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On curvilinear subschemes of $\mathbb{P}^2$

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Abstract

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Let Z be a curvilinear subscheme of $\mathbb{P}^2$, i.e. a zero-dimensional scheme whose embedding dimension at every point of their support is ≤ 1. We find bounds for the minimum degree of the plane curves on which Z imposes independent conditions and we show that the Hilbert function of Z is maximal for a "generic choice of Z".

1. Introduction

The starting problems for this paper are the following:

Let $P_1, \ldots, P_s$ be smooth points of a plane curve C, and let $m_1, \ldots, m_s$ be s positive integers. How many conditions are imposed to plane curves of a given degree by requiring that they intersect C with multiplicity m_i at each P_i?

Given positive integers $s, m_1, \ldots, m_s, d$, such that d divides $\sum_{i=1}^{s} m_i$, is it possible to find an integral curve C of degree d, s distinct simple points $P_1, \ldots, P_s$ on C and another curve C' such that C' cuts on C the divisor $\sum_{i=1}^{s} m_i P_i$?

An answer to the first problem is given by Proposition 3.3, and its sharpness is related to the second problem (which already came out in [3]). In the case when $m_i \leq 2$, we are able to give an answer also to the second question, see Proposition 3.6.

Those questions led us to the study of the postulation of *curvilinear* subschemes of $\mathbb{P}^2$, i.e. of zero-dimensional schemes whose embedding dimension at every point of their support is ≤ 1.

The relevance of such schemes lies in at least two facts:

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(a) they are the only non-reduced zero-dimensional schemes which lie on non-singular plane curves (see [4, Theorem 1.2]), i.e. they can be viewed as divisors on some smooth plane curve;

(b) the generic non-reduced zero-dimensional subscheme of $\mathbb{P}^2$ is curvilinear (this is a consequence of [1], see Section 4 below).

The paper is organized as follows: Section 2 is dedicated to preliminaries; in Section 3 we look for bounds for the value $\tau(Z)$, the minimum degree of the plane curves which the scheme Z imposes independent conditions. We give several bounds related to the geometry of the scheme Z (see Propositions 3.3, 3.7, 3.8 and 3.9) and a relation between $\tau(Z)$ and $\tau(Z_{\text{red}})$, see Theorem 3.11. The methods used here are a generalization of those used for schemes of "fat points" (see e.g. [3, 6]). More precisely, if one compares Corollary 3.2 below with Corollary 3.2 of [3], one can see the analogies in the numerology between the two results. On the other hand, a comparison between the proof of Theorem 3.1 here and Theorem 3.1 in [3] shows the different, and less immediate, situation in the curvilinear case.

In the last section we study the "generic situation", i.e. given s positive integers $m_1, \ldots, m_s$, we consider the curvilinear schemes Z with support at s distinct points $P_1, \ldots, P_s$ and multiplicity m_i at each P_i. We show that the Hilbert function of Z is maximal for a "generic choice of Z" (see Theorem 4.1). Note that this implies that a generic non-reduced element of $\text{Hilb}^N\mathbb{P}^2$ ($N = \sum_{i=1}^{s} m_i$) has maximal Hilbert function.

A question from P. Ellia started us on this work. We would like to thank him for his interest and also to thank Tony Geramita for several useful talks.

2. Preliminaries and notation

Let $\mathbb{P}^2$ be the projective plane over an algebraically closed field k, and let Z be a zero-dimensional subscheme of $\mathbb{P}^2$. Let $\mathscr{I} \subseteq \mathcal{O}_{\mathbb{P}^2}$ and $I \subseteq R = k[x_0, x_1, x_2]$ be, respectively, the ideal sheaf and the homogeneous ideal corresponding to Z.

For any positive integer t we have that $\dim I_t = h^0(\mathscr{I}(t))$, and we will refer to I_t also as "the linear system of all the plane curves of degree t containing Z", since this is, from a geometrical point of view, what the forms in I_t correspond to.

From the exact sequence $O \to \mathscr{I} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Z \to O$, by twisting with $\mathcal{O}_{\mathbb{P}^2}(t)$ and taking cohomology, one gets

$$\dim I_t = h^0(\mathscr{I}(t)) = \binom{t+2}{2} - N + h^1(\mathscr{I}(t)),$$

where $N = h^0(\mathcal{O}_Z)$ is the *degree* of Z, while $h^1(\mathscr{I}(t))$ is called the *superabundance* of the linear system given by I_t. Recall that when $h^1(\mathscr{I}(t)) > 0$, the system I_t is said to be *superabundant*, and that when $h^0(\mathscr{I}(t)).h^1(\mathscr{I}(t)) = 0$, the system is said to be *regular*.

Finally the function

$$H(Z, t) = \dim_k R_t/I_t = \binom{t+2}{2} - h^0(\mathcal{I}(t)) \tag{1}$$

is called the *Hilbert function* of Z. We can view $H(Z, t)$ as the number of conditions that Z imposes to curves of degree t.

Our aim in the next section will be to find upper bounds for the integer $\tau(Z)$, or τ for short, defined as follows:

$$\tau(Z) = \min\{t \mid h^1(\mathcal{I}(t)) = 0\}. \tag{2}$$

The number $\tau + 1$ (often denoted by σ in the literature) is the least integer for which the difference function $\Delta H(Z, t) = H(Z, t+1) - H(Z, t)$ vanishes.

