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When are torsionless modules projective?

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Abstract

In this paper, we study the problem when a finitely generated torsionless module is projective. Let Λ be an Artinian local algebra with radical square zero. Then a finitely generated torsionless Λ-module M is projective if $\mathrm{Ext}^1_\Lambda(M, M) = 0$. For a commutative Artinian ring Λ, a finitely generated torsionless Λ-module M is projective if the following conditions are satisfied: (1) $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i = 1, 2, 3$; and (2) $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for $i = 1, 2$. As a consequence of this result, we have that for a commutative Artinian ring Λ, a finitely generated Gorenstein projective Λ-module is projective if and only if it is selforthogonal.

Keywords: Torsionless modules; Projective modules; Gorenstein projective modules; Artinian algebras; Commutative Artinian rings

1. Introduction

M. Ramras in [G, p. 380] raised an open question: For a left and right Noetherian ring Λ, when is every finitely generated reflexive Λ-module projective? He proved in [R] that if Λ is a commutative Noetherian local ring and M is a finitely generated Λ-module such that the sequence of Betti numbers of M is strictly increasing, then the condition M is torsionless with $\mathrm{Ext}^1_\Lambda(M, \Lambda) = 0$ implies M is projective. Menzin in [M] proved that if Λ is an Artinian local algebra with radical square zero, then for Λ not Gorenstein all finitely generated reflexive modules are projective. Recently, Braun in [B] proved that for a commutative Noetherian ring Λ, a finitely generated Λ-module M is projective if it satisfies the following conditions: (1) The

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projective dimension of M is finite; (2) $\mathrm{End}_\Lambda(M)$ is a projective Λ-module; and (3) M is reflexive or $\mathrm{Ext}^1_\Lambda(M, M) = 0$. In this paper, we will study a stronger problem: When is a finitely generated torsionless module projective?

As a common generalization of the notion of projective modules, Auslander and Bridger in [AuB] introduced the notion of finitely generated modules of Gorenstein dimension zero. Such a kind of modules is called Gorenstein projective following Enochs and Jenda's terminology in [EJ]. It is well known that a projective module is Gorenstein projective. Then it is natural to ask when the converse holds true, or equivalently, what is the difference between the projectivity and Gorenstein projectivity of modules? In views of the properties of projective modules and Gorenstein projective modules, we conjecture that the difference between these two classes of modules is the selforthogonality of modules.

Gorenstein Projective Conjecture (GPC). Over an Artinian algebra, a finitely generated Gorenstein projective module M is projective if and only if it is selforthogonal.

It is trivial that the necessity in **GPC** is always true. So the sufficiency is essential in **GPC**. Observe that **GPC** is related to the question mentioned above. On the other hand, part of motivation for studying **GPC** is that it is a special case of the well-known generalized Nakayama conjecture (**GNC**) (it still remains open), which states that for an Artinian algebra Λ and a finitely generated Λ-module M, the condition $\mathrm{Ext}^i_\Lambda(M \oplus \Lambda, M \oplus \Lambda) = 0$ for any $i \geqslant 1$ implies M is projective (see [AuR1]). In this paper, we will prove that **GPC** is true if Λ is commutative, that is, if Λ is a commutative Artinian ring.

In Section 2, we collect some known facts for later use. In Section 3, we prove that for an Artinian local algebra Λ with radical square zero, a finitely generated torsionless Λ-module M is projective if $\mathrm{Ext}^1_\Lambda(M, M) = 0$. For any Artinian algebra, we also give some criteria for judging an indecomposable torsionless module being projective. In particular, we provide some support to **GNC**. In Section 4, we prove that if Λ is a commutative Artinian ring, then a finitely generated torsionless Λ-module M is projective provided that the following conditions are satisfied: (1) $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i = 1, 2, 3$; and (2) $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for $i = 1, 2$. As an immediate consequence, we have that for a commutative Artinian ring Λ, a finitely generated Gorenstein projective Λ-module is projective if and only if it is selforthogonal, that is, **GPC** is true for commutative Artinian rings.

2. Preliminaries

In this section, we give some notions and notations in our terminology and collect some facts for later use. For a ring Λ, we use $\operatorname{mod}\Lambda$ and $J(\Lambda)$ to denote the category of finitely generated left Λ-modules and the Jacobson radical of Λ, respectively. We use $(-)^*$ to denote $\mathrm{Hom}_\Lambda(-, \Lambda)$. All modules considered are finitely generated.

