V_{2n}(\Omega) + \int_{\varphi \leqslant -t_0} \frac{e^{-2\varphi}}{(1-\varphi)^\lambda}\,dV_{2n} \\
&\leqslant V_{2n}(\Omega) + \chi(t_0)V_{2n}(\Omega) + \int_{t_0}^{+\infty} \chi'(t) V_{2n}\bigl(\{\varphi \leqslant -t\}\bigr)\,dt.
\end{aligned}$$

By Corollary 4.2, we have that

$$\int_{\Omega} \frac{e^{-2\varphi}}{(1-\varphi)^{\lambda}}\, dV_{2n} \leqslant \big(1+\chi(t_0)\big)V_{2n}(\Omega) + 2c_n\delta_{\Omega}^{2n}\int_0^{+\infty}(1+t)^{n-\lambda-1}\,dt. \qquad \square$$

5.2. Local integrability of $\exp(-2\varphi)$

Theorem 5.5. *Suppose* $\varphi \in \mathcal{E}(\Omega)$ *and* $a \in \Omega$. *If* $\int_{\{a\}}(dd^c\varphi)^n < n^n$, *then* $\exp(-2\varphi)$ *is locally integrable near* a.

Proof. By definition, functions in $\mathcal{E}(\Omega)$ are locally in $\mathcal{F}(\Omega)$ so we can assume that $\varphi \in \mathcal{F}(\Omega)$. Set for $j \geqslant 1$,

$$\psi_j := \sup\big\{u \in PSH(\Omega);\ u \leqslant 0,\ u \leqslant \varphi \text{ on } B_j\big\},$$

where $B_j := \mathbb{B}(a, 1/j)$ is the ball of center a and radius $1/j$.

Then $\psi_j \in \mathcal{F}(\Omega)$, $\psi_j \geqslant \varphi$ and $\psi_j = \varphi$ on B_j. Moreover, $\operatorname{supp}(dd^c\psi_j)^n \Subset B_{j-1}$. Denote by $G(z,a)$ the pluricomplex Green function for Ω with logarithmic pole at a and choose $\delta > 0$ so small that

$$\int_{\Omega}\big(-\max\{\delta G(z,a), -1\}\big)\big(dd^c\varphi\big)^n < n^n.$$

Using integration by parts in $\mathcal{F}(\Omega)$ we see that

$$\int_{\Omega}\big(-\max\{\delta G(z,a), -1\}\big)\big(dd^c\psi_j\big)^n \leqslant \int_{\Omega}\big(-\max\{\delta G(z,a), -1\}\big)\big(dd^c\varphi\big)^n < n^n.$$

If we choose k so large that $B_{k-1} \Subset \{\delta G(z,a) < -1\}$, it follows that $\int_{\Omega}(dd^c\psi_k)^n = \int_{B_{k-1}}(dd^c\psi_k)^n < n^n$.

Now since $\psi_k = \varphi$ on B_k, it follows from Corollary 5.2 that

$$\int_{B_k} e^{-2\varphi}\, dV_n = \int_{B_k} e^{-2\psi_k}\, dV_n \leqslant \int_{\Omega} e^{-2\psi_k}\, dV_n < +\infty. \qquad \square$$

Remark 1. Note that the theorem is optimal as the functions $(n-\varepsilon)\log|z-a|$ $(\varepsilon > 0)$ show.

Remark 2. The following example shows that there is no local version of Theorem 5.4. Indeed, fix $0 < \alpha < 1/n$ and consider the function defined on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ by

$$\varphi(z) := n\ln|z| - \big(-\ln|z|\big)^{\alpha}, \quad z \in \mathbb{B}_n.$$

Then $\varphi \in \mathcal{E}(\mathbb{B}_n)$ and $(dd^c\varphi)^n(\{0\}) = n^n$. However for any $\lambda > 0$ the function $(1-\varphi)^{-\lambda}e^{-2\varphi}$ is not locally integrable near 0.