In this paper we study a particular case of the above situation: let $P_1, P_2, \ldots, P_s$ be s distinct points in $\mathbb{P}^2$, let $m_1, \ldots, m_s$ be non-negative integers and $C_1, \ldots, C_s$ be curves in $\mathbb{P}^2$ so that P_i is a non-singular point for C_i, let c_i be a polynomial defining C_i and $I = \bigcap_{i=1}^{s} ((c_i) + \mathfrak{p}_i^{m_i})$, where $\mathfrak{p}_i$ is the homogeneous prime ideal which corresponds to P_i. I defines a scheme $Z \subseteq \mathbb{P}^2$ such that $\operatorname{edim} \mathcal{O}_{Z,P_i} \leq 1$ (edim = embedding dimension) for any P_i; we will refer to Z as to a *curvilinear scheme*, and we will write

$$Z = (P_1, P_2, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s).$$

We recall that, by [4, Theorem 1.2], these are the only zero-dimensional schemes such that there exists a non-singular curve C in $\mathbb{P}^2$ containing them: Z can be viewed as the Cartier divisor on C given by $Z = \sum_{i=1}^{s} m_i P_i$.

We recall that if D is a Cartier divisor on an integral curve C and K is the canonical divisor on C, we say that

$$h^0(\mathcal{O}_C(K - D)) = h^1(\mathcal{O}_C(D))$$

is the *index of speciality* of D.

3. Bounds for $\tau(Z)$

Given a linear system I_t associated to some curvilinear scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$, our method to study its regularity consists of finding a suitable curve $\Gamma \subseteq \mathbb{P}^2$ and "splitting" the problem into studying first the index of speciality of $Z \cap \Gamma$ on Γ and then the superabundance of the linear system I'_{t-d} (where $d = \deg \Gamma$), associated to the residual Z' of Z with respect to Γ.

Our main result (which generalizes methods used with schemes of "fat points", see e.g. [3, 6]) is the following (notation as in Section 2):

Theorem 3.1. *Let $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ be a curvilinear scheme in $\mathbb{P}^2$ and let Γ be an integral curve of degree d which is smooth at each $P_i \in \Gamma$.*

Let $e_i = i(\Gamma, C_i; P_i)$, *i.e. the intersection multiplicity of* Γ *and* C_i *at* P_i *and* $n_i = \min\{e_i, m_i\}$. *Let E be the (Cartier) divisor* $\sum_{i=1}^{s} n_i P_i$ *on* Γ *and H a generic line section of* Γ.

Set $Z' = (P_1, \ldots, P_s; m_1 - n_1, \ldots, m_s - n_s; C_1, \ldots, C_s)$ *and let* $\mathscr{I}, \mathscr{I}'$ *be the ideal sheaves corresponding to* Z, Z', *respectively.*

(a) *If* $t \geq d$, *then*

$$h^1(\mathcal{O}_\Gamma(tH - E)) \leq h^1(\mathscr{I}(t)) \leq h^1(\mathcal{O}_\Gamma(tH - E)) + h^1(\mathscr{I}'(t-d)).$$

(b) *If* $t < d$, *then*

$$h^1(\mathcal{O}_\Gamma(tH - E)) \leq h^1(\mathscr{I}(t)) + \binom{d-t-1}{2}$$

$$\leq h^1(\mathcal{O}_\Gamma(tH - E)) + \sum_{i=1}^{s} (m_i - n_i).$$

Proof. Let I, I' be the homogeneous ideals of Z, Z' in R, respectively, and let g be a polynomial defining Γ. Multiplication by g gives an injection $I' \to I$, hence we get the following short exact sequence of sheaves:

$$O \to \mathscr{I}'(-d) \xrightarrow{g} \mathscr{I} \to \mathscr{J} \to O. \tag{3}$$

We want to check that $\mathscr{J}$ is canonically isomorphic to $\mathcal{O}_\Gamma(-E)$. Let $\mathfrak{q}_i = \mathfrak{p}_i \mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_i}$, then $I\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}_i} = ((\bar{c}_i) + \mathfrak{q}_i^{m_i})$, $I'\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}_i} = ((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i})$, where $\bar{g}, \bar{c}_i$ are local equations of Γ, C_i in $\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}_i}$.

To show that $\mathscr{J} \cong \mathcal{O}_\Gamma(-E)$ means showing that, for each P_i,

$$\mathscr{J}_{P_i} = \frac{(\bar{c}_i) + \mathfrak{q}_i^{m_i}}{\bar{g}((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i})} \quad \text{is isomorphic to} \quad \frac{(\bar{g}) + \mathfrak{q}_i^{n_i}}{(\bar{g})}. \tag{4}$$

Let us consider the case $e_i = \infty$ first. We will have $(\bar{g}) = (\bar{c}_i)$, so $m_i = n_i$ and (4) is trivially true.

Now assume $e_i < \infty$. Since $(\bar{c}_i)$ is a prime ideal and $\bar{g} \notin (\bar{c}_i)$, then $(\overline{gc}_i) = (\bar{g}) \cap (\bar{c}_i)$. Moreover, since the intersection multiplicity of Γ and C_i at P_i is e_i, then $\mathfrak{q}_i^{e_i} + (\bar{g}) = (\bar{c}_i) + (\bar{g}) = \mathfrak{q}_i^{e_i} + (\bar{c}_i)$.

