On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems

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1. Introduction

Wave propagation in nonlinear periodic lattices is associated with a host of exciting phenomena that have no counterpart whatsoever in bulk media. Perhaps, the most intriguing entities that can exist in such systems are discrete self-localized state – better known as discrete solitons. By their very nature, these intrinsically localized models represent collective excitations of the chain as a whole, and are the outcome of the balance between nonlinear and linear coupling effects [7]. One of the simplest lattice models, which deserves special attention, is represented by Discrete Nonlinear Schrödinger (DNLS for short) equations [8]. In the past decade, the existence of discrete solitons of the DNLS equations has drawn a great deal of interest. To mention a few, see [1,2,4,6,15,17]. Among the methods used are the principle of anticontinuity [2,17], variational methods [1,4], centre manifold reduction [15], and Nehari manifold approach [21]. However, most of the existing literature is devoted...
to the DNLS equations with constant coefficients. Results on such DNLS equations have been summarized in the reviews [3,8,9]. And the experimental observations of two-dimensional discrete solitons have been reported in [5,10,11,16].

Recently, the DNLS equations with periodic coefficients have been considered in the physics literature, for example, [24]. Moreover, Gorbach and Johansson [14] reported results on numerical simulation of discrete gap solitons (a special discrete soliton which is defined later) in a particular periodic DNLS equation.

Assume that $T$ is a positive integer. In this paper, we will consider the following discrete nonlinear periodic system

$$Lu_n - \omega u_n = \sigma f_n(u_n), \quad n \in \mathbb{Z}, \quad (1.1)$$

where $f_n(u)$ is continuous in $u$ and with saturable nonlinearity for each $n \in \mathbb{Z}$, $f_{n+T}(u) = f_n(u)$, and $L$ is a Jacobi operator [25] given by

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

here $\{a_n\}, \{b_n\}$ are real valued $T$-periodic sequences.

Since $f_n(0) = 0$, $u_n \equiv 0$ is a solution of (1.1), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1.1) is homoclinic (to 0) if

$$\lim_{|n| \to \infty} u_n = 0. \quad (1.2)$$

In addition, if $u_n \not\equiv 0$, then $u$ is called a nontrivial homoclinic solution. We are interested in the existence of the nontrivial homoclinic solutions for (1.1). This problem appears when we look for the discrete solitons of the periodic DNLS equation

$$i \dot{\psi}_n = -\Delta \psi_n + \varepsilon_n \psi_n - \frac{\sigma \chi_n |\psi_n|^2 \psi_n}{1 + c_n |\psi_n|^2}, \quad n \in \mathbb{Z}, \quad (1.3)$$

where $\sigma = \pm 1$, $\Delta \psi_n = \psi_{n+1} + \psi_{n-1} - 2 \psi_n$ is the discrete Laplacian in one spatial dimension, the given sequences $\{\varepsilon_n\}, \{\chi_n\}, \{c_n\}$ are assumed to be $T$-periodic, $\{\chi_n\}$ and $\{c_n\}$ are positive, and the nonlinear function $f_n(u) = \chi_n u^3 (1 + c_n u^2)^{-1}$ is a representative of nonlinearities with saturation. The DNLS equations with saturable nonlinearities can describe optical pulse propagation in various doped fibres [12,13] and were studied in [21,26]. Since solitons are spatially localized time-periodic solutions and decay to zero at infinity. Thus $\psi_n$ has the form

$$\psi_n = u_n e^{-i\omega t},$$

and

$$\lim_{|n| \to \infty} \psi_n = 0,$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.3) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \frac{\sigma \chi_n u_n^3}{1 + c_n u_n^2}, \quad n \in \mathbb{Z}, \quad (1.4)$$

and (1.2) holds. Therefore, the problem on the existence of solitons of (1.3) has been reduced to that on the existence of homoclinic solutions of (1.4), which is a special case of (1.1) with $a_n \equiv -1$, $b_n = 2 + \varepsilon_n$, and $f_n(u) = \chi_n u^3 (1 + c_n u^2)^{-1}$.
We notice that, (1.1) was considered in [19,20] when \( f_n(u) = \chi_n u^3 \). Here, \( f_n(u) \) is superlinear at both 0 and \( \infty \), which played an important role in the existence of homoclinic solutions of (1.1).

When \( f_n(u) = \chi_n u^3(1 + c_n u^2)^{-1} \), (1.1) was considered in [23]. We will point out in Section 3, even in this special case, our results greatly improve those in [23].