Let Λ be an Artinian algebra and

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

a minimal projective resolution of a module M in $\operatorname{mod}\Lambda$. We call $\operatorname{Coker} f^*$ the *transpose* of M, and denote it by $\operatorname{Tr} M$. Let $M \in \operatorname{mod}\Lambda$ and $\sigma_M : M \to M^{**}$ defined by $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism. M is called *torsionless* if

σ_M is a monomorphism; M is called *reflexive* if σ_M is an isomorphism (see [AuB]). By [Au, Proposition 6.3], we have an exact sequence:

$$0 \to \operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda^{op}) \to M \xrightarrow{\sigma_M} M^{**} \to \operatorname{Ext}^2_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda^{op}) \to 0.$$

On the other hand, it is easy to see that $\operatorname{Tr}\operatorname{Tr} M$ and M are projectively equivalent. So, we have that M (resp. $\operatorname{Tr} M$) is torsionless if and only if $\operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda^{op}) = 0$ (resp. $\operatorname{Ext}^1_\Lambda(M, \Lambda) = 0$); and M (resp. $\operatorname{Tr} M$) is reflexive if and only if $\operatorname{Ext}^i_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda^{op}) = 0$ (resp. $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$) for $i = 1, 2$.

We use $\operatorname{mod}_P \Lambda$ to denote the subcategory of $\operatorname{mod} \Lambda$ consisting of modules without non-zero projective summands. For M and N in $\operatorname{mod} \Lambda$, we use $\underline{\operatorname{Hom}}_\Lambda(M, N)$ (resp. $\overline{\operatorname{Hom}}_\Lambda(M, N)$) to denote the set of the equivalence classes of module homomorphisms modulo those factoring through a projective (resp. injective) Λ-module. For an Artinian algebra Λ, we denote by $\mathbb{D}$ the ordinary duality of Λ, that is, $\mathbb{D}(-) = \operatorname{Hom}_R(-, I(R/J(R)))$, where R is the center of Λ which is a commutative Artinian ring, and $I(R/J(R))$ is the injective envelope of $R/J(R)$.

Lemma 2.1. *(See [AuR2, Theorem 3.3].) Let Λ be an Artinian algebra, $M \in \operatorname{mod}_P \Lambda$ and $X \in \operatorname{mod} \Lambda$. Then there is an isomorphism*:

$$\overline{\operatorname{Hom}}_\Lambda(X, \mathbb{D}\operatorname{Tr} M) \to \operatorname{Hom}_{\underline{\operatorname{End}}(M)^{op}}\big(\operatorname{Ext}^1_\Lambda(M, X), \operatorname{Ext}^1_\Lambda(M, \mathbb{D}\operatorname{Tr} M)\big).$$

Recall from [AF] that a module M in $\operatorname{mod} \Lambda$ is called *faithful* if the annihilator of M in Λ is zero.

Lemma 2.2. *(See [AF, p. 217].) Let Λ be a left Artinian ring and $M \in \operatorname{mod} \Lambda$. Then the following statements are equivalent.*

(1) *M is faithful.*
(2) *M cogenerates every projective module.*
(3) *M generates every injective module.*

Definition 2.3. (See [AuB] or [EJ].) Let Λ be a left and right Noetherian ring. A module M in $\operatorname{mod} \Lambda$ is called *Gorenstein dimension zero* (or *Gorenstein projective*) if the following conditions are satisfied: (1) M is reflexive; (2) $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0 = \operatorname{Ext}^i_{\Lambda^{op}}(M^*, \Lambda^{op})$ for any $i \geqslant 1$.

Recall that a module in $\operatorname{mod} \Lambda$ is called *selforthogonal* if $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for any $i \geqslant 1$. Then it is trivial that **GPC** is a special case of **GNC**.

3. The case for Artinian algebras

In this section, Λ is an Artinian algebra. The following lemma plays a crucial role in this section.

