Note

An extremal problem on potentially $K_{r,s}$-graphic sequences

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Abstract

We consider a variation of a classical Turán-type extremal problem (F. Chung, R. Graham, Erdős on Graphs: His Legacy of Unsolved Problems, AK Peters Ltd., Wellesley, 1998, Chapter 3) as follows: Determine the smallest even integer $\sigma(K_{r,s},n)$ such that every $n$-term graphic sequence $\pi=(d_1,d_2,\ldots,d_n)$ with term sum $\sigma(\pi)=d_1+d_2+\cdots+d_n \geq \sigma(K_{r,s},n)$ is potentially $K_{r,s}$-graphic, where $K_{r,s}$ is a $r \times s$ complete bipartite graph, i.e., $\pi$ has a realization $G$ containing $K_{r,s}$ as its subgraph. In this paper, we first give sufficient conditions for a graphic sequence being potentially $K_{r,s}$-graphic, and then we determine $\sigma(K_{r,s},n)$ for $r=3,4$.

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1. Introduction

The set of all sequences $\pi=(d_1,d_2,\ldots,d_n)$ of nonnegative integers with $d_i \leq n-1$ for each $i$ is denoted by $NS_n$. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of some simple graph $G$ on $n$ vertices, and such graph $G$ is called a realization of $\pi$. The set of all graphic $n$-term sequences in nonincreasing order is denoted by $G_n$. For a given graph $H$, a sequence $\pi \in G_n$ is said to be forcibly (resp. potentially) $H$-graphic if each realization of $\pi$ contains $H$ as its subgraph (resp. there exists a realization of $\pi$ containing $H$ as its subgraph).

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It is well known [1] that one of classical extremal problems in extremal graph theory is to determine the smallest integer \( t(H,n) \) such that every graph \( G \) on \( n \) vertices with edge number \( e(G) \geq t(H,n) \) contains \( H \) as its subgraph. The number \( t(H,n) \) is called the Turán number of \( H \). The classical Turán theorem [1] determined the Turán number \( t(K_r,n) \) for \( K_r \), a complete graph on \( r \) vertices. About the Turán number \( t(K_{2r},n) \), Kővári et al. [11] and Erdős and Rényi [6], gave the general upper and lower bounds, respectively, as follows: \( c_1 n^{2-(2/r)} \leq t(K_{2r},n) \leq c_2 n^{2-(1/r)} \). In [2, Chapter 3], a conjecture has been made that \( t(K_{2r},n) \geq cn^{2-(1/r)} \) for \( r \geq 4 \). Erdős et al. [7] proved \( t(K_{2r},n) \sim \frac{1}{2} n^{3/2} \). Recently, Füredi [8] proved \( t(K_{3,3},n) \sim \frac{1}{2} n^{5/3} \). Nothing new is known about \( t(K_{4,4},n) \). But Erdős [3] has asked if the ratio \( t(K_{4,4},n)/t(K_{3,3},n) \) tends to infinity or not.

In terms of graphic sequences, the number \( 2t(K_r,n) \) is the smallest even integer such that each graphic \( n \)-term sequence \( \pi \) with \( \sigma(\pi) \geq 2t(K_r,n) \) is forcibly \( K_r \)-graphic. Erdős et al. [5] considered a variation of Turán theorem as follows: determine the smallest even integer \( \sigma(K_r,n) \) such that every positive graphic \( n \)-term sequence \( \pi = (d_1, \ldots, d_n) \) with term sum \( \sigma(\pi) = d_1 + \cdots + d_n \geq \sigma(K_r,n) \) is potentially \( K_r \)-graphic. They conjectured that \( \sigma(K_r,n) = (r-2)(2n-r+1) + 2 \) for enough large \( n \) and proved that the conjecture holds for \( r = 3 \) and \( n \geq 6 \). Recently, Li and Song [13] and Gould et al. [9] proved that the conjecture holds for \( r = 4 \) and \( n \geq 8 \) independently. Li and Song [12] and Li et al. [14] also proved that the conjecture is positive for \( r = 5 \) and \( n \geq 10 \) and for \( r \geq 6 \) and \( n \geq (r-1)^2 + 3 \). The problem about determining \( \sigma(K_{k+1},n) \) is completely solved.