It follows that for $e_i > m_i$, we have $n_i = m_i$, and

$$\bar{g}((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i}) = (\bar{g}) = (\bar{g}) \cap ((\bar{g}) + \mathfrak{q}_i^{e_i} + \mathfrak{q}_i^{n_i}) = (\bar{g}) \cap ((\bar{c}_i) + \mathfrak{q}_i^{m_i}).$$

For $e_i \leq m_i$, we have $n_i = e_i$ and

$$\bar{g}((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i}) = (\overline{gc}_i) + \bar{g}\mathfrak{q}_i^{m_i - n_i} = (\bar{g}) \cap (\bar{c}_i) + (\bar{g}) \cap \bar{g}\mathfrak{q}_i^{m_i - n_i}$$

$$= (\bar{g}) \cap ((\bar{c}_i) + \bar{g}\mathfrak{q}_i^{m_i - n_i}) = (\bar{g}) \cap ((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i}(\bar{g}, \bar{c}_i))$$

$$= (\bar{g}) \cap ((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i}(\mathfrak{q}_i^{n_i} + (\bar{c}_i)) = (\bar{g}) \cap ((\bar{c}_i) + \mathfrak{q}_i^{m_i}).$$

Hence for any $e_i < \infty$, we get

$$\mathscr{I}_{P_i} = \frac{(\bar{c}_i) + \mathfrak{q}_i^{m_i}}{\bar{g}((\bar{c}_i) + \mathfrak{q}_i^{m_i - n_i})} = \frac{(\bar{c}_i) + \mathfrak{q}_i^{m_i}}{\bar{g} \cap ((\bar{c}_i) + \mathfrak{q}_i^{m_i}}$$

$$\cong \frac{(\bar{g}) + (\bar{c}_i) + \mathfrak{q}_i^{m_i}}{(\bar{g})} = \frac{(\bar{g}) + \mathfrak{q}_i^{m_i} + \mathfrak{q}_i^{e_i}}{(\bar{g})} = \frac{(\bar{g}) + \mathfrak{q}_i^{n_i}}{(\bar{g})}.$$

So $\mathscr{I} \cong \mathcal{O}_\Gamma(-E)$, which is the sheaf of ideals of E in $\mathcal{O}_\Gamma$. Now let us twist the sequence (3) by $\mathcal{O}_{\mathbb{P}^2}(t)$ and consider the following long exact sequence of cohomology:

$$\cdots \to H^1(\mathbb{P}^2, \mathscr{I}'(t-d)) \to H^1(\mathbb{P}^2, \mathscr{I}(t)) \to H^1(\mathbb{P}^2, \mathscr{I}(t))$$
$$\to H^2(\mathbb{P}^2, \mathscr{I}'(t-d)) \to H^2(\mathbb{P}^2, \mathscr{I}(t)) \to \cdots.$$

We have $H^i(\mathbb{P}^2, \mathscr{I}(t)) = H^i(\Gamma, \mathscr{I}(t))$, and $\mathscr{I}(t) = \mathcal{O}_\Gamma(tH - E)$; now points (a) and (b) follow immediately from the above exact sequence by noticing (use the exact sequence $O \to \mathscr{I}' \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{Z'} \to O$) that if $t \geq d$ then $H^2(\mathbb{P}^2, \mathscr{I}'(t-d)) = 0$, while when $t < d$ we have

$$h^1((\mathbb{P}^2, \mathscr{I}'(t-d)) = h^0(\mathbb{P}^2, \mathcal{O}_{Z'})) = \sum_{i=1}^{s} (m_i - n_i) \quad \text{and}$$

$$h^2((\mathbb{P}^2, \mathscr{I}'(t-d)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-t-3)) = \binom{d-t-1}{2}. \qquad \square$$

As an immediate consequence of this theorem we have the following corollary, which gives a way, by induction, to look for bounds for $\tau(Z)$:

Corollary 3.2. *Let Z, Z' and Γ be as in Theorem* 3.1. *Let p be the arithmetic genus of Γ and suppose that*

(i) $t \geq d$,
(ii) $td - \sum_{i=1}^{s} n_i \geq 2p - 1$,
(iii) $h^1(\mathscr{I}'(t-d)) = 0$.
Then $h^1(\mathscr{I}(t)) = 0$.

Proof. The conclusion follows immediately from Theorem 3.1, since (ii) implies that $h^1(\mathcal{O}_\Gamma(tH - E)) = 0$. $\square$

Now we are able to give an answer to the first problem we saw in the Introduction.

Proposition 3.3. *Let $C \subseteq \mathbb{P}^2$ be an integral curve of degree d. Let $P_1, \ldots, P_s$ be smooth distinct points of C and consider the curvilinear scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C, \ldots, C) \subseteq C$. Then*

$$\tau(Z) \leq \left\lceil \frac{\sum_{i=1}^{s} m_i}{d} \right\rceil + d - 2.$$

Proof. Let $t = [\sum_{i=1}^{s} m_i/d] + d - 2$ and p be the arithmetic genus of C. It suffices to show that $h^1(\mathscr{I}(t)) = 0$.

We will use Theorem 3.1 with $\Gamma = C_1 = \cdots = C_s = C$ and $n_i = m_i$; we have that

$$\deg(tH - E) = td - \sum_{i=1}^{s} n_i = d\left(\left[\sum_{i=1}^{s} m_i/d\right] + d - 2\right) - \sum_{i=1}^{s} n_i$$

$$\geq d^2 - 3d + 1 = 2p - 1,$$

hence $h^1(\mathcal{O}_C(tH - E)) = 0$.