Lemma 3.1. *Let $M \in \operatorname{mod} \Lambda$ be an indecomposable module. If there exists an exact sequence $M^t \xrightarrow{f} N \to 0$ and $\underline{\operatorname{Hom}}_\Lambda(M^t, N) = 0$ for some $t \geqslant 1$ and $N \in \operatorname{mod} \Lambda$, then M is projective.*

Proof. Let $(P(N), g)$ be the projective cover of N. Because $\underline{\mathrm{Hom}}_\Lambda(M^t, N) = 0$, we get a homomorphism $h : M^t \to P(N)$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\, f & \longrightarrow & M^t & \xrightarrow{f} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & \mathrm{Ker}\, g & \longrightarrow & P(N) & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where h' is an induced homomorphism. Since g is a superfluous epimorphism, h is epimorphic and splitable. So $P(N)$ is isomorphic to a direct summand of M^t. Since M is indecomposable, $P(N) \cong M^s$ for some $s \geqslant 1$ and M is projective. □

Lemma 3.2. *Let Λ be a radical square zero algebra and $M \in \mathrm{mod}\,\Lambda$ an indecomposable module. If M is torsionless and not simple, then M is projective.*

Proof. Suppose $M \neq 0$. Then $M \neq J(\Lambda)M$ and there exists a simple Λ-module S such that $M/J(\Lambda)M \to S \to 0$ is exact. Since M is indecomposable, we have a non-split epimorphism $f : M \to S$.

We claim that $\overline{\mathrm{Hom}}_\Lambda(M, S) = 0$. If S is injective, then it is clear that $\overline{\mathrm{Hom}}_\Lambda(M, S) = 0$. If S is not injective, then, by [AuR2, Proposition 4.3], we have an almost split sequence $0 \to S \to E \to \mathrm{Tr}\,\mathbb{D}S \to 0$. Notice that $J(\Lambda)^2 = 0$ by assumption, so E is projective by [AuR2, Proposition 5.7]. Since M is not simple, $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}\,\mathbb{D}S, M) = 0$ by [AuR2, Theorem 5.5]. So $\overline{\mathrm{Hom}}_\Lambda(M, S) = 0$ by Lemma 2.1. The claim is proved.

Since M is torsionless, there exists a projective $P \in \mathrm{mod}\,\Lambda$ such that $0 \to M \to P$ is exact. Then it is easy to see that $\underline{\mathrm{Hom}}_\Lambda(M, S) = 0$ and there exists an exact sequence $M \to S \to 0$. By Lemma 3.1, M is projective. □

Lemma 3.3. *Let Λ be a local algebra with radical square zero and $M \in \mathrm{mod}\,\Lambda$ an indecomposable module. If M is torsionless and $\mathrm{Ext}^1_\Lambda(M, M) = 0$, then M is projective.*

Proof. If M is not simple, M is projective by Lemma 3.2. If M is simple, then the condition $\mathrm{Ext}^1_\Lambda(M, M) = 0$ implies M is projective by [XC, Lemma 3]. □

The following is the main result in this section.

Theorem 3.4. *Let Λ be a local algebra with radical square zero. Then a torsionless module $M \in \mathrm{mod}\,\Lambda$ is projective if $\mathrm{Ext}^1_\Lambda(M, M) = 0$.*

Proof. If $M \in \mathrm{mod}\,\Lambda$ is torsionless and $\mathrm{Ext}^1_\Lambda(M, M) = 0$, then N is torsionless and $\mathrm{Ext}^1_\Lambda(N, N) = 0$ for any direct summand N of M. Thus the assertion follows immediately from Lemma 3.3. □

In the following, we give some criteria for judging an indecomposable torsionless module being projective.

Proposition 3.5. *Let $M \in \mathrm{mod}\,\Lambda$ be faithful and indecomposable. Then M is projective if M is torsionless.*

Proof. By Lemma 2.1, for any $n \geqslant 1$, we have an isomorphism:

$$\overline{\mathrm{Hom}}_{\Lambda^{op}}\big(\Lambda^{op}, \mathbb{D}\big(M^n\big)\big) \cong \mathrm{Hom}_{\underline{\mathrm{End}}(\mathrm{Tr}(M^n))}\big(\mathrm{Ext}^1_{\Lambda^{op}}\big(\mathrm{Tr}\big(M^n\big), \Lambda^{op}\big), \mathrm{Ext}^1_{\Lambda^{op}}\big(\mathrm{Tr}\big(M^n\big), \mathbb{D}\big(M^n\big)\big)\big).$$