Since the operator \( L \) is a bounded and self-adjoint operator in the space \( l^2 \) of two-sided infinite sequences, we consider (1.1) as a nonlinear equation in \( l^2 \) with (1.2) being satisfied automatically. The spectrum \( \sigma(L) \) of \( L \) has a band structure, i.e., \( \sigma(L) \) is a union of a finite number of closed intervals [25]. Thus the complement \( \mathbb{R} \setminus \sigma(L) \) consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite which are denoted by \( (-\infty, \beta) \) and \( (\alpha, +\infty) \), respectively. In this paper, we consider the homoclinic solutions of (1.1) in \( l^2 \) for the case where \( \omega \in (-\infty, \beta) \) and \( \sigma = 1 \). The case where \( \omega \in (\alpha, \infty) \) and \( \sigma = -1 \) is omitted, since in this case, we can replace \( L \) by \( -L \).

For the special case (1.4) of (1.1), the homoclinic solutions correspond to the gap solitons of (1.3). Generally, a soliton of (1.3) with the temporal frequency \( \omega \) belonging to a spectral gap is called a gap soliton.

The main idea in this paper is as follows. First, we consider (1.1) in a finite \( 2kT \)-periodic sequence space, and replace \( \omega \) by \( \omega_k \) which is near to \( \omega \) and is not a spectrum of the corresponding operator. By using critical point theory, we obtain the existence of \( 2kT \)-periodic solutions. Then, we show these solutions have upper and lower bounds. Finally, by an approximation technique, we prove that the limit of these solutions exists and is the solution of (1.1) in \( l^2 \). We mention that, critical point theory is a powerful tool to deal with the periodic solutions and the boundary value problems of differential equations [18,22] and is used to study periodic solutions and boundary value problems of discrete systems in recent years [27,28].

The remaining of this paper is organized as follows. First, in Section 2, we establish the variational framework associated with (1.1) and transfer the problem of the existence of homoclinic solutions of (1.1) into that of seeking a nonzero critical point of the functional.

2. Preliminaries

In this section, we first establish the variational framework associated with (1.1).

On the Hilbert space \( E = l^2 \), we consider the functional

\[
J(u) = \frac{1}{2} (Lu - \omega u, u) - \sum_{n=-\infty}^{\infty} F_n(u),
\]

where \((\cdot, \cdot)\) is the inner product in \( l^2 \), and

\[
F_n(u) = \int_{0}^{u} f_n(s) \, ds
\]

is the primitive function of \( f_n(u) \). The corresponding norm in \( E \) is denoted by \( \| \cdot \| \). Then \( J \in C^1(E, \mathbb{R}) \) and

\[
\langle J'(u), v \rangle = (Lu - \omega u, v) - \sum_{n=-\infty}^{\infty} f_n(u_n) v_n, \quad u, v \in E.
\]

Eq. (2.1) implies that (1.1) is the corresponding Euler–Lagrange equation for \( J \). Therefore, we have reduced the problem of finding a nontrivial homoclinic solution of (1.1) to that of seeking a nonzero critical point of the functional \( J \) on \( E \).
Let $S$ be the set of all two-sided sequences, that is,

$$S = \{u = \{u_n\} \mid u_n \in \mathbb{R}, \ n \in \mathbb{Z}\}.$$  

Then $S$ is a vector space with $au + bv = \{au_n + bv_n\}$ for $u, v \in S$, $a, b \in \mathbb{R}$.

For any fixed positive integer $k$, we define the subspace $E_k$ of $S$ as

$$E_k = \{u = \{u_n\} \in S \mid u_{n+2kT} = u_n \text{ for } n \in \mathbb{Z}\}.$$  

Obviously, $E_k$ is isomorphic to $\mathbb{R}^{2kT}$ and hence $E_k$ can be equipped with the inner product $(\cdot, \cdot)_k$ and norm $\|\cdot\|_k$ as

$$(u, v)_k = \sum_{n=-kT}^{kT-1} u_n v_n \quad \text{for } u, v \in E_k,$$

and

$$\|u\|_k = \left( \sum_{n=-kT}^{kT-1} u_n^2 \right)^{1/2} \quad \text{for } u \in E_k,$$

respectively. We also define a norm $\|\cdot\|_{k\infty}$ in $E_k$ by

$$\|u\|_{k\infty} = \max\{|u_n| : -kT \leq n \leq kT - 1\} \quad \text{for } u \in E_k.$$

Consider the functional $J_k$ on $E_k$ defined by

$$J_k(u) = \frac{1}{2} (Lu - \omega u, u)_k - \sum_{n=-kT}^{kT-1} F_n(u_n). \quad (2.2)$$

Then

$$\langle J_k'(u), v \rangle = (Lu - \omega u, v)_k - \sum_{n=-kT}^{kT-1} f_n(u_n)v_n, \quad u, v \in E_k. \quad (2.3)$$

Since the coefficients of the operator $L$ are $T$-periodic, it is easy to see that the critical points of $J_k$ in $E_k$ are exactly $2kT$-periodic solutions of system (1.1).