6. Applications

6.1. An inequality between volume and capacity

Our first application of Theorem 4.1 is a useful inequality between volume and Monge–Ampère capacity improving a previous result in [5].

Proposition 6.1. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. Then for any Borel subset $E \subset \Omega$, we have that*

$$V_{2n}(E) \leqslant c_n \delta_\Omega^{2n} \big(1 + Cap_\Omega(E)^{-1/n}\big)^{n-1} \exp\big(-2n Cap_\Omega(E)^{-1/n}\big), \tag{6.1}$$

where $\delta_\Omega := \operatorname{diam}(\Omega)$ is the diameter of Ω and c_n is the constant defined by (4.9).

Proof. We first assume that $E \Subset \Omega$. Then its plurisubharmonic relative extremal function satisfies $h_E^* \in \mathcal{E}_0(\Omega)$. Therefore applying the last corollary, we obtain

$$V_{2n}(E) \leqslant V_{2n}\big(\big\{h_E^* \leqslant -1\big\}\big) \leqslant c_n \delta_\Omega^{2n} \big(1 + \mu^{-1}\big)^{n-1} \exp\big(-2n\mu^{-1}\big),$$

where $\mu^n = \int_\Omega (dd^c h_E^*)^n$. Then the estimate of the theorem follows since $\int_\Omega (dd^c h_E^*)^n = Cap(E, \Omega)$ by [3].

Now assume that $Cap_\Omega(E) < +\infty$. Then approximating E by a non-decreasing sequence of relatively compact subsets of Ω, it follows from continuity properties of the Monge–Ampère operator in $\mathcal{F}(\Omega)$ that $h_E^* \in \mathcal{F}(\Omega)$ and the formula $\int_\Omega (dd^c h_E^*)^n = Cap_\Omega(E)$ still holds in this case. The proof of the inequality follows then in the same way. □

Observe that actually the estimates (4.10) and (6.1) are equivalent since for a function $\varphi \in \mathcal{F}(\Omega)$ we know that $Cap(\{\varphi \leqslant -s\}) \leqslant s^{-n} \int_\Omega (dd^c \varphi)^n$ (see [12]).

Remark. Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain such that $\Omega \cap \mathbb{R}^n \neq \emptyset$. Then the same method can be used to prove an estimate of the n-dimensional volume of Borel subsets of $\Omega \cap \mathbb{R}^n$ in terms of their capacity with respect to Ω. Namely, if $K \subset \Omega \cap \mathbb{R}^n$ is a Borel subset, then its n-dimensional volume satisfies the inequality

$$V_n(K) \leqslant b_n \delta_\Omega^{2n} \big(1 + Cap_\Omega(K)^{-1/n}\big)^{n-1} \exp\big(-n Cap_\Omega(K)^{-1/n}\big),$$

where $b_n > 0$ is a uniform constant which can be made explicit. The proof uses induction as before and the following real version of the inequality (4.4) (see [27]): if $K \subset [-1, +1]$ is a real compact set of length $V_1(K)$ and logarithmic capacity $c(K)$ then

$$V_1(K)/4 \leqslant c(K) \leqslant 2 \exp\big(-1/Cap_{\mathbb{D}}(K)\big).$$

(See [5] where such kind of estimates were considered.)

6.2. Integral estimates for capacity of slices

Given a Borel subset $E \subset \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$, we define its n-dimensional slices as follows. For a given $\zeta \in \mathbb{C}^m$ we set

$$E_\zeta := \{z \in \mathbb{C}^n;\ (z, \zeta) \in E\}.$$

It is easy to see that if E is pluripolar then its slices E_ζ are pluripolar sets in $\mathbb{C}^n$ except for a pluripolar set of ζ's in $\mathbb{C}^m$. The converse is not true as the following example of Kiselman [22]

$$S := \{(z, w) \in \mathbb{C}^2;\ \mathrm{Im}(z + w^2) = \mathrm{Re}(z + w + w^2) = 0\}$$

shows. Indeed S is a smooth totally real analytic 2-manifold in $\mathbb{C}^2$ whose intersection with any complex line consists of at most 4 points.