Notice that M is torsionless, so $\mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}(M^n), \Lambda^{op}) \cong \mathrm{Ext}^1_{\Lambda^{op}}((\mathrm{Tr}\, M)^n, \Lambda^{op}) \cong (\mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}\, M, \Lambda^{op}))^n = 0$, and hence $\overline{\mathrm{Hom}}_{\Lambda^{op}}(\Lambda^{op}, \mathbb{D}(M^n)) = 0$ and $\underline{\mathrm{Hom}}_{\Lambda}(M^n, \mathbb{D}\Lambda^{op}) = 0$. On the other hand, because M is faithful, by Lemma 2.2, there exists an $n \geqslant 1$ such that $M^n \to \mathbb{D}\Lambda^{op} \to 0$ is exact. So, by Lemma 3.1, we have that M is projective. □

Proposition 3.6. *Let $M \in \mathrm{mod}\, \Lambda$ be faithful and indecomposable. Then M is a projective if* $\mathrm{Tr}\, M$ *is torsionless* (*equivalently,* $\mathrm{Ext}^1_{\Lambda}(M, \Lambda) = 0$) *and* $\mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}\, M, \mathrm{Tr}\, M) = 0$.

Proof. Since $\mathrm{Tr}\, M$ is torsionless, there exists a monomorphism $0 \to \mathrm{Tr}\, M \to (\Lambda^{op})^n$. Then $\mathbb{D}(\Lambda^{op})^n \to \mathbb{D}\,\mathrm{Tr}\, M \to 0$ is exact. Because M is faithful, there exists an $m \geqslant 1$ such that $M^m \to \mathbb{D}(\Lambda^{op})^n \to 0$ is exact. So we have an exact sequence $M^m \to \mathbb{D}\,\mathrm{Tr}\, M \to 0$. On the other hand, since $\mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}(M^m), \mathrm{Tr}\, M) \cong \mathrm{Ext}^1_{\Lambda^{op}}((\mathrm{Tr}\, M)^m, \mathrm{Tr}\, M) \cong (\mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}\, M, \mathrm{Tr}\, M))^m = 0$ by assumption, $\overline{\mathrm{Hom}}_{\Lambda^{op}}(\mathrm{Tr}\, M, \mathbb{D}(M^m)) = 0$ by Lemma 2.1. So $\underline{\mathrm{Hom}}_{\Lambda}(M^m, \mathbb{D}\,\mathrm{Tr}\, M) = 0$, and hence M is projective by Lemma 3.1. □

Recall from [AuR1] that the generalized Nakayama conjecture (**GNC**) states that a module $M \in \mathrm{mod}\, \Lambda$ is projective if $\mathrm{Ext}^i_{\Lambda}(M \oplus \Lambda, M \oplus \Lambda) = 0$ for any $i \geqslant 1$, which still remains open. The following result provides some support to this conjecture.

Proposition 3.7. *Let S be a faithful and simple in* $\mathrm{mod}\, \Lambda$. *If* $\mathrm{Ext}^1_{\Lambda}(S \oplus \Lambda, S \oplus \Lambda) = 0$*, then S is projective.*

Proof. Since $\mathrm{Ext}^1_{\Lambda}(S, S) = 0$ by assumption, $\overline{\mathrm{Hom}}_{\Lambda}(S, \mathbb{D}\,\mathrm{Tr}\, S) = 0$ by Lemma 2.1. So $\underline{\mathrm{Hom}}_{\Lambda^{op}}(\mathrm{Tr}\, S, \mathbb{D}S) = 0$.

If $\mathrm{Hom}_{\Lambda}(S, \mathbb{D}\,\mathrm{Tr}\, S) \neq 0$, then we have an epimorphism $\mathrm{Tr}\, S \to \mathbb{D}S \to 0$ since S is simple. By Lemma 3.1, $\mathrm{Tr}\, S$ is projective and $\mathrm{Tr}\, S = 0$. So S is projective.

If $\mathrm{Hom}_{\Lambda}(S, \mathbb{D}\,\mathrm{Tr}\, S) = 0$, then $\mathrm{Hom}_{\Lambda}(S^m, \mathbb{D}\,\mathrm{Tr}\, S) = 0$ for any $m \geqslant 1$. Because $\mathrm{Ext}^1_{\Lambda}(S, \Lambda) = 0$ by assumption, $\mathrm{Tr}\, S$ is torsionless and there exists a monomorphism $0 \to \mathrm{Tr}\, S \to (\Lambda^{op})^n$. So $\mathbb{D}(\Lambda^{op})^n \to \mathbb{D}\,\mathrm{Tr}\, S \to 0$ is exact. Because S is faithful, there exists an $m \geqslant 1$ such that $S^m \to \mathbb{D}(\Lambda^{op})^n \to 0$ is exact. So we have an epimorphism $S^m \to \mathbb{D}\,\mathrm{Tr}\, S \to 0$. It implies that $\mathbb{D}\,\mathrm{Tr}\, S = 0$ and $\mathrm{Tr}\, S = 0$. Thus S is projective. □