Let $L_k$ be the operator $L$ acting in $E_k$. It follows from the spectral theory of Jacobi operators [25] that $\sigma(L_k) \subset \sigma(L)$ and hence $\|L_k\| \leq \|L\|$.

Let $\delta$ be the distance from $\omega$ to the spectrum $\sigma(L)$, that is,

$$\delta = \beta - \omega.$$

Then, we have

$$(Lu - \omega u, u) \geq \delta \|u\|^2 \quad \text{for } u \in E, \quad (2.4)$$

and

$$(L_ku - \omega u)_k \geq \delta \|u\|^2_k \quad \text{for } u \in E_k. \quad (2.5)$$
In order to obtain the existence of critical points of $J_k$ on $E_k$, for the convenience of the readers, we cite some basic notations and some known results from critical point theory.

Let $H$ be a Hilbert space and $C^1(H, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on $H$.

**Definition 2.1.** Let $J \in C^1(H, \mathbb{R})$. A sequence $\{x_j\} \subset H$ is called a Palais–Smale sequence (P.S. sequence for short) for $J$ if $\{J(x_j)\}$ is bounded and $J'(x_j) \to 0$ as $j \to \infty$. We say $J$ satisfies the Palais–Smale condition (P.S. condition for short) if any P.S. sequence for $J$ possesses a convergent subsequence.

Let $B_r$ be the open ball in $H$ with radius $r$ and center 0, and let $\partial B_r$ denote its boundary. The following lemma is taken from [22] and will play an important role in the proofs of our main results.

**Lemma 2.1 (Mountain Pass Lemma).** Let $H$ be a real Hilbert space and $J \in C^1(H, \mathbb{R})$ satisfies the P.S. condition. If $J(0) = 0$ and the following conditions hold.

1. There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \geq a$.
2. There exist an $e \in H \setminus B_\rho$ such that $J(e) < 0$.

Then $J$ possesses a critical value $c \geq a$. Moreover, $c$ can be characterized as

$$c = \inf_{h \in \Gamma} \max_{s \in [0, 1]} J(h(s)),$$

where

$$\Gamma = \{h \in C([0, 1], H) : h(0) = 0, h(1) = e\}.$$

### 3. Main results

In this section, we will establish some sufficient conditions on the existence of nontrivial solutions of (1.1) in $l^2$.

**Theorem 3.1.** Assume that $\sigma = 1$, $\omega \in (-\infty, \beta)$, $f_n$ is continuous in $u$, $f_{n+1}(u) = f_n(u)$ for any $n \in \mathbb{Z}$ and $u \in \mathbb{R}$, $f_n(u) = o(u)$ as $u \to 0$. And the following conditions hold.

1. $f_n(u)/u$ is strictly increasing in $(0, \infty)$ and strictly decreasing in $(-\infty, 0)$. Moreover, $\lim_{|u| \to \infty} f_n(u)/u = d_n < \infty$.
2. $f_n(u)u - 2F_n(u) \to \infty$ as $|u| \to \infty$, and

$$\limsup_{u \to 0} \sup_{n} \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} = p_n < \infty.$$

If $d_n > \beta - \omega$ for $n \in \mathbb{Z}$, then system (1.1) has at least a nontrivial solution $u$ in $l^2$. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants $C$ and $\tau$ such that

$$|u_n| \leq Ce^{-\tau|n|}, \quad n \in \mathbb{Z}. \quad (3.1)$$

Theorem 3.1 gives a sufficient condition on the existence of nontrivial solutions of (1.1) in $l^2$. We notice that, with part of these conditions being violated, (1.1) has no nontrivial solutions in $l^2$. In fact, we have the following proposition.

**Proposition 3.2.** Assume that $\sigma = 1$, $\omega \in (-\infty, \beta)$, $f_n$ satisfies the conditions in Theorem 1.1. If $d_n \leq \beta - \omega$ for $n \in \mathbb{Z}$, then system (1.1) has no nontrivial solutions in $l^2$. 