Now $t \geq d - 2$, so for $t = d - 1,\ d - 2$, by Theorem 3.1(b), we get $h^1(\mathscr{I}(t)) = 0$. Assume $t \geq d$. By Theorem 3.1(a) we get $h^1(\mathscr{I}(t)) \leq h^1(\mathscr{I}'(t - d))$, where $\mathscr{I}'$ is the ideal sheaf of the scheme $Z' = (P_1, \ldots, P_s; O, \ldots, O; C, \ldots, C)$, so $\mathscr{I}' = \mathcal{O}_{\mathbb{P}^2}$ and $h^1(\mathscr{I}'(t - d)) = 0$. It follows again that $h^1(\mathscr{I}(t)) = 0$. □

It is an open problem whether the bound given by Theorem 3.3 is sharp or not, i.e. to find, given $d, m_1, \ldots, m_s$, a curve C and a curvilinear scheme $Z \subseteq C$ as in the theorem such that we have exactly

$$\tau(Z) = \left[\sum_{i=1}^{s} m_i/d\right] + d - 2$$

(note that it is enough to do that when d divides $\sum_{i=1}^{s} m_i$).

The question reduces to the second problem we saw at the beginning, and it is exactly the same as the one that arises in the study of schemes of "fat points" (defined by homogeneous ideals of type $I = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_s^{m_s}$).

Problem 3.4. Given positive integers $s, m_1, \ldots, m_s, d$, is it possible, when d divides $\sum_{i=1}^{s} m_i$, to find an integral curve C of degree d, s simple distinct points $P_1, \ldots, P_s$ on C and another curve C' such that C' cuts on C the divisor $\sum_{i=1}^{s} m_i P_i$?

For details on this problem, see [3, Section 5]. Let us note that the question is obvious for $d = 1, 2$ and it has been solved (if $\operatorname{char} k = 0$) when $d = 3$ (see [3, Proposition 5.6].

We are able to give a positive answer also when $m_i \leq 2, i = 1, \ldots, s$ and $\operatorname{char} k = 0$; at this aim we will need the following easy lemma (we give a proof for lack of references):

Lemma 3.5. *Let $m_1, \ldots, m_s, d$ be positive integers. Let $d = 1$ or 2, and e be such that $\sum_{i=1}^{s} m_i = ed$. Then, for every $P_1, \ldots, P_s$ on a smooth curve $C \subseteq \mathbb{P}^2 = \mathbb{P}_k^2$ of degree d, with* char $k = 0$, *there exists a smooth curve C' of degree e such that $C \cdot C' = \sum_{i=1}^{s} m_i P_i$.*

Proof. We can suppose $e \geq d$, otherwise the conclusion is trivial.

Since C is rational, any divisor $\sum_{i=1}^{s} m_i P_i$ is cut on C by curves of degree e. In order to check that we have a smooth one among them, let C' be such that $C \cdot C' = \sum_{i=1}^{s} m_i P_i$ and consider the curve C'' given by C and $(e - d)$ lines not passing through the P_is. The generic curve in the linear system generated by C' and C'' is smooth by Bertini's Theorem and it cuts the divisor $\sum_{i=1}^{s} m_i P_i$ on C, as requested. □

Now we can give the solution to Problem 3.4 in the case $m_i \leq 2$:

Proposition 3.6. *Let x, y, d, e be positive integers such that $2x + y = de$, and let k be an algebrically closed field with $\operatorname{char} k = 0$. Then it is possible to find a smooth curve $C_d \subseteq \mathbb{P}_k^2$ of degree d, $x + y$ simple points $P_1, \ldots, P_x, Q_1, \ldots, Q_y$ on C_d and a smooth curve C' of degree e such that $C_d \cdot C'$ is the divisor $\sum_{i=1}^{x} 2P_i + \sum_{j=1}^{y} Q_j$.*

Proof. We may assume that $e \geq d$. Let $x = dq + r$, $y = dq' + r'$, with $0 \leq r, r' \leq d - 1$. Then we have $de = 2x + y = d(2q + q') + 2r + r'$, and let $2r + r' = dz$. Since $2r + r' \leq 3d - 3$, then $0 \leq z \leq 2$.

Let $C_z \subseteq \mathbb{P}^2$ be a smooth curve of degree z ($C_z = \emptyset$, if $z = 0$), and consider $r + r'$ distinct points $P_1, \ldots, P_r; Q_1, \ldots, Q_{r'}$ on C_z. By Lemma 3.5, there exists a smooth curve $C_d \subseteq \mathbb{P}^2$, of degree d, such that

$$C_d \cdot C_z = 2P_1 + \cdots + 2P_r + Q_1 + \cdots + Q_{r'}.$$

Now let $C_q \subseteq \mathbb{P}^2$ be a curve of degree q such that

$$C_d \cdot C_q = P_{r+1} + \cdots + P_x,$$

and $C_{q'} \subseteq \mathbb{P}^2$ a curve of degree q' such that $C_d \cdot C_{q'} = Q_{r'+1} + \cdots + Q_y$, where $P_{r+1}, \ldots, P_x, Q_{r'+1}, \ldots, Q_y$ are distinct points, and they are also distinct from $P_1, \ldots, P_r; Q_1, \ldots, Q_{r'}$.