4. The case for commutative Artinian rings

In this section, Λ is a commutative Artinian ring. According to the localization theory of commutative ring, by Theorem 3.4, we have the following

Theorem 4.1. *If Λ is radical square zero, then a torsionless module $M \in \mathrm{mod}\, \Lambda$ is projective if* $\mathrm{Ext}^1_{\Lambda}(M, M) = 0$.

Let M and N be in $\mathrm{mod}\, \Lambda$. We define a homomorphism $\zeta : M \otimes_{\Lambda} N \to \mathrm{Hom}_{\Lambda}(M^*, N)$ of Λ-modules by $\zeta(m \otimes n)(g) = g(m)n$ for any $m \otimes n \in M \otimes_{\Lambda} N$ and $g \in M^*$. Then we obtain a natural transformation $\zeta(-) : M \otimes_{\Lambda} - \to \mathrm{Hom}_{\Lambda}(M^*, -)$ of functors from $\mathrm{mod}\, \Lambda$ to itself.

Lemma 4.2. *(See [AuB, Proposition 2.6].) For any $M \in \operatorname{mod}\Lambda$, there exists an exact sequence of functors from* $\operatorname{mod}\Lambda$ *to itself*:

$$0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, -) \to M \otimes_\Lambda - \xrightarrow{\zeta(-)} \operatorname{Hom}_\Lambda(M^*, -) \to \operatorname{Ext}^2_\Lambda(\operatorname{Tr} M, -) \to 0.$$

Definition 4.3. (See [AuR3].) Assume that $\mathscr{X}$ is a full subcategory of $\operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}\Lambda$, $X \in \mathscr{X}$. The morphism $f: X \to Y$ is said to be a *right $\mathscr{X}$-approximation* of Y if $\operatorname{Hom}_\Lambda(X', X) \to \operatorname{Hom}_\Lambda(X', Y) \to 0$ is exact for any $X' \in \mathscr{X}$. The morphism $f: X \to Y$ is said to be *right minimal* if an endomorphism $g: X \to X$ is an automorphism whenever $f = fg$. The subcategory $\mathscr{X}$ is said to be *contravariantly finite* in $\operatorname{mod}\Lambda$ if every $Y \in \operatorname{mod}\Lambda$ has a right $\mathscr{X}$-approximation. The notions of *(minimal) left $\mathscr{X}$-approximations* and *covariantly finite subcategories* of $\operatorname{mod}\Lambda$ may be defined dually. The subcategory $\mathscr{X}$ is said to be *functorially finite* in $\operatorname{mod}\Lambda$ if it is both contravariantly finite and covariantly finite in $\operatorname{mod}\Lambda$.

For a module $M \in \operatorname{mod}\Lambda$, we denote ${}^{\perp_1}M = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(X, M) = 0\}$.

Lemma 4.4. *(See [T, Lemma 6.9].) Let $M \in \operatorname{mod}\Lambda$ with $M \in {}^{\perp_1}M$. Then for any $N \in \operatorname{mod}\Lambda$, there exists an exact sequence $0 \to F \to E \to N \to 0$ with $F = M^{(n)}$ and $E \in {}^{\perp_1}M$, where n is the number of generators of $\operatorname{Ext}^1_\Lambda(N, M)$ as an $\operatorname{End}(M)$-module. Hence ${}^{\perp_1}M$ is contravariantly finite.*

Lemma 4.5. *(See [AuS, Proposition 7.1].) ${}^{\perp_1}\Lambda$ is functorially finite in* $\operatorname{mod}\Lambda$.

Now we give the main result in this section.

Theorem 4.6. *A torsionless module $M \in \operatorname{mod}\Lambda$ is projective if the following conditions are satisfied*:

(1) $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ *for* $i = 1, 2, 3$.
(2) $\operatorname{Ext}^i_\Lambda(M, M) = 0$ *for* $i = 1, 2$.

Proof. Without loss of generality, we can assume that Λ is local with unique maximal ideal $\mathfrak{m}$ and residue field $k (= R/\mathfrak{m})$.