A similar problem is to determine the smallest even integer \( \sigma(K_r,s,n) \) such that each graphic \( n \)-term sequence \( \pi \) with \( \sigma(\pi) \geq \sigma(K_r,s,n) \) is potentially \( K_{r,s} \)-graphic. Gould et al. [9] determined the number \( \sigma(K_{2,2},n) \). In this paper, we first give certain sufficient conditions for a graphic sequence being potentially \( K_{r,s} \)-graphic. And then we use these sufficient conditions to determine \( \sigma(K_{r,r},n) \) for \( r = 3, 4 \).

2. On the potentially \( K_{r,r} \)-graphic sequences

In this section, we first give sufficient conditions for a graphic sequence being potentially \( K_{r,s} \)-graphic. We need some notations and known results.

Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n \). If \( \pi \) has a realization \( H \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \), such that \( d_i \) is the degree of \( v_i \) in \( H \), and \( H \) contains \( K_{r,s} \) as its subgraph, where \( X = \{v_1, \ldots, v_r\} \), \( Y = \{u_{r+1}, \ldots, u_{r+s}\} \subseteq \{v_{r+1}, \ldots, v_n\} \) is the bipartite partition of the vertex set of \( K_{r,s} \), then \( \pi \) is called potentially \( B_{r,s} \)-graphic. Furthermore, if \( \{u_{r+1}, \ldots, u_{r+s}\} = \{v_{r+1}, \ldots, v_{r+s}\} \), then \( \pi \) is said to be potentially \( A_{r,s} \)-graphic.

**Theorem 2.1** (Erdős and Gallai [4]). Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \) be a nonincreasing sequence and with even \( \sigma(\pi) \). Then \( \pi \in G_n \) if and only if for any \( t \), \( 1 \leq t \leq n - 1 \),

\[
\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.
\]
For a nonincreasing sequence \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \), let

\[
\pi'' = \begin{cases} 
(d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k} - 1, d_{d_k+2}, \ldots, d_n) & \text{if } d_k \geq k, \\
(d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{d_k-1}, d_{d_k+1}, \ldots, d_n) & \text{if } d_k < k.
\end{cases}
\]

Denote \( \pi' = (d'_1, d'_2, \ldots, d'_{n-1}) \), where \( d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1} \) is the rearrangement of the \((n-1)\) terms in \( \pi'' \). Then \( \pi' \) is called the residual sequence obtained by laying off \( d_k \) from \( \pi \).

**Theorem 2.2** (Kleitman and Wang [10]). Let \( \pi \in NS_n \) be a nonincreasing sequence. Then \( \pi \in G_n \) if and only if \( \pi' \in G_{n-1} \).

**Proposition 2.3.** Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n \), where \( n - 1 \geq d_1 \geq \cdots \geq d_r \geq s \), \( n - 1 \geq d_{r+1} \geq \cdots \geq d_{r+s} \geq r \) and \( n - 1 \geq d_{r+s+1} \geq \cdots \geq d_n \geq r \). Denote

\[
\pi'_1 = \begin{cases} 
(d_2 - 1, \ldots, d_r - 1, d_{r+1}, \ldots, d_{d_1}, \ldots, d_n) & \text{if } d_1 \leq n - r, \\
d_2 - 1, \ldots, d_{d_1-r+n-1} - 1, \\
d_{d_1+r-n+2}, \ldots, d_r, d_{r+1} - 1, \ldots, d_n - 1) & \text{if } d_1 > n - r
\end{cases}
\]