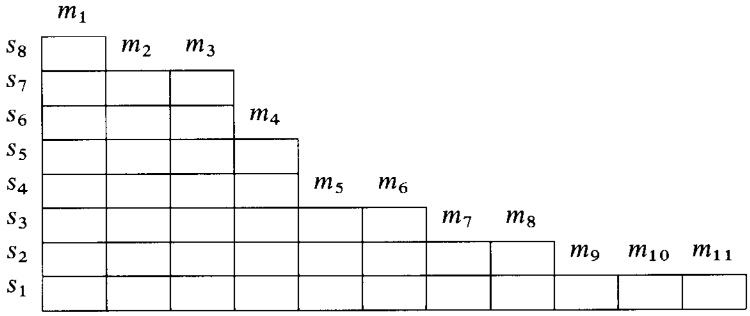
Let $C_e \subseteq \mathbb{P}^2$ be the curve composed by twice C_q, by $C_{q'}$ and C_z, i.e. $C_e = 2C_q C_{q'} C_z$. Then $C_e \cdot C_d = 2P_1 + \cdots + 2P_x + Q_1 + \cdots + Q_y$.

In order to find a smooth curve C'_e which cut the same divisor on C_d, consider the curve C formed by C_d and $e - d$ generic lines (not passing through any of the P_1s and Q_1s and not intersecting on C_e). By Bertini's Theorem, C and C_e generate a linear system whose generic element C' is irreducible (C and C_e have no common components) and smooth (C is smooth at every base point of the system).

Hence C_d and C' are as required. □

Now we want to consider the case in which we have a curvilinear scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ and an integral curve passing through the P_is and transversal to C_i, for each i.

In order to do that, we need to define, given $m_1 \geq \cdots \geq m_s$, the numbers $s_n = \max\{i \in \mathbb{N} \mid m_i \geq n\}$. The s_ns can be better visualized by considering the block diagram representing $m_1, \dots, m_s$ (the next figure shows the case $(m_1, \dots, m_s) = (8, 7, 7, 5, 3, 3, 2, 2, 1, 1, 1)$, where $s_1 = s = 11$, $s_2 = 8$, $s_3 = 6$, $s_4 = s_5 = 4$, $s_6 = s_7 = 3$, $s_8 = 1$):



Proposition 3.7. *Let* $Z = (P_1, \dots, P_s; m_1, \dots, m_s; C_1, \dots, C_s)$ *be a curvilinear scheme, with* $m_1 \geq \dots \geq m_s > 0$, *and such that* $P_1, \dots, P_s$ *are simple points for an integral curve* C *of degree* d, *with* $i(C, C_i; P_i) = 1$, *for every* i. *Let* s_n *be as above, and define* t *as follows*:

(a) *for* $s \leq d$, $t = \left(\sum_{i=1}^{s} m_i\right) - 1$,

(b) *for* $s \geq d$,

$$t = \max\left\{[s_1/d] + d - 2,\ [s_2/d] + 2d - 2,\ \dots,\ [s_{m_d}/d] + m_d d - 2, \left(\sum_{i=1}^{d} m_i\right) - 1\right\}.$$

Then we have $\tau(Z) \leq t$.

Proof. In case $s \leq d$, the bound is classically known, see also Remark 3.13 (note that for $s = d$, t is well defined).

Assume $s > d$. By induction on m_1: for $m_1 = 1$ the conclusion follows from Proposition 3.3, since $t = [s/d] + d - 2$.

For $m_1 > 1$, let $Z' = (P_1, \dots, P_s; m_1 - 1, \dots, m_s - 1; C_1, \dots, C_s)$; it is easy, but quite tedious, to verify that by inductive hypothesis we get $h^1(\mathcal{I}'(t - d)) = 0$. Moreover, $t \geq \sum_{i=1}^{d} m_i - 1 \geq d$ and

$$td - s \geq ([s/d] + d - 2)d - s \geq d^2 - 3d + 1 = 2p - 1.$$

Therefore, by Corollary 3.2, we get the conclusion. □

In the case of Proposition 3.7 it is not too hard to check that the bound for $\tau(Z)$ is sharp. Namely, we have (notation as in Proposition 3.7) the following:

Proposition 3.8. *Let $C \subseteq \mathbb{P}^2$ be an integral curve, L a line which cuts C in $Q_1, \ldots, Q_d$ distinct points, and F_n an integral curve of degree $[s_n/d]$ cutting C in $R_1, \ldots, R_{d[s_n/d]}$ distinct points.*

Let $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ be such that

(a) if $s \leq d$, then $P_i = Q_i$ $(1 \leq i \leq s)$ and $C_1 = \cdots = C_s = L$,

(b) if $s > d$ and $t = \sum_{i=1}^{d} m_i - 1$, then $P_i = Q_i$ $(1 \leq i \leq d)$, and $C_1 = \cdots = C_d = L$,

(c) if $s > d$ and $t = [s_n/d] + nd - 2$ $(1 \leq n \leq m_d)$, then $P_i = R_i$ $(1 \leq i \leq d[s_n/d])$ and $C_1 = \cdots = C_{d[s_n/d]} = F_n$.

Then $\tau(Z) = t$.

Proof. It suffices to show that $h^1(\mathscr{I}(t-1)) \neq 0$.

In cases (a) and (b), apply Theorem 3.1 with $\Gamma = L$. If P is a point on L, we have

$$h^1(\mathscr{I}(t-1)) \geq h^1\left(\mathcal{O}_L\left((t-1)P - \sum_{i \leq d} m_i Q_i\right)\right) \neq 0$$

since $\deg \mathcal{O}_L((t-1)P - \sum_{i \leq d} m_i Q_i) = -2$.