From Lemma 4.4, we know that there exists a right ${}^{\perp_1}M$-approximation: $0 \to M^n \to E' \to k \to 0$ for the simple Λ-module k, where n is the number of the generators of $\operatorname{Ext}^1_\Lambda(k, M)$ as an $\operatorname{End}(M)$-module. If $n = 0$, then $\operatorname{Ext}^1_\Lambda(k, M) = 0$. So M is injective. But M is torsionless by assumption, thus M is projective.

Now suppose $n \geqslant 1$. Consider the minimal right ${}^{\perp_1}M$-approximation of k: $0 \to M^m \to E \to k \to 0$. By applying the functor $\operatorname{Tr} M \otimes_\Lambda -$ to it, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & \operatorname{Tr} M \otimes_\Lambda M^m & \longrightarrow & \operatorname{Tr} M \otimes_\Lambda E & \longrightarrow & \operatorname{Tr} M \otimes_\Lambda k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta(M^m)} & & \downarrow{\scriptstyle \zeta(E)} & & \downarrow{\scriptstyle \zeta(k)} & & \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda((\operatorname{Tr} M)^*, M^m) & \xrightarrow{\alpha} & \operatorname{Hom}_\Lambda((\operatorname{Tr} M)^*, E) & \xrightarrow{\beta} & \operatorname{Hom}_\Lambda((\operatorname{Tr} M)^*, k). & &
\end{array}$$

Since $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for $i = 1, 2$, $\zeta(M^m)$ is an isomorphism by Lemma 4.2.

Consider the homomorphism $\zeta(k) : \mathrm{Tr}\, M \otimes_\Lambda k \to \mathrm{Hom}_\Lambda((\mathrm{Tr}\, M)^*, k)$ via $\zeta(k)(a \otimes \bar{r})(f) = f(a)\bar{r}$ for any $a \in \mathrm{Tr}\, M$, $f \in (\mathrm{Tr}\, M)^*$ and $\bar{r} \in k$. Because $\mathrm{Tr}\, M$ has no projective summands, f is not epimorphic. Notice that $(\Lambda, \mathfrak{m}, k)$ is local, $f(\mathrm{Tr}\, M) \subseteq \mathfrak{m}$. It follows that $\zeta(k)(a \otimes \bar{r})(f) = f(a)\bar{r} = 0$ and $\zeta(k) = 0$. Then we have that $\beta \circ \zeta(E) = 0$, and thus there exists a homomorphism $\gamma : \mathrm{Tr}\, M \otimes_\Lambda E \to \mathrm{Hom}_\Lambda((\mathrm{Tr}\, M)^*, M^m)$ such that $\alpha \circ \gamma = \zeta(E)$. Since α is monomorphic, the sequence $0 \to \mathrm{Tr}\, M \otimes_\Lambda M^m \to \mathrm{Tr}\, M \otimes_\Lambda E \to \mathrm{Tr}\, M \otimes_\Lambda k \to 0$ (the upper row in the above diagram) is exact and split. Then we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathrm{Tr}\, M \otimes_\Lambda k)^* & \longrightarrow & (\mathrm{Tr}\, M \otimes_\Lambda E)^* & \longrightarrow & (\mathrm{Tr}\, M \otimes_\Lambda M^m)^* & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}_\Lambda(k, (\mathrm{Tr}\, M)^*) & \longrightarrow & \mathrm{Hom}_\Lambda(E, (\mathrm{Tr}\, M)^*) & \longrightarrow & \mathrm{Hom}_\Lambda(M^m, (\mathrm{Tr}\, M)^*) & \longrightarrow & 0.
\end{array}$$

By the exactness of the bottom row in the above diagram, we have an exact sequence $0 \to \mathrm{Ext}^1_\Lambda(k, (\mathrm{Tr}\, M)^*) \to \mathrm{Ext}^1_\Lambda(E, (\mathrm{Tr}\, M)^*)$. By the claim below, $\mathrm{Ext}^1_\Lambda(E, (\mathrm{Tr}\, M)^*) = 0$, so $\mathrm{Ext}^1_\Lambda(k, (\mathrm{Tr}\, M)^*) = 0$ and $(\mathrm{Tr}\, M)^*$ is injective. Notice that $\mathrm{Ker}\, f \cong (\mathrm{Tr}\, M)^*$ in the minimal projective resolution $P_1 \xrightarrow{f} P_0 \to M \to 0$, so the projective dimension of M is at most 1. On the other hand, $\mathrm{Ext}^1_\Lambda(M, \Lambda) = 0$ by assumption, then it is easy to see that M is projective.