and \( \pi''_1 = (d^{(1)}_2, \ldots, d^{(1)}_r, d^{(1)}_{r+1}, \ldots, d^{(1)}_r, d^{(1)}_{r+s}, d^{(1)}_{r+s+1}, \ldots, d^{(1)}_n) \), where \( d^{(1)}_2 \geq \cdots \geq d^{(1)}_r \) is the rearrangement of the first \( r - 1 \) terms in \( \pi'_1 \), \( d^{(1)}_{r+1} = d_{r+1} - 1, \ldots, d^{(1)}_{r+s} = d_{r+s} - 1 \), and \( d^{(1)}_{r+s+1} \geq \cdots \geq d^{(1)}_n \) is the rearrangement of the final \( n - r - s \) terms in \( \pi'_1 \). If \( \pi''_1 \) has a realization \( H \) with vertex set \( \{v_2, \ldots, v_n\} \) such that \( d^{(1)}_i \) is the degree of \( v_i \) in \( H \) and \( H \) contains \( K_{r-1,s} \) as its subgraph, where \( X = \{v_2, \ldots, v_r\}, Y = \{v_{r+1}, \ldots, v_{r+s}\} \) is the bipartite partition of the vertex set of \( K_{r-1,s} \), then \( \pi \) is potentially \( Ar_{s,r} \)-graphic.

**Proof.** It follows from the definition of \( \pi''_1 \) that the Proposition holds. \( \square \)

For the sequence \( \pi''_1 \), if \( d^{(1)}_2 \geq \cdots \geq d^{(1)}_r \geq s \), we can define similarly the sequence \( \pi''_2 \) as follows: Define

\[
\pi'_2 = \begin{cases} 
(d^{(1)}_3, \ldots, d^{(1)}_r, d^{(1)}_{r+1} - 1, \ldots, d^{(1)}_{d^{(1)}_r} - 1, d^{(1)}_{r+d^{(1)}_r}, \ldots, d^{(1)}_n) & \text{if } d^{(1)}_2 \leq n - r, \\
(d^{(1)}_3 - 1, \ldots, d^{(1)}_{d^{(1)}_r-r+n-2} - 1, \\
(d^{(1)}_{d^{(1)}_r} - 1, \ldots, d^{(1)}_{d^{(1)}_r-r+n-2}) & \text{if } d^{(1)}_2 > n - r
\end{cases}
\]

and \( \pi''_2 = (d^{(2)}_3, \ldots, d^{(2)}_r, d^{(2)}_{r+1}, \ldots, d^{(2)}_{r+s}, d^{(2)}_{r+s+1}, \ldots, d^{(2)}_n) \), where \( d^{(2)}_3 \geq \cdots \geq d^{(2)}_r \) is the rearrangement of the first \( r - 2 \) terms in \( \pi'_2 \), \( d^{(2)}_{r+1} = d^{(1)}_{r+1} - 1, \ldots, d^{(2)}_{r+s} = d^{(1)}_{r+s} - 1 \), and \( d^{(2)}_{r+s+1} \geq \cdots \geq d^{(2)}_n \) is the rearrangement of the final \( n - r - s \) terms in \( \pi'_2 \). For any \( 3 \leq k \leq r \), if \( d^{(k-1)}_k \geq \cdots \geq d^{(k-1)}_r \geq s \), the definitions of \( \pi'_k \) and \( \pi''_k \) are similar.
Proposition 2.4. Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n \) be a sequence in Proposition 2.3, and \( \pi'' \) can be defined as above. If \( \pi'' \) is graphic, then \( \pi \) is potentially \( A_{r,s} \)-graphic.

Proof. The result follows from the definition of \( \pi'' \). \( \square \)

Lemma 2.5. Let \( \pi = (d_1, \ldots, d_n) \in NS_n \), \( m = \max \{d_1, \ldots, d_n\} \), and \( \sigma(\pi) \) be even. Denote \( \tilde{\pi} = (\tilde{d}_1, \ldots, \tilde{d}_n) \), where \( m = \tilde{d}_1 \geq \cdots \geq \tilde{d}_n \) is the rearrangement of \( d_1, \ldots, d_n \). If there exists an integer \( n_1 \leq n \) such that \( \tilde{d}_{n_1} \geq h+1 \) and \( n_1 \geq \frac{1}{h}(\frac{(m+h+1)^2}{4}) \), then \( \pi \in G_n \).