Proof. By way of contradiction, we assume that (1.1) has a nontrivial solution \( u = \{u_n\} \in l^2 \). Then \( u \) is a nonzero critical point of \( J \), and

\[
\{J'(u), u\} = ((L - \omega)u, u) - \sum_{n=-\infty}^{\infty} f_n(u_n)u_n = 0.
\]

Noticing (2.4) and (H1), the above equality implies that

\[
\delta \|u\|^2 \leq ((L - \omega)u, u) = \sum_{n=-\infty}^{\infty} f_n(u_n)u_n < \sum_{n=-\infty}^{\infty} d_nu_n^2,
\]

where \( \delta = \beta - \omega \). This is impossible as \( d_n \leq \delta \) for \( n \in \mathbb{Z} \) and the proof is complete. \( \square \)

Remark 3.1. By combining Theorem 3.1 and Proposition 3.2, we find that: if \( \sigma = 1 \), \( \omega \in (-\infty, \beta) \), \( f_n \) satisfies the conditions in Theorem 1.1, and \( d_n \equiv d \) for \( n \in \mathbb{Z} \), then (1.1) has at least a nontrivial solution \( u \) in \( l^2 \) if and only if \( d > \beta - \omega \). In this sense, \( d_n > \beta - \omega \) for \( n \in \mathbb{Z} \) is a “sharp” condition on the existence of nontrivial solution of (1.1) in \( l^2 \).

Remark 3.2. Compared with the Nehari Manifold approach used in [20], in this paper, we need not to assume that \( f_n(u) \) is differentiable in \( u \) for each \( n \in \mathbb{Z} \).

Remark 3.3. It is easy to see that the function \( f_n \) defined by

\[
f_n(u) = \frac{\chi_n u_n^3}{1 + c_n u_n^2},
\]

satisfies all conditions in Theorem 1.1 with \( d_n = \chi_n/c_n \) and \( p_n \equiv 0 \), where \( \{\chi_n\} \) and \( \{c_n\} \) are positive real valued \( T \)-periodic sequences. This case was studied by [23], we find that Theorem 3.1 greatly improves Theorem 1.1 in [23].

Example 3.1. Consider the following nonlinear difference equation

\[
-\Delta u_n - \omega u_n = \frac{\chi_n u_n^3}{1 + c_n u_n^2}, \quad n \in \mathbb{Z},
\]

which is deduced by the periodic DNLS equation (1.3) with \( \varepsilon_n \equiv 0 \) and \( \sigma = 1 \), here \( \{\chi_n\} \) and \( \{c_n\} \) are positive and \( T \)-periodic sequences. Since the spectrum of \(-\Delta\) is \([0, 4]\). According to Theorem 3.1, Proposition 3.2 and Remark 3.3, we have: if \( \omega < 0 \) and \( \chi_n/c_n + \omega > 0 \) for \( n \in \mathbb{Z} \), then (3.2) has at least a nontrivial solution in \( l^2 \); if \( \omega < 0 \) and \( \chi_n/c_n + \omega \leq 0 \) for \( n \in \mathbb{Z} \), then (3.2) has no nontrivial solution in \( l^2 \).

In order to complete the proof of Theorem 3.1, in the following, for each \( k \in \mathbb{N} \), we define the linear operator \( \tilde{L}_k \) as

\[
\tilde{L}_k u_n = L_k u_n - d_n u_n, \quad u \in E_k.
\]

We let

\[
d^* = \max_{n \in \mathbb{Z}} \{d_n\}, \quad d_* = \min_{n \in \mathbb{Z}} \{d_n\},
\]

then \( d_* > \delta \).
First, we need to establish some lemmas.

**Lemma 3.1.** Assume that \( k \in \mathbb{N} \) and the conditions of Theorem 3.1 hold. If \( \omega \notin \sigma(\tilde{L}_k) \), then the functional \( J_k \) satisfies the P.S. condition.

**Proof.** Let \( \{u^{(j)}\} \subset E_k \) be a P.S. sequence for \( J_k \). We need to show that \( \{u^{(j)}\} \) has a convergent subsequence. Since \( E_k \) is finite dimensional, it suffices to show that \( \|u^{(j)}\| \) is bounded. There is no harm in assuming that \( \|J'_k(u^{(j)})\| \leq 1 \) for \( j \in \mathbb{N} \).