Claim. $\mathrm{Ext}^1_\Lambda(E, (\mathrm{Tr}\, M)^*) = 0$.

Consider the exact sequence $0 \to (\mathrm{Tr}\, M)^* \to P_1 \to P_0 \to M \to 0$. Since $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i = 1, 2, 3$, $(\mathrm{Tr}\, M)^* \in {}^{\perp_1}\Lambda$. Let $0 \to (\mathrm{Tr}\, M)^* \to Z \xrightarrow{h} E \to 0$ be any exact sequence in $\mathrm{Ext}^1_\Lambda(E, (\mathrm{Tr}\, M)^*)$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Tr}\, M)^* & \longrightarrow & X & \longrightarrow & M^m & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Tr}\, M)^* & \longrightarrow & Z & \longrightarrow & E & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & k & = & k & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since ${}^{\perp_1}\Lambda$ is closed under extensions, X is in ${}^{\perp_1}\Lambda$. On the other hand, ${}^{\perp_1}\Lambda$ is covariantly finite by Lemma 4.5, therefore $\mathrm{Hom}_\Lambda(M^m, -)|_{{}^{\perp_1}\Lambda} \to \mathrm{Ext}^1_\Lambda(k, -)|_{{}^{\perp_1}\Lambda} \to 0$ is exact. So there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^m & \longrightarrow & E & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & k & \longrightarrow & 0. \end{array}$$

Then we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^m & \longrightarrow & E & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & M^m & \longrightarrow & E & \longrightarrow & k & \longrightarrow & 0. \end{array}$$

Because $0 \to M^m \to E \to k \to 0$ is right minimal, it follows that the composition $E \to Z \xrightarrow{h} E$ is an isomorphism. Thus the exact sequence $0 \to (\mathrm{Tr}\, M)^* \to Z \xrightarrow{h} E \to 0$ splits and therefore $\mathrm{Ext}^1_\Lambda(E, (\mathrm{Tr}\, M)^*) = 0$. □

As an immediate consequence of Theorem 4.6, we get the following result, which means that **GPC** is true for commutative Artinian rings.

Theorem 4.7. *A Gorenstein projective module in* $\mathrm{mod}\, \Lambda$ *is projective if and only if it is self-orthogonal.*

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References

[AF] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, second ed., Grad. Texts in Math., vol. 13, Springer-Verlag, Berlin, 1992.

[Au] M. Auslander, Coherent functors, in: Proc. of the Conf. on Categorial Algebra, La Jolla, 1965, Springer-Verlag, Berlin, 1966, pp. 189–231.

[AuB] M. Auslander, M. Bridger, Stable module theory, Mem. Amer. Math. Soc. 94 (1969).

[AuR1] M. Auslander, I. Reiten, On a generalized version of the Nakayama conjecture, Proc. Amer. Math. Soc. 52 (1975) 69–74.

[AuR2] M. Auslander, I. Reiten, Representation theory of Artin algebra III, almost split sequences, Comm. Algebra 3 (1975) 239–294.

[AuR3] M. Auslander, I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1991) 111–152.

[AuS] M. Auslander, S.O. Smalø, Almost split sequences in subcategories, J. Algebra 69 (1981) 426–454.
[B] A. Braun, On a question of M. Auslander, J. Algebra 276 (2004) 674–684.
[EJ] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995) 611–633.
[G] R. Gordon (Ed.), Ring Theory, Proc. Conf. on Ring Theory, Held in Park City, March 2–6, 1971, Academic Press, New York, 1972.
[M] M.S. Menzin, The condition $\mathrm{Ext}^i(M, R) = 0$ for modules over local Artin algebras $(R, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$, Proc. Amer. Math. Soc. 43 (1974) 47–52.
[R] M. Ramras, Betti numbers and reflexive modules, in: R. Gordon (Ed.), Ring Theory, Proc. Conf. on Ring Theory, Held in Park City, March 2–6, 1971, Academic Press, New York, 1972, pp. 297–308.
[T] J. Trlifaj, Whitehead test modules, Trans. Amer. Math. Soc. 348 (1996) 1521–1554.
[XC] J.Z. Xu, F.C. Cheng, Homological dimensions over non-commutative semi-local rings, J. Algebra 169 (1994) 679–685.