Proof. It is enough to prove that \( \tilde{\pi} \in G_n \). Since \( m \geq \frac{1}{h}(\frac{(m+h+1)^2}{4}) \geq \frac{(m+h+1)}{h} = m + \frac{1}{h} \), we have \( n_1 \geq m + 1 \). By Theorem 2.1, we only need to verify that (1) holds for \( \tilde{\pi} \). If \( 1 \leq i \leq h \), then \( \sum_{i=1}^{n_1} \tilde{d}_i \leq \sum_{i=1}^{n} d_i \leq m(t(n_1-1)) = mt(n_1-1) = (t-1) + (n_1-t) = t(n_1-t) + \sum_{i=t+1}^{n_1} \min\{\tilde{d}_i, t\} \). Assume \( h < t \leq m \). If \( \sum_{i=1}^{n_1} \tilde{d}_i > t(n_1-t) + \sum_{i=t+1}^{n_1} \min\{\tilde{d}_i, t\} \), then \( tm > t(n_1-t) + h(n_1-t) \), i.e., \( tm > t(n_1-t) + h(n_1-t) \). Hence \( n_1 < \frac{1}{h}(\frac{(m+h+1)^2}{4}) \geq \frac{1}{h}(\frac{(m+h+1)^2}{4}) \), which is impossible. So, (1) holds for \( h < t \leq m \). Moreover, (1) holds clearly for \( m < t < n_1 - 1 \). Thus (1) holds for \( \tilde{\pi} \). \( \square \)

The sequences \( \tilde{\pi} = (\tilde{d}_1, \ldots, \tilde{d}_n) \) is called the rearrangement sequence of \( \pi \).

Lemma 2.6. If \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, \ldots, d_n) \) is potentially \( B_{r,s} \)-graphic, then \( \pi \) is potentially \( A_{r,s} \)-graphic.

Proof. Since \( \pi \) is potentially \( B_{r,s} \)-graphic, we can choose a realization \( H \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \) such that the following conditions are satisfied: (i) \( d_i \) is the degree of \( v_i \) in \( H \) for each \( i \), (ii) \( H \) contains \( K_{r,s} \) as its subgraph, where \( X = \{v_1, \ldots, v_r\} \), \( Y = \{u_{r+1}, \ldots, u_{r+s}\} \subseteq \{v_{r+1}, \ldots, v_n\} \) is the bipartite partition of the vertex set of \( K_{r,s} \), (iii) \( |Z \cap Y| \) is maximum, where \( Z = \{v_{r+1}, \ldots, v_{r+s}\} \). Clearly, if \( |Z \cap Y| = s \), i.e., \( Z = Y \), then \( \pi \) is potentially \( A_{r,s} \)-graphic. If \( |Z \cap Y| < s \), then \( Y - Z \) and \( Z - Y \) are nonempty. Hence there exist \( v \in Z - Y \) and \( u \in Y - Z \) such that \( d(v) \geq d(u) \). By the choice of \( H \) and \( Y \), we have \( X \notin N(v) \), where \( N(v) \) is the set of neighbors of \( v \) in \( H \). Now assume that \( X \cap N(v) = \{v_1, \ldots, v_t\} \), where \( 0 \leq t \leq r - 1 \). Since \( d(v) \geq d(u) \), \( |N(v) - (X \cup \{u\})| \geq |N(u) - (X \cup \{v\})| + r - t \). Hence, there exist \( w_1, \ldots, w_r \in N(v) - (X \cup \{u\}) \) such that \( w_i \notin E(H) \) for \( i = 1, \ldots, r - t \). Then \( H' = H - \{v_i : i = t + 1, \ldots, r\} + \{w_i : i = 1, \ldots, r - t\} - \{w_i : i = 1, \ldots, r - t\} + \{v_i : i = t + 1, \ldots, r\} \) realizes \( \pi \), and contains \( K_{r,s} \) as its subgraph, where \( X, Y - \{u\} + \{v\} \) is the bipartite partition of the vertex set of \( K_{r,s} \). Clearly \( |(Y - u + v) \cap Z| > |Z \cap Y| \), contradicts the choice of \( H \) and \( Y \). Thus, \( \pi \) is potentially \( A_{r,s} \)-graphic. \( \square \)