Let \( W_k^+ \) and \( W_k^- \) be the spectral subspaces of the operator \( \tilde{L}_k \) that correspond to eigenvalues \( \lambda > \omega \) and \( \lambda < \omega \), respectively. Since \( \omega \notin \sigma(\tilde{L}_k) \), we have \( E_k = W_k^+ \oplus W_k^- \), and there is a positive constant \( \eta_k \) such that

\[
\pm(\tilde{L}_k u - \omega u, u)_k \geq \eta_k \|u\|^2_k \quad \text{for } u \in W_k^+.
\]

(3.4)

For each \( j \in \mathbb{N} \), let \( u^{(j)} = u^{(j)+} + u^{(j)-} \), where \( u^{(j)+} \in W_k^+ \) and \( u^{(j)-} \in W_k^- \). Then

\[
\langle J'_k(u^{(j)}), u^{(j)+} \rangle = (L_k u^{(j)} - \omega u^{(j)}, u^{(j)+})_k - \sum_{n=-kT}^{kT-1} f_n(u_n^{(j)}) u_n^{(j)+} + \sum_{n=-kT}^{kT-1} (d_n u_n^{(j)} - f_n(u_n^{(j)})) u_n^{(j)+}.
\]

(3.5)

By (H1), there is a constant \( M > 0 \) such that

\[
0 < d_n - \frac{f_n(u)}{u} \leq \frac{1}{2} \eta_k \quad \text{for } |u| \geq M \text{ and } n \in \mathbb{Z}.
\]

(3.6)

Let

\[
Q_k^{(j)} = \{ n \in \mathbb{Z} : |u_n^{(j)}| < M, -kT \leq n \leq kT - 1 \},
\]

\[
R_k^{(j)} = \{ n \in \mathbb{Z} : |u_n^{(j)}| \geq M, -kT \leq n \leq kT - 1 \}.
\]

Then,

\[
\sum_{n=-kT}^{kT-1} (d_n u_n^{(j)} - f_n(u_n^{(j)}))^2 = \sum_{n \in Q_k^{(j)}} (d_n u_n^{(j)} - f_n(u_n^{(j)}))^2 + \sum_{n \in R_k^{(j)}} (d_n u_n^{(j)} - f_n(u_n^{(j)}))^2 \\
\leq \sum_{n \in Q_k^{(j)}} d_n^2 M^2 + \sum_{n \in R_k^{(j)}} \left( \frac{1}{2} \eta_k u_n^{(j)} \right)^2 \\
\leq 2kT (d^2) M^2 + \frac{1}{4} \eta_k^2 \|u^{(j)}\|^2_k,
\]

\[\]
which implies that
\[
\left( \frac{1}{2} \left( \sum_{n=-kT}^{kT-1} (d_n u_n^j - f_n(u_n^j))^2 \right) \right) \leq \sqrt{2kT} d^* M + \frac{1}{2} \eta_k \|u^j\|_k. \tag{3.7}
\]

This, combined with (3.5) and Cauchy inequality, gives us
\[
(\tilde{L}_k u^{(j)+} - \omega u^{(j)+}, u^{(j)+})_k
\]
\[
= (J_k(u^{(j)}), u^{(j)+}) - \sum_{n=-kT}^{kT-1} (d_n u_n^j - f_n(u_n)) u_n^j
\]
\[
\leq \|u^{(j)+}\|_k + \left( \sum_{n=-kT}^{kT-1} (d_n u_n^j - f_n(u_n))^2 \right) \frac{1}{2} \|u^{(j)+}\|_k
\]
\[
\leq (1 + \sqrt{2kT} d^* M)\|u^{(j)+}\|_k + \frac{1}{2} \eta_k \|u^{(j)}\|_k \|u^{(j)+}\|_k. \tag{3.8}
\]

Using (3.4), the above inequality implies
\[
\eta_k \|u^{(j)+}\|^2_k \leq (1 + \sqrt{2kT} d^* M)\|u^{(j)+}\|_k + \frac{1}{2} \eta_k \|u^{(j)}\|_k \|u^{(j)+}\|_k. \tag{3.9}
\]

Similarly, we have
\[
\eta_k \|u^{(j)-}\|^2_k \leq (1 + \sqrt{2kT} d^* M)\|u^{(j)-}\|_k + \frac{1}{2} \eta_k \|u^{(j)}\|_k \|u^{(j)-}\|_k. \tag{3.10}
\]

Noticing that \(\|u^{(j)}\|^2_k = \|u^{(j)+}\|_k^2 + \|u^{(j)-}\|_k^2\) and \(\|u^{(j)+}\|_k + \|u^{(j)-}\|_k \leq \sqrt{2}\|u^{(j)}\|_k\), by (3.9) and (3.10), we get
\[
\eta_k \|u^{(j)}\|^2_k \leq \sqrt{2} \left(1 + \sqrt{2kT} d^* M\right)\|u^{(j)}\|_k + \frac{\sqrt{2}}{2} \eta_k \|u^{(j)}\|^2_k,
\]
which immediately implies that \(\|u^{(j)}\|_k\) is bounded. This completes the proof. \(\square\)

**Lemma 3.2.** Assume that \(\sigma = 1\), \(f_n\) satisfies the conditions in Theorem 1.1, and \(d_n > \beta - \omega_0 > 0\). Then there exists \(k_0 \in \mathbb{N}\) and a positive constant \(\gamma\) such that \(J_k\) has at least a nonzero critical point \(u^{(k)}\) in \(E_k\) for each \(k > k_0\), if \(\omega \in [\omega_0 - \gamma, \omega_0 + \gamma]\) and \(\omega \notin \sigma(\tilde{L}_k)\). Moreover, there exist positive constants \(\xi\) and \(\eta\) such that
\[
\xi \leq \|u^{(k)}\|_{k_\infty} \leq \eta \tag{3.11}
\]
for \(k > k_0\).