Here we want to give a quantitative estimate in terms of Monge–Ampère capacity of the size of the slices of a Borel set. We consider an increasing function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $F(0^+) = 0$ and polynomial growth at infinity i.e. there is a constant $N > 0$ such that $F(t) = O(t^N)$. To such a function we associate the following function ($m \in \mathbb{N}^*$)

$$\tilde{F}_m(x) := \int_{\mathbb{R}^{+m}} F(t_1 \cdots t_m \cdot x) e^{-2(t_1 + \cdots + t_m)}\, dt_1 \cdots dt_m. \tag{6.2}$$

Then we can prove the following estimate on the Monge–Ampère capacity of slices of Borel sets.

Proposition 6.2. *Let $\Omega \subset \mathbb{C}^n$ and $D \Subset \mathbb{C}^m$ be two hyperconvex domains and $\tilde{\Omega} := \Omega \times D$. Let $F : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ be an increasing function as above. Then for any Borel subset $E \subset \tilde{\Omega}$ we have that*

$$\int_D F(Cap_\Omega(E_\zeta))\, dV_{2m}(\zeta) \leqslant (8\pi)^m \delta_D^{2m} \tilde{F}_m(Cap_{\tilde{\Omega}}(E)). \tag{6.3}$$

Proof. It is enough to assume that $Cap_{\tilde{\Omega}}(E) < +\infty$. As in the proof of Proposition 6.1, if h_E^* is the plurisubharmonic extremal function of the condenser $(E, \tilde{\Omega})$, we see that $h_E^* \in \mathcal{F}(\tilde{\Omega})$.

First suppose that $m = 1$ and $D = \mathbb{D} \subset \mathbb{C}$ is the unit disc in $\mathbb{C}$. For each $\zeta \in \mathbb{D}$, let $h_{E_\zeta}^*$ be the plurisubharmonic extremal function of the condenser (E_ζ, Ω). It follows from the definitions that for any $\zeta \in \mathbb{D}$, the partial function $h_E^*(\cdot, \zeta)$ satisfies the inequality

$$h_E^*(\cdot, \zeta) \leqslant h_{E_\zeta}^*, \quad \text{on } \Omega.$$

Moreover by Theorem 3.1, for almost all $\zeta \in \mathbb{D}$, these functions are in $\mathcal{E}_1(\Omega)$ and then by the proof of Lemma 2.1, we have

$$Cap_\Omega(E_\zeta) = \int_\Omega (-h_{E_\zeta}^*)(dd^c h_{E_\zeta}^*)^n$$

$$\leqslant \int_{\Omega} \left(-h_E^*(\cdot,\zeta)\right) dd^c h_E^*(\cdot,\zeta)^n$$
$$=: -u(\zeta),$$

where $u(\zeta)$ is precisely the partial energy function associated to the function $h_E^* \in \mathcal{F}(\tilde{\Omega})$. The measurability of the function $\zeta \longmapsto Cap_{\Omega}(E_\zeta)$ follows from a combination of Theorems IX:4 and IX:5 in [6]. Therefore we have that

$$\int_{\mathbb{D}} F\left(Cap_{\Omega}(E_\zeta)\right) dV_2(\zeta) \leqslant \int_{\mathbb{D}} F\left(-u(\zeta)\right) dV_2(\zeta)$$
$$= \int_0^{+\infty} V_2\left(\{u \leqslant -s\}\right) dF(s). \tag{6.4}$$

By Theorem 3.1, we also have that $u \in \mathcal{F}(\mathbb{D})$ and

$$\int_{\mathbb{D}} dd^c u = \int_{\tilde{\Omega}} \left(dd^c h_E^*\right)^{n+1} = Cap_{\tilde{\Omega}}(E).$$

Then applying (4.1) in the one-dimensional case, it follows from (6.4) that

$$\int_{\mathbb{D}} F\left(Cap_{\Omega}(E_\zeta)\right) dV_2(\zeta) \leqslant 4\pi \int_0^{+\infty} \exp\left(-2s/Cap_{\tilde{\Omega}}(E)\right) dF(s). \tag{6.5}$$