Lemma 2.7. Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, \ldots, d_n) \in G_n \), where \( n \geq r + s + 1 \) and \( d_n \geq r \), and let \( \pi'' = (d'_1, d'_2, \ldots, d'_{n-1}) \) be the residual sequence obtained by laying off \( d_n \) from \( \pi \). Then \( \pi \) is potentially \( B_{r,s} \)-graphic if \( \pi'' \) is potentially \( B_{r,s} \)-graphic.
**Proof.** If there exists \( t \) such that \( r \leq t \leq d_n \) and \( d_i > d_{i+1} \), then \( d'_i = d_i - 1, \ldots, d'_j = d_j - 1 \). Since \( \pi' \) is potentially \( B_{r,s} \)-graphic, \( \pi \) is potentially \( B_{r,s} \)-graphic too. Hence, we assume that \( d_1 \geq \cdots \geq d_r > d_{r+1} = \cdots = d_{d_r} = \cdots = d_{d_{r+k-1}} \geq \cdots \geq d_{d_r+1} \), where \( 0 \leq \ell \leq r - 1 \) and \( h \geq 1 \). If \( h \geq r - \ell \), then \( \pi' \) satisfies \( d'_i = d_i - 1, \ldots, d'_j = d_j - 1, d'_{r+1} = d_{r+1}, \ldots, d'_{r+\ell} = d_{r+\ell} \). Since \( \pi' \) is potentially \( B_{r,s} \)-graphic, there exists a realization \( H \) with vertex set \( \{v'_1, v'_2, \ldots, v'_{d_n - 1}\} \) such that \( d'_i \) is the degree of \( v'_i \) in \( H \) and \( H \) contains \( K_{r,s} \) as its subgraph, where \( \{v'_1, \ldots, v'_{d_n - 1}\}, \{u_{r+1}, \ldots, u_{r+s}\} \subseteq \{v'_{d_n - 1}, \ldots, v'_{d_n - \ell - 1}\} \) is the bipartite partition of the vertex set of \( K_{r,s} \). Hence, \( \pi \) has a realization \( G \) with vertex set \( \{v_1, v_2, \ldots, v_{d_n}\} \) containing \( K_{r,s} \) as its subgraph such that \( d_i \) is the degree of \( v_i \) in \( G \) and \( \{v_1, \ldots, v_r, v_{d_r+1}, \ldots, v_{d_n+\ell - r}\} \) is one part of the bipartite partition of the vertex set of \( K_{r,s} \). Since \( d'_{r+1} = \cdots = d'_{d_r} = d_{r+1} = \cdots = d_{d_{r+k-1}} \), \( \pi \) is still nonincreasing after interchanging \( d_{r+i} \) with \( d_{d_r+i} \) for \( i = 1, \ldots, r - \ell \). Thus, \( \pi \) is potentially \( B_{r,s} \)-graphic.

**Lemma 2.8.** Let \( r, s \geq 2 \), and \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, \ldots, d_n) \in G_n \), where \( n - r - 1 \geq d_1 \geq \cdots \geq d_{r-1} \geq d_r = d_{r+1} = \cdots = d_{d_r} = d_{r+1} = \cdots = d_{d_r + 1} \geq d_{d_r + 2} \geq \cdots \geq d_n \geq r \geq 1 \) and \( d_{r+s} = r + s - 1 \). Then \( \pi \) is potentially \( A_{r,s} \)-graphic.

**Proof.** By Proposition 2.4, we only need to prove that the type 1 criteria sequences \( \pi'' \) is graphic. It is easy to see from the definition of \( \pi'' \) that \( \sigma(\pi'') \) is even, \( d^{(r)}_{r+i} = \cdots = d^{(r)}_{r+i} = s - 1 \) and \( s \leq d^{(r)}_{r+i} \leq r + s - 1 \). Denote \( t_i = \max \{t \in \{1, \ldots, n\} : d_{r+i} \geq r + s - 1 - i\} \) for \( 0 \leq i \leq r \).

First, we prove the claim: For \( 1 \leq k \leq r \), if \( d^{(k)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\), then \( d^{(k)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\). Use induction on \( k \). If \( k = 1 \), then by \( d_1 \geq \cdots \geq d_r = d_{r+1} = \cdots = d_{d_r} = d_{r+1} = \cdots = d_{d_r + 1} \), we have \( d^{(1)}_{r+i} = r + s - 1 - i \) for \( 1 \leq i \leq r \). Now assume the claim holds for \( k - 1 \). We will prove it holds for \( k \). Since \( d^{(k-1)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\), we have \( d^{(k-1)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\). Then by the induction hypothesis, \( d^{(k)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\). Hence \( d^{(k)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq k - 1)\). Thus, the claim holds for \( k \).