**Proof.** Let
\[
\gamma = \min \left\{ \frac{d_n - \beta + \omega_0}{2}, \frac{\beta - \omega_0}{2} \right\},
\]
and $\varepsilon_0 \in (0, 1)$ satisfying
\[
\beta - \omega_0 + \gamma + 3\varepsilon_0 - (d_* - \varepsilon_0)(1 - \varepsilon_0) < -\varepsilon_0.
\] (3.12)

Since $L$ is a bounded self-adjoint linear operator and $\beta \in \sigma(L)$, there exists $e = \{e_n\} \in l^2$ with $\|e\| = 1$ such that $(Le, e) < \beta + \varepsilon_0$. Let $k_0$ be large enough such that
\[
k_0 T - 1 \sum_{n=-k_0 T}^{k_0 T-1} (L - \omega)e_n, e_n \frac{\varepsilon_0}{1 - \varepsilon_0} + \sum_{n=-k_0 T}^{k_0 T-1} e_n^2 \geq 1 - \varepsilon_0.
\]
and
\[
|a_T^{-1}(ek_0 T e_{k_0 T} - e_{-k_0 T} e_{-k_0 T})| \leq \varepsilon_0.
\]

In the following, we use Lemma 2.1 to finish the proof.

By Lemma 3.1, $J_k$ satisfies the P.S. condition. Now we show that $J_k$ satisfies the condition $(J_1)$ in Lemma 2.1. In fact, there exists a positive constant $\rho$ such that
\[
0 \leq F_n(u) \leq \frac{1}{8}(\beta - \omega_0)u^2 \quad \text{for } n \in \mathbb{Z} \text{ and } |u| \leq \rho.
\] (3.13)

Then, for $\|u\|_k \leq \rho$,
\[
J_k(u) = \frac{1}{2}((L_k - \omega)u, u)_k - \sum_{n=-k_0 T}^{k_0 T-1} F_n(u_n)
\]
\[
\geq \frac{1}{2}(\beta - \omega)\|u\|_k^2 - \sum_{n=-k_0 T}^{k_0 T-1} F_n(u_n)
\]
\[
\geq \frac{1}{2}(\beta - \omega - \gamma)\|u\|_k^2 - \frac{1}{8}(\beta - \omega_0)\sum_{n=-k_0 T}^{k_0 T-1} u_n^2
\]
\[
\geq \frac{1}{8}(\beta - \omega_0)\|u\|_k^2.
\]
Taking $a = \frac{1}{8}(\beta - \omega_0)\rho^2$. Then $J|_{\partial B_\rho} \geq a$ and hence $J_k$ satisfies condition $(J_1)$ of Lemma 2.1.

Next we prove that $J$ satisfies condition $(J_2)$ of Lemma 2.1 if $k \geq k_0 + 1$. In fact, for $k \geq k_0 + 1$, define $e^{(k)} \in E_k$ by
\[
e_n^{(k)} = \begin{cases}
e_n, & -k_0 T \leq n \leq k_0 T - 1, \\
0, & -kT \leq n \leq -k_0 T - 1 \text{ or } k_0 T \leq n \leq kT - 1.
\end{cases}
\]

Then
\[
((L_k - \omega)e^{(k)}, e^{(k)})_k = \sum_{n=-k_0 T}^{k_0 T-1} ((L - \omega)e_n, e_n) - a_T^{-1}(ek_0 T e_{k_0 T} - e_{-k_0 T} e_{-k_0 T})
\]
\[
< \beta - \omega_0 + \gamma + 3\varepsilon_0,
\]
and for \( r > 0 \),

\[
J_k(\rho e^{(k)}) = \frac{r^2}{2}((L_k - \omega)e^{(k)}, e^{(k)})_k - \sum_{n=-kT}^{kT-1} F_n(\rho e^{(k)}) \\
\leq \frac{r^2}{2} (\beta - \omega_0 + \gamma + 3\varepsilon_0) - \sum_{n=-kT}^{kT-1} F_n(\rho e^{(k)}) .
\] (3.14)