Since $F(t) = O(t^N)$ as $t \to +\infty$, setting $t = s/Cap_{\tilde{\Omega}}(E)$ and integrating by parts in the right-hand side of the estimate (6.5), we obtain

$$\int_{\mathbb{D}} F\left(Cap_{\Omega}(E_\zeta)\right) dV_2(\zeta) \leqslant 8\pi \int_0^{+\infty} F\left(t \cdot Cap_{\tilde{\Omega}}(E)\right) e^{-2t}\, dt.$$

Now if $D = \mathbb{D}^2 \Subset \mathbb{C}^2$ is the unit bidisc, we can iterate the previous inequality. Observe that for any $\zeta = (\zeta_1, \zeta_2) \in \mathbb{D}^2$, we have that $E_{\zeta_2} \subset \Omega \times \mathbb{D}$ and $E_\zeta = (E_{\zeta_2})_{\zeta_1}$.

Therefore using the previous estimate twice, we get

$$\int_{\mathbb{D}^2} F\left(Cap_{\Omega}(E_\zeta)\right) dV_4(\zeta) = \int_{\mathbb{D}} dV_2(\zeta_2) \int_{\mathbb{D}} F\left(Cap_{\Omega}\left((E_{\zeta_2})_{\zeta_1}\right)\right) dV_2(\zeta_1)$$
$$\leqslant 8\pi \int_{\mathbb{D}} dV_2(\zeta_2) \int_0^{+\infty} F\left(t \cdot Cap_{\Omega \times \mathbb{D}}(E_{\zeta_2})\right) e^{-2t}\, dt$$

$$\leqslant (8\pi)^2 \int_0^{+\infty}\int_0^{+\infty} F\big(s \cdot t \cdot Cap_{\Omega\times\mathbb{D}}(E)\big) e^{-2(s+t)}\, ds\, dt.$$

Now for $m \geqslant 3$ we obtain by induction on m,

$$\int_{\mathbb{D}^m} F\big(Cap_\Omega(E_\zeta)\big)\, dV_{2m}(\zeta) \leqslant (8\pi)^m \tilde{F}_m\big(Cap_{\Omega\times\mathbb{D}^m}(E)\big).$$

In the general case we can always assume that $D \subset \mathbb{D}^n$ and then $Cap_{\Omega\times\mathbb{D}^n}(E) \leqslant Cap_{\Omega\times D}(E)$ and the required estimate follows. $\square$

As a simple example of application of the last result we get

Corollary 6.3. *Let $\Omega \subset \mathbb{C}^n$ and $D \Subset \mathbb{C}^m$ be two hyperconvex domains and $\tilde{\Omega} := \Omega \times D$. Assume that $E \subset \tilde{\Omega}$ is a Borel subset such that $Cap_{\tilde{\Omega}}(E) < +\infty$. Then for any real number $p > 0$ we have that*

$$\int_D \big(Cap_\Omega(E_\zeta)\big)^p\, dV_{2m}(\zeta) \leqslant (4\pi)^m \delta_D^{2m} 2^{-mp} \Gamma(p+1)^m Cap_{\tilde{\Omega}}(E)^p, \tag{6.6}$$

where Γ is the Euler function defined by $\Gamma(p) := \int_0^{+\infty} t^{p-1}e^{-t}\, dt$.

Proposition 6.2 can be used to construct new measures dominated by capacity. This notion is important in the study of the range of the complex Monge–Ampère operator on various classes of plurisubharmonic functions on bounded hyperconvex domains (see [4,7,25]).

A positive Borel measure ν on a hyperconvex domain $\Omega \Subset \mathbb{C}^n$ is said to be dominated by the Monge–Ampère capacity on Ω if there exists an increasing function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $F(0^+) := \lim_{t\to 0^+} F(t) = 0$ such that

$$\nu(K) \leqslant F\big(Cap_\Omega(K)\big),$$

for any Borel subset $K \subset \Omega$. In such a case, we will say that ν is F-dominated by the Monge–Ampère capacity on Ω.