Now suppose that \( d^{(r)}_{r+i} = r + s - i - 1 \) \((0 \leq i \leq r - 1)\). Denote \( k_0 = \lceil \frac{r + s - 1}{r} \rceil \). If \( d_{r+i+k_0} \leq r + s - i - 3 \) \((0 \leq i \leq r - 1)\), then by \( d_{r+i+k_0} = d_{r+i+k_0} = \cdots = d_{r+i+k_0} = d_{r+i+k_0} = \cdots = d_{r+i+k_0} \), we have \( s + k_0 + 1 \geq d_{i} \geq d_{i-3} \). Moreover, for \( t \geq k_0 + 1 \), \( r + s - i - 3 \geq d_{r+i+k_0} = d^{(1)}_{r+i+k_0} = d^{(2)}_{r+i+k_0} = \cdots = d^{(r)}_{r+i+k_0} \), \((r + s - 1 - (r + s - i - 2)) \geq i \geq 0 \) contradicts. Hence \( d_{r+i+k_0} = \cdots = d_{r+i+k_0} > r + s - i - 2 \). Then \( t_{i+1} \geq k_0 + 1 \). By the claim of \( d^{(r)}_{r+i+k_0} = r + s - i - 2 \) \((0 \leq i \leq r - 1)\).

If \( r = 2 \), then \( \pi'' \) satisfies \( d^{(2)}_0 = \cdots = d^{(2)}_{s+1} = s - 1 \) and \( s \leq d^{(2)}_{s+1} = s + 1 \). Assume \( d^{(2)}_{s+1} = s + 1 \). Then \( i = 0, x_0 = 0 \geq 3 \) and \( d^{(2)}_{s+3} \geq s \). If \( s = 2 \), then it is easy to verify
that $\pi''_2$ is graphic. If $s \geq 3$, then the rearrangement sequence $\pi''_3 = (d_{2}^{(2)}, \ldots, d_{n}^{(2)})$ of $\pi''_3$ satisfies $d_{2}^{(2)} = s + 1, d_{2+s+1}^{(2)} \geq s - 1$ and $s + 3 \geq \frac{1}{4}((s+3-1+1)^2 - 1)$. By Lemma 2.5, $\pi''_3$ is graphic. Now assume $d_{2+s}^{(2)} = s$. Then $i = 1, x_0 + 1 \geq 2$ and $d_{2+s+i}^{(2)} \geq s - 1$. Clearly, $\pi''_3$ satisfies $d_{3}^{(2)} = s, d_{2+s+i}^{(2)} \geq s - 1$ and $s + 2 \geq \frac{(s+2-1+1)^2}{4(s-1)}$. By Lemma 2.5, $\pi''_3$ is also graphic. If $r = 3$, then by using the similar way, we can prove that $\pi''_3$ is graphic. Now assume $r \geq 4$. If $i = r - 1$ or $i = r - 2$, similar to above discussion, $\pi''_r$ is graphic. If $0 \leq i \leq r - 3$, then $r + s - 1 - i = d_{r+i}^{(r)} \geq \cdots \geq d_{r+s+i}^{(r)} \geq r + s - i \geq 2 \geq s$ and $x_0 \geq \frac{(r-1)^2}{r+1}$. If $s \leq r - 2$, then $(i + 1)[\frac{(r+s-1-i+r+s-2-i)^2}{4(r+s-2-i)} - 1] \leq (r + s - i)(i + 1) \leq (r - 2)(r + 1) \leq r(r - 1)$, i.e., $x_0 + 1 \leq \frac{(r-1)^2}{r+1} \leq \frac{(r+s-1-i+r+s-2-i)^2}{4(r+s-2-i)}$. By Lemma 2.5, $\pi''_r$ is graphic. If $s \geq r - 1$, then $(i + 1)[\frac{(r+s-1-i+r+s-2-i)^2}{4(r+s-2-i)} - 1] = \frac{(r+s)^2}{4(r-1)} + (i+1)(r-i-1) \leq \frac{(r+2)^2}{2(r-2)} + \frac{r^2}{4} \leq r(r-1)$, i.e., $\frac{(r-1)^2}{r+1} + s \leq \frac{(r+s-1-i+r+s-2-i)^2}{4(r+s-2-i)}$. By Lemma 2.5, $\pi''_r$ is also graphic. □

The following Theorems are our main results in this section.