By the fact that \( \lim_{|u| \to \infty} F_n(u)/u^2 = d_n/2 \) and the periodicity of \( F_n \) in \( n \), there exists a positive constant \( r_0 \) such that

\[
F_n(\rho e_n) \geq d_n - \varepsilon_0 - \varepsilon_0 \sum_{n=-k0T}^{n} (d_n - \varepsilon_0) e_n^2 ,
\] (3.15)

for \( r \geq r_0 \). Thus,

\[
J_k(\rho_0 e^{(k)}) \leq -\frac{\varepsilon_0 r_0^2}{2} < 0.
\]

Now that we have verified all assumptions of Lemma 2.1, we know that \( J_k \) possesses a critical value \( \alpha_k \geq a \) with

\[
\alpha_k = \inf_{h \in I_k} \max_{s \in [0,1]} J_k(h(s))
\]

where

\[
I_k = \{ h \in C([0,1], E_k) : h(0) = 0, h(1) = \rho_0 e^{(k)} \}.
\]

A critical point \( u^{(k)} \) of \( J_k \) corresponding to \( \alpha_k \) is nonzero as \( \alpha_k \geq a > 0 \).

For \( k \geq k_0 + 1 \), we define \( h_k \in I_k \) as \( h_k(s) = s \rho_0 e^{(k)} \) for \( s \in [0,1] \), then

\[
J_k(u^{(k)}) \leq \max\{ J(s \rho_0 e^{(k)} ) : s \in [0,1] \}
= \max\{ J(s \rho_0 e^{(k_0+1)} ) : s \in [0,1] \}
\leq \max\{ J(s \rho_0 e^{(k_0+1)} ) : s \in [0,1], \omega \in [\omega_0 - \gamma, \omega_0 + \gamma] \}.
\]

Let

\[
M_0 \triangleq \max\{ J(s \rho_0 e^{(k_0+1)} ) : s \in [0,1], \omega \in [\omega_0 - \gamma, \omega_0 + \gamma] \}.
\]
then by (2.2) and (2.3),
\[
J_k(u^{(k)}) = \sum_{n=-kT}^{kT-1} \left( \frac{1}{2} f_n(u_n^{(k)}) u_n^{(k)} - F_n(u_n^{(k)}) \right) \leq M_0.
\] (3.16)

From (H2), there exists a positive constant \( \eta \) such that
\[
\frac{1}{2} f_n(u) u - F_n(u) > M_0 \quad \text{for } n \in \mathbb{Z} \text{ and } |u| > \eta,
\]
then (3.16) implies that
\[
|u_n^{(k)}| \leq \eta \quad \text{for } n \in \mathbb{Z}, \text{ that is }
\|

\|u^{(k)}\|_{k\infty} \leq \eta.
\]

On the other hand, from (2.3) and (2.5), we have
\[
\frac{1}{2} (\beta - \omega_0) \|u^{(k)}\|_k^2 \leq (\beta - \omega) \|u^{(k)}\|_k^2 \\
\leq ((L_k - \omega) u^{(k)}, u^{(k)})_k \\
= \sum_{n=-kT}^{kT-1} f_n(u_n^{(k)}) u_n^{(k)}
\] (3.18)
for \( \omega \in [\omega_0 - \gamma, \omega_0 + \gamma] \). And there exists a positive number \( \xi \), such that
\[
0 \leq f_n(u) u \leq \frac{1}{4} (\beta - \omega_0) u^2 \quad \text{for } |u| \leq \xi.
\]
This combining with (3.18) gives
\[
\|u^{(k)}\|_{k\infty} \geq \xi.
\] (3.19)

The proof is complete. \( \square \)

Now, we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Replace \( \omega_0 \) by \( \omega \), and let \( k_0 \) be the integer obtained in Lemma 3.2. For every \( k > k_0 \), choose a \( \omega_k \in [\omega - \gamma, \omega + \gamma] \) such that \( |\omega_k - \omega| < 1/k \) and \( \omega_k \notin \sigma(L_k) \), then \( J_k \) has a critical point \( u^{(k)} = \{u_n^{(k)}\} \in E_k \) corresponding to \( \omega = \omega_k \), which is obtained by Lemma 3.2. Moreover, there exists an \( n_k \in \mathbb{Z} \) such that
\[
\xi \leq |u_n^{(k)}| \leq \eta.
\] (3.20)

Note that
\[
a_n u_{n+1}^{(k)} + a_{n-1} u_{n-1}^{(k)} + (b_n - \omega_k) u_n^{(k)} = f_n(u_n^{(k)}), \quad n \in \mathbb{Z}.
\] (3.21)