It is clear from Proposition 6.1 that the Lebesgue measure V_{2n} is dominated by the Monge–Ampère capacity on any bounded hyperconvex domain $\Omega \Subset \mathbb{C}^n$.

As a consequence of Proposition 6.2, we obtain the following generalization of Proposition 6.1.

Corollary 6.4. *Let $\Omega \Subset \mathbb{C}^n$ and $D \Subset \mathbb{C}^m$ be bounded hyperconvex domains. If ν is a Borel measure F-dominated by Monge–Ampère capacity on Ω, then the product measure $\nu \otimes V_{2m}$ is G-dominated by Monge–Ampère capacity on $\Omega \times D$, where $G := (8\pi)^m \delta_D^{2m} \tilde{F}_m$ and $\tilde{F}_m$ is defined by the formula* (6.2).

6.3. A local transcendental inequality

Here we want to give a transcendental version of a local algebra inequality (see [15,21]) following an argument of Demailly [18].

Let us first recall the definition of complex integrability exponents introduced by Demailly and Kollár [19]. Let φ be a plurisubharmonic function on an open set $\Omega \subset \mathbb{C}^n$ and $a \in \Omega$. We define the complex singularity exponent of φ at the point a to be the positive real number

$$c_a(\varphi) := \sup\bigl\{c > 0;\ \exists U \text{ neighborhood of } a,\ \exp(-2c\varphi) \in L^1_{loc}(U)\bigr\}.$$

Actually the real number $\lambda_a(\varphi) := 1/c_a(f)$ is a kind of multiplicity which measures the "strength" of the singularity of φ at the point a (see [19]).

Recall that the Lelong number of φ at the point a is defined by the formula

$$\nu_a(\varphi) := \sup\bigl\{\nu > 0;\ \varphi(z) \leqslant \nu \log|z-a| + O(1),\ 0 < |z-a| \ll 1\bigr\}.$$

By Skoda's integrability theorem [28], it follows that

$$\frac{1}{\nu_a(\varphi)} \leqslant c_a(\varphi) \leqslant \frac{n}{\nu_a(\varphi)}.$$

Our Theorem 5.5 can be rephrased in the following way.

Proposition 6.5. *Let $\varphi \in \mathcal{E}(\Omega)$, then for any $a \in \Omega$, we have that*

$$c_a(\varphi) \geqslant \frac{n}{\mu_a(\varphi)},$$

where $\mu_a(\varphi)$ is defined by the formula

$$\mu_a(\varphi)^n := \int_{\{a\}} \bigl(dd^c\varphi\bigr)^n.$$

Therefore

$$\frac{n}{\mu_a(\varphi)} \leqslant c_a(\varphi) \leqslant \frac{n}{\nu_a(\varphi)}.$$

As pointed out by Demailly [18], this inequality implies an important inequality between two algebraic invariants associated to an ideal $\mathcal{I}$ of germs of holomorphic functions with an isolated singularity at the origin in $\mathbb{C}^n$. Let $\mathcal{I}$ be the ideal generated by the holomorphic germs $g_1, \ldots, g_N$ near the origin, then its log canonical threshold at the origin can be defined to be $lc(\mathcal{I}) := c_0(\varphi)$, where $\varphi := (1/2)\log(\sum_j |g_j|^2)$ (see [19]). There is another numerical invariant $e(\mathcal{I})$, called the Hilbert–Samuel multiplicity of the ideal $\mathcal{I}$ (see [20] for the definition), which turns out to be equal to $\mu_0(\varphi)^n$ [18].

Thus our last result combined with Lemma 2.1 in [18] implies the following result from local algebra due to Corti [15] in dimension 2 and de Fernex, Ein and Mustaţă [20] in higher dimensions.

Corollary 6.6. *Let $\mathcal{I}$ be an ideal as above. Then we have that*

$$lc(\mathcal{I}) \geqslant \frac{n}{(e(\mathcal{I}))^{1/n}}.$$

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