**Theorem 2.9.** Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, \ldots, d_n) \in G_n$, where $d_{r+s} \geq r + s - 1$ and $d_n \geq r$. Then $\pi$ is potentially $B_{r,s}$-graphic.

**Proof.** Use induction on $r + s$. If $r = 1$ (resp. $s = 1$), then $d_1 \geq s$ (resp. $d_{r+1} \geq r$). Since the sequence $\pi'$ obtained by laying off $d_1$ (resp. $d_{r+1}$) from $\pi$ is graphic, $\pi$ is potentially $B_{1,s}$-graphic (resp. $B_{r,1}$-graphic). Now assume the theorem holds for $r + s - 1$ and $r + s \geq 2$. We will prove by using induction on $n$ the theorem holds for $r + s$. If $n = r + s$, then $\pi = ((r + s - 1) + s)$, where $(r + s - 1)$ means that $\pi$ has exactly $r + s$ terms from $r + s - 1$. Clearly, $\pi$ is potentially $B_{r,s}$-graphic. Suppose that the theorem holds for $n - 1 \geq r + s$, and $\pi = (d_1, \ldots, d_n) \in G_n$, where $d_{r+s} \geq r + s - 1$ and $d_n \geq r$. If $d_{r+s} \geq r + s$, then the sequence $\pi'$ obtained by laying off $d_{r+s}$ from $\pi$ satisfies $d_{r+s} \geq r + s - 1$ and $d_{n-r+s} \geq r$. By the induction hypothesis and Lemma 2.7, $\pi'$ and $\pi''_r$ both are potentially $B_{r,s}$-graphic. If $d_{r+s} = r + s - 1$ and $d_n \geq r + s$, then the sequence $\pi'$ obtained by laying off $d_{r+s}$ from $\pi$ satisfies $d_{r+s} = r + s - 1$ and $d_n \geq r + s - 2$ and $d_{r+s-1} \geq r$. By the induction hypothesis, $\pi'$ is potentially $B_{r,s-1}$-graphic, hence $\pi$ is potentially $B_{r,s}$-graphic.

Now assume that $d_1 = \cdots = d_{r+s} = r + s - 1$. If $d_1 \geq n - r$, or there exists $t \in \{r+s, \ldots, r+d_1\}$ such that $d_t > d_{t+1}$, then $\pi''_3$ satisfies $n - 2 \geq d_{d_1}^{(1)} \geq \cdots \geq d_{d_1}^{(1)} \geq d_{d_1}^{(1)} = \cdots = d_{d_1}^{(1)} = d_{r+s-1}^{(1)} \geq d_{r+s-2}^{(1)} \geq \cdots \geq d_{n}^{(1)} \geq r - 1$, where $d_{d_1}^{(1)} = r + s - 2$. It is easy to verify by Theorem 2.1 that $\pi''_3 \in G_{n-1}$. Hence by the induction hypothesis and Lemma 2.6, $\pi''_3$ is potentially $A_{r-1,s}$-graphic. Thus, $\pi$ is potentially $A_{r,s}$-graphic.

Finally, we further assume that $n - r - 1 \geq d_1 \geq \cdots \geq d_r = d_{r+1} = \cdots = d_{r+s} = r + s - 1$. By Lemma 2.8, $\pi$ is potentially $A_{r,s}$-graphic. □

The proofs of the following Theorems 2.10 and 2.11 involve applying Theorems 2.9, 2.1 and induction on $r + s$, and verifying $\pi''$ being graphic, the details are technical and lengthy, and are omitted here.
Theorem 2.10. Let $\pi=(d_1, \ldots, d_r, d_{r+1}, \ldots, d_n) \in G_n$, where $d_r \geq r + s - 1$, $d_{r+s} \leq r + s - 2$ and $d_n \geq r$. If $n \geq (r+2)(s-1)$, then $\pi$ is potentially $B_{r,s}$-graphic.