In case (c), let $q = d[s_n/d]$ and $Z'' = (P_1, \ldots, P_q; n, \ldots, n; F_n, \ldots, F_n)$. Since $Z'' \subseteq Z$, it will be enough to show that $\tau(Z'') = t$, i.e. that $h^1(\mathscr{I}''(t-1)) \neq 0$. Now n times the curve C gives rise to a curve which cuts on F_n the Cartier divisor $\sum_{i=1}^{q} nP_i$, i.e. Z'' itself. By a well-known theorem of Segre (see [9] or [7, Corollary 3.4]), it follows that Z'' does not impose independent conditions on the curves of degree $nd + [s_n/d] - 3 = t - 1$, i.e. $h^1(\mathscr{I}''(t-1)) \neq 0$. □

The next result gives a bound on τ in the hypothesis that the scheme Z is in "general position", i.e. that it does not intersect any line in more than two points (may be infinitely near):

Proposition 3.9. *Let $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ be a curvilinear scheme such that the multiplicity of intersection of Z with every line $L \subseteq \mathbb{P}^2$ is ≤ 2. Then*

$$\tau(Z) \leq \left[\sum_{i=1}^{s} m_i/2\right].$$

Proof. By induction on $\sum_{i=1}^{s} m_i$: the conclusion is obvious for $\sum m_i = 1, 2$, so assume $\sum m_i > 2$.

Applying Corollary 3.2 with $\Gamma = L$, a line with $\deg(L \cap Z) = 2$, the conclusion follows. □

Remark 3.10. It is not hard to check that the above bound is sharp and it is achieved when Z lies on a non-singular conic.

Finally we want to relate $\tau(Z)$ with $\tau(Z_{\text{red}})$, where Z_{red} is the reduced scheme $P_1 \cup \cdots \cup P_s$.

Theorem 3.11. *Let* $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ *be a curvilinear scheme with* $m_{1i} \geq \cdots \geq m_s > 0$ *and let* $\rho = \tau(Z_{\mathrm{red}})$, *then*

$$\tau(Z) \leq m_1 + \cdots + m_{\rho+1} - 1.$$

Proof. It is well known that $\rho \leq s - 1$, so the above summation has exactly $\rho + 1$ terms.

If $s = 1$, then $\rho = 0$, so the conclusion is obvious. Assume $s > 1$ and define Z' as follows:

$$Z' = (P_1, \ldots, P_s; m_1, \ldots, m_{s-1}, m_s - 1; C_1, \ldots, C_s) \quad \text{for } m_s > 1,$$

$$Z' = (P_1, \ldots, P_s; m_1, \ldots, m_{s-1}; C_1, \ldots, C_{s-1}) \quad \text{for } m_s = 1.$$

Let $\mathscr{I}, \mathscr{I}'$ be the ideal sheaves corresponding to Z and Z', respectively, and let t be a positive integer. We have (see Section 2):

$$h^0(\mathscr{I}(t)) = \binom{t+2}{2} - \sum_{i=1}^{s} m_i + h^1(\mathscr{I}(t)),$$

$$h^0(\mathscr{I}'(t)) = \binom{t+2}{2} - \sum_{i=1}^{s} m_i + 1 + h^1(\mathscr{I}'(t)).$$

It follows that

$$h^0(\mathscr{I}'(t)) - h^0(\mathscr{I}(t)) = 1 + h^1(\mathscr{I}'(t)) - h^1(\mathscr{I}(t)). \tag{5}$$

Now we prove the theorem by induction on $\sum m_i$. Since the result is obvious for $\sum m_i = 1, 2$, assume $\sum m_i > 2$.

Let $\rho' = \tau(Z'_{\mathrm{red}})$, $t = m_1 + \cdots + m_{\rho+1} - 1$. By the inductive hypothesis $\tau(Z') \leq m_1 + \cdots + m_{\rho'+1} - 1 \leq t$. It follows that

$$h^1(\mathscr{I}'(t)) = 0. \tag{6}$$

In order to conclude, by virtue of (5) and (6) it will be enough to prove that $h^0(\mathscr{I}'(t)) - h^0(\mathscr{I}(t)) \geq 1$, i.e. that there is a curve of degree t containing Z' and not Z.

Since $\tau(Z_{\mathrm{red}}) = \rho$, there exists a curve C of degree ρ through $P_1, \ldots, P_{s-1}$, such that $P_s \notin C$. Let L_i be a line through P_i $(1 \leq i \leq \rho)$, not passing through P_s. Let L_s be a line through P_s meeting transversely C_s, and let L be a line not passing through P_s. Since $m_{\rho+1}$ times C, $(m_i - m_{\rho+1})$ times L_i $(1 \leq i \leq \rho)$, $(m_s - 1)$ times L_s, and $(m_{\rho+1} - m_s)$ times L give rise to a curve of degree t passing through Z' but not through Z, we are done. □

Remark 3.12. In this case too it is quite easy to show that the above bound is sharp: take $P_1, \ldots, P_{\rho+1}$ lying on a line L, $C_1, \ldots, C_{\rho+1} = L$, and $m_1, \ldots, m_{\rho+1} > 0$. In this case $\tau(Z) = m_1 + \cdots + m_{\rho+1} - 1$. The proof is quite immediate, using Theorem 3.1 with $\Gamma = L$.

Another example is given by seven distinct points on a conic C; the scheme $Z = (P_1, \ldots, P_7; 2, \ldots, 2; C, \ldots, C)$ has $\tau(Z) = 7$ and here $\rho = 3$, so the bound is sharp.