By the periodicity of the coefficients in (3.21), we see that \( \{u_{n+T}^{(k)}\} \) is also a solution of (3.21). Making some shifts if necessary, without loss of generality, we can assume that \( 0 \leq n_k \leq T - 1 \) in (3.20). Moreover, passing to a subsequence of \( \{u^{(k)}\} \) if necessary, we can also assume that \( n_k = n^* \) for \( k \geq k_0 \) and
some integer $n^*$ such that $0 \leq n^* \leq T - 1$. It follows from (3.20) that we can choose a subsequence, still denoted by $\{u^{(k)}\}$, such that
\[
u_n^{(k)} \to u_n \quad \text{as} \quad k \to \infty \quad \text{for} \quad n \in \mathbb{Z}.
\]

Then $u = \{u_n\}$ is a nonzero sequence as (3.20) implies $|u_n| \geq \xi$. It remains to show that $u = \{u_n\} \in l^2$ and is a solution of (1.1).

First, we show that $u \in l^2$. In view of (2.5), we have
\[
\frac{1}{2} \left( \beta - \omega \right) \left\| u^{(k)} \right\|_k^2 \leq (\beta - \omega_k) \left\| u^{(k)} \right\|_k^2 \\
\leq \left( (L_k - \omega_k) u^{(k)} , u^{(k)} \right) \\
= \sum_{n=-kT}^{kT-1} f_n(u^{(k)}) u_n^{(k)} \\
\leq \left( \sum_{n=-kT}^{kT-1} (f_n(u^{(k)}))^2 \right) \frac{1}{2} \left\| u^{(k)} \right\|_k.
\]

which implies that
\[
\frac{1}{4} (\beta - \omega)^2 \left\| u^{(k)} \right\|_k^2 \leq \sum_{n=-kT}^{kT-1} (f_n(u^{(k)}))^2. \tag{3.22}
\]

From (H2), there exists a positive constant $\zeta$ such that
\[
f_n^2(u) \leq \zeta \left( f_n(u)u - 2F_n(u) \right), \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad |u| \leq \eta. \tag{3.23}
\]

Combining (3.11), (3.16), (3.22) and (3.23), we find that
\[
\left\| u^{(k)} \right\|_k^2 \leq 8 \zeta M_0 (\beta - \omega)^{-2}. \tag{3.24}
\]

For each $s \in \mathbb{N}$, let $k > \max\{s, k_0\}$. Then it follows from (3.24) that
\[
\sum_{n=-s}^{s} (u_n^{(k)})^2 \leq \left\| u^{(k)} \right\|_k^2 \leq 8 \zeta M_0 (\beta - \omega)^{-2}. \tag{3.24}
\]

Letting $k \to \infty$ gives us $\sum_{n=-s}^{s} u_n^2 \leq 8 \zeta M_0 (\beta - \omega)^{-2}$. By the arbitrariness of $s$, we know that $u = \{u_n\} \in l^2$.

Next, we show that $u$ satisfies (1.1). Indeed, for each $n \in \mathbb{Z}$, letting $k \to \infty$ in (3.21) gives us
\[
a_n u_{n+1} + a_{n-1} u_{n-1} + (b_n - \omega) u_n = f_n(u_n),
\]

that is, $u = \{u_n\}$ satisfies (1.1).

Finally, we show that $u$ satisfies that (3.1).
In fact, for \( n \in \mathbb{Z} \), let
\[
 w_n = \begin{cases} 
 -\frac{L(u_n)}{u_n}, & \text{if } u_n \neq 0, \\
 0, & \text{if } u_n = 0,
\end{cases}
\]
then
\[
 \tilde{L}u_n = \omega u_n, \tag{3.25}
\]
where
\[
 \tilde{L}u_n = Lu_n + w_n u_n.
\]
Clearly, \( \lim_{|n| \to \infty} w_n = 0 \). Thus, the multiplication by \( w_n \) is a compact operator in \( l^2 \), which implies that
\[
 \sigma_{\text{ess}}(\tilde{L}) = \sigma_{\text{ess}}(L),
\]
where \( \sigma_{\text{ess}} \) stands for the essential spectrum. (3.25) means that \( u = \{u_n\} \) is an eigenfunction that corresponds to the eigenvalue of finite multiplicity \( \omega \notin \sigma_{\text{ess}}(\tilde{L}) \) of the operator \( \tilde{L} \). (3.1) follows from the standard theorem on exponential decay for such eigenfunctions [25].

Now the proof of Theorem 3.1 is complete. \( \square \)

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References