Remark 3.13. A consequence of Theorem 3.11 and of the previous remark is the well-known fact that the more general bound for τ that can be given for a curvilinear scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ is $\tau(Z) \leq \sum_{i=1}^{s} m_i - 1$, which is achieved when Z is on a line.

4. Hilbert function of a generic curvilinear scheme

In this section we want to check that, with respect to their postulation, curvilinear schemes behave, in general, as reduced points do, i.e. they have maximal Hilbert function.

Namely, let $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$ and d, k be integers defined by $N = \binom{d+1}{2} + k$, $0 \leq k \leq d$; then we want

$$H(Z, t) = \min\left\{\binom{t+2}{2}, N\right\}$$

which is equivalent to

$$h^0(\mathcal{I}(d-1)) = h^1(\mathcal{I}(d)) = 0. \tag{7}$$

Let us work on a fixed plane curve first.

Theorem 4.1. *Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d. Given s positive integers $m_1, \ldots, m_s$ with*

$$\binom{d+1}{2} \leq \sum_{i=1}^{s} m_i < \binom{d+2}{2},$$

then for a generic choice of $P_1, \ldots, P_s$ on C, the curvilinear scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C, \ldots, C) \subseteq \mathbb{P}^2$ has maximal Hilbert function.

Proof. We have seen that what we require from Z amounts to (7), i.e. that the divisor $dH - \sum_{i=1}^{s} m_i P_i$ on C is non-special, while the divisor $(d-1)H - \sum_{i=1}^{s} m_i P_i$ is non-effective.

We will work by induction on d. If $d = 1, 2$ obviously we have $h^0(\mathcal{I}(d-1)) = 0$, while $h^1(\mathcal{I}(d)) = 0$ follows from Proposition 3.3.

Assume $d \geq 3$ and apply induction on s. Let $s = 1$, $n = d(d-1)$, $r = d(d-1) - p$ and let g_n^r, g_{n+d}^{r+d} be the linear series cut on C by the curves of degrees $d-1, d$ respectively. Let

$$\Phi = \{Q \in C \mid g_n^r - (r+1)Q \neq \emptyset\},$$

$$\Psi = \{Q \in C \mid g_{n+d}^{r+d} - (r+d+1)Q \neq \emptyset\}.$$

Φ and Ψ have only a finite number of points, see e.g. [10, p. 131]. Let $P \in C$, $P \notin \Phi$, $P \notin \Psi$.

We claim that the divisor $Z = m_1 P$ of C has maximal Hilbert function. In fact, $\binom{d+1}{2} + k = m_1$, then $r + 1 \le \binom{d+1}{2} + k = m_1 \le r + d$; hence, since $P \notin \Phi$, then $g_n^r - m_i P = \emptyset$, i.e. $h^0(\mathscr{I}(d-1)) = 0$.

But also $g_{n+d}^{r+d} - (r + d + 1)P = \emptyset$, so the divisor $(r + d + 1)P$ imposes independent conditions to curves of degree d. It follows that $m_1 P$ also imposes independent conditions to curves of degree d, i.e. that $h^1(\mathscr{I}(d)) = 0$.

Now let $s > 1$. We consider the two possible cases:

$$\text{(a)} \quad \sum_{i=1}^{s-1} m_i < \binom{d+1}{2}, \qquad \text{(b)} \quad \sum_{i=1}^{s-1} m_i \ge \binom{d+1}{2}.$$

Case (a). Let $x = \binom{d+1}{2} - \sum_{i=1}^{s-1} m_i$. Since $\sum_{i=1}^{s-1} m_i + x = \binom{d+1}{2}$, by the inductive hypothesis for generic $P_1, \ldots, P_{s-1}$ on C we have that the scheme $Z' = (P_1, \ldots, P_{s-1}; m_1, \ldots, m_{s-2}, m_{s-1} + x; C, \ldots, C)$ has maximal Hilbert function. Let $\mathscr{I}'$ be the ideal sheaf corresponding to Z'. It follows that $h^0(\mathscr{I}'(d-1)) = 0 = h^1(\mathscr{I}'(d))$. Hence, since $\deg Z' = \binom{d+1}{2}$, we get also $h^1(\mathscr{I}'(d-1)) = 0$.

Let $Z'' = (P_1, \ldots, P_{s-1}; m_1, \ldots, m_{s-1}; C, \ldots, C)$ and $\mathscr{I}''$ the corresponding ideal sheaf. Since $Z'' \subseteq Z'$, we have $h^1(\mathscr{I}''(d)) = 0 = h^1(\mathscr{I}''(d-1))$.

Let $n = d(d-1) - \sum_{i=1}^{s-1} m_i$, $r = n - p$, and g_n^r, g_{n+d}^{r+d} be the linear series cut on C by the curves through Z'' of degrees $d-1, d$ respectively. Then let

$$\Phi = \{Q \in C \mid g_n^r - (r+1)Q \ne \emptyset\},$$

$$\Psi = \{Q \in C \mid g_{n+d}^{r+d} - (r+d+1)Q \ne \emptyset\}.$$

These two sets (again by [10]) have a finite number of points. Let $P_s \in C$, with $P_s \notin \{P_1, \ldots, P_{s-1}\} \cup \Phi \cup \Psi$. We claim that the scheme $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C, \ldots, C)$ has maximal Hilbert function.

Since $r + 1 = d(d-1) - p - \sum_{i=1}^{s-1} m_i + 1 \le m_s \le r + d + 1 = d^2 - p - \sum_{i=1}^{s-1} m_i + 1$ and $P_s \notin \Phi$, then $g_n^r - (r+1)P_s = \emptyset$, so $h^0(\mathscr{I}(d-1)) = 0$. Then, $P_s \notin \Psi$, so $g_{n+d}^{r+d} - (r+d+1)P_s = \emptyset$, and $(r+d+1)P_s$ imposes independent conditions to curves of degree d passing through Z''. It follows that $m_s P_s$ also imposes independent conditions to such curves, i.e. $h^1(\mathscr{I}(d)) = 0$.

Case (b). By the inductive hypothesis, for generic $P_1, \ldots, P_{s-1}$ on C the scheme $Z'' = (P_1, \ldots, P_{s-1}; m_1, \ldots, m_{s-1}; C, \ldots, C)$ has maximal Hilbert function.

Let $\mathscr{I}''$ be the ideal sheaf of Z''; it follows that

$$h^0(\mathscr{I}''(d-1)) = 0 = h^1(\mathscr{I}''(d)).$$

Let g_{n+d}^{r+d}, Ψ be as in case (a). We have again that Ψ has a finite number of points, and we will choose P_s on C, $P_s \notin \{P_1, \ldots, P_{s-1}\} \cup \Psi$.

We claim that $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C, \ldots, C)$ has maximal Hilbert function. In fact, $Z'' \subseteq Z$, so $h^0(\mathscr{I}''(d-1)) = 0$ implies that also $h^0(\mathscr{I}(d-1)) = 0$. In order to get $h^1(\mathscr{I}(d)) = 0$, one works as in the previous case. $\square$

Now, in order to consider "generic" curvilinear subschemes of $\mathbb{P}^2$, some considerations have to be made.

Let $M = (m_1, \ldots, m_s) \in (\mathbb{N}^*)^s$, with $|M| = \sum_{i=1}^{s} m_i = N$, and $m_1 \geq m_2 \geq \cdots \geq m_s$. Let $H^M \subseteq \mathrm{Hilb}^N \mathbb{P}^2$ be the (locally closed) subscheme parameterizing the zero-dimensional schemes Z with support at s distinct points $P_1, \ldots, P_s$ and degree m_i at each P_i.

It is known that H^M is irreducible of dimension $\sum_{i=1}^{s} (m_i + 1)$ and that its generic point represents a curvilinear scheme. The local result can be found in [1], and it can be globalized via the morphism $\mathrm{Hilb}^N \mathbb{P}^2 \to \mathrm{Symm}^N \mathbb{P}^2$ (see [8]). Moreover, we have that $\bar{H}^{M'} \subseteq \bar{H}^M$ if $M' = (m'_1, \ldots, m'_{s'})$, with $s' \leq s$ and $m'_i \geq m_i$, and "$\bar{\ }$" represents the closure in $\mathrm{Hilb}^N \mathbb{P}^2$ (see [2, Section 2]).

So, given $M = (m_1, \ldots, m_s)$, when we say that a property is true for a "generic curvilinear scheme" $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C_1, \ldots, C_s)$, we mean that the property is true in an open subset of the dense set representing curvilinear schemes inside H^M.

In particular, we want to show that the Hilbert function for a generic curvilinear scheme $Z \subseteq \mathbb{P}^2$ of degree $N = \sum_{i=1}^{s} m_i$ is the maximal one (i.e. that I_t is regular for every t), as it happens for the generic (reduced) element of $\mathrm{Hilb}^N \mathbb{P}^2$ (see e.g. [5]). This is now an easy consequence of Theorem 4.1, since, by semicontinuity of the cohomology on H^M, it is enough to find one curvilinear scheme Z for which this happens.

Note also that, since $\mathrm{Hilb}^N \mathbb{P}^2 \to \mathrm{Symm}^N \mathbb{P}^2$ is a morphism, we can rephrase what we have seen in term of the genericity of the P_is (instead of the genericity of Z in H^M), i.e., we have the following:

Proposition 4.2. *Given s positive integers* $m_1, \ldots, m_s$, *for a generic choice of* $P_1, \ldots, P_s$ *in* $\mathbb{P}^2$, *there exists a curvilinear scheme* $Z = (P_1, \ldots, P_s; m_1, \ldots, m_s; C, \ldots, C) \subseteq \mathbb{P}^2$ *with maximal Hilbert function.* □

Remark 4.3. Actually, in Theorem 4.1 we have done more than just finding a scheme with maximal Hilbert function: we have shown that we can find such a scheme on any smooth plane curve C of the appropriate degree. If we had wanted only to show the existence of such schemes in H^M, the following argument would have sufficed. Let us consider schemes concentrated at one N-ple point in $\bar{H}^M$. It is easy to find one of them with maximal Hilbert function (e.g. by an appropriate determinantal ideal), hence to conclude by semicontinuity on $\bar{H}^M$.

Moreover, in the same way, we can always find a scheme supported at one point and with a degree N structure (for all $N \in \mathbb{N}$), such that not only it has maximal Hilbert function, but its generators are the expected ones (i.e. the generators in the minimum degree span the maximum number of independent forms in the next degree when multiplied by linear forms). Since also "having the expected generation" is an open property in $\mathrm{Hilb}^N \mathbb{P}^2$, we have just shown the following:

Corollary 4.4. *For any* $N \in \mathbb{N}$, *the generic non-reduced scheme in* $\mathrm{Hilb}^N \mathbb{P}^2$ *has maximal Hilbert function and the expected generators.*

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