Theorem 2.11. Let $s \geq r \geq 3$, $n \geq (r+s)^2/4 + (r+s)/2$, and $\pi=(d_1, \ldots, d_r, d_{r+1}, \ldots, d_r+\cdots+d_n) \in G_n$, where $d_r \leq r+s-2$ and $d_n \geq r$. If there exists $t \in \{1,2,\ldots,\left\lceil r/2 \right\rceil - 1\}$ such that $d_{r+t} \geq r+s-1-t$ and $d_{r+t} \geq r+t$, then $\pi$ is potentially $K_{r,s}$-graphic.

3. The numbers $\sigma(K_{r,r}, n)$ for $r=3,4$

Lemma 3.1. Let $\pi=((n-1)^{-1}, (2r-2)^{2}, d_{r+1}, \ldots, d_n) \in G_n$, where $n \geq 2r$, $d_n \geq r$. Denote $\pi^*=(d_{r+2}, \ldots, d_n)=(d_{r+2} - r - 1, d_{r+3} - r - 1, \ldots, d_{r+t} - r - 1, d_{r+t+1} - r + 1, \ldots, d_n - r + 1)$. If $\pi^*$ is graphic, then $\pi$ is potentially $K_{r,r}$-graphic.

Proof. Suppose that $H$ is a realization of $\pi^*$ with vertex set $V(H) = \{v_{r+2}, v_{r+3}, \ldots, v_{r+t}\}$, where the degree of $v_{r+i}$ is $d_{r+i}^*$. Adding vertices $v_1, v_2, \ldots, v_{r+1}$ to $H$ in the following way: for any $i=1,2,\ldots,r-1$, join $v_i$ to $v_j$, $j=i+1,\ldots,n$, and for $i=r, r+1$, join $v_i$ to $v_j$, $j=r+2,\ldots,2r$. We obtain a realization $G$ of $\pi$. Clearly, the subgraph of $G$ induced by $\{v_1, v_2, \ldots, v_{2r}\}$ contains a $r \times r$ bipartite complete graph $K_{r,r}$. □

The sequence $\pi^*$ is called the type 2 criteria sequence of $\pi$.

Theorem 3.2. Let $n \geq 6$. Then $\sigma(K_{3,3}, n) \geq \begin{cases} 5n - 3 & \text{if } n \text{ is odd,} \\ 5n - 4 & \text{if } n \text{ is even.} \end{cases}$

Proof. If $n$ is odd, then $\pi=(((n-1)^2, 4^3, 3^{n-5})$ is clearly graphic and is not potentially $K_{3,3}$-graphic. Thus $\sigma(K_{3,3}, n) \geq \sigma(\pi) + 2 = 5n - 3$.

If $n$ is even, then $\pi=(((n-1)^2, 4^3, 3^{n-6}, 2^1)$ is also graphic, but not potentially $K_{3,3}$-graphic. Thus $\sigma(K_{3,3}, n) \geq \sigma(\pi) + 2 = 5n - 4$. □

Lemma 3.3. Let $n \geq 10$ and $\pi=(d_1, d_2, \ldots, d_n) \in G_n$, where $d_3 \geq 5$ and $d_n \geq 3$. Then $\pi$ is potentially $K_{3,3}$-graphic.

Proof. The lemma follows from Theorems 2.10 and 2.11. □

Lemma 3.4. Let $n \geq 9$ and $\pi=(d_1, d_2, \ldots, d_n) \in G_n$. If $d_3=4$, $d_n \geq 3$ and $\sigma(\pi) \geq 5n - 4$, then $\pi$ is potentially $K_{3,3}$-graphic.

Proof. It is easy to see that $\pi=(d_1, d_2, 4^4, d_7, \ldots, d_n)$. If $d_1=d_2=n-1$, then the type 2 criteria sequence $\pi^*$ of $\pi$ is $(0^2, d_3 - 2, \ldots, d_n - 2)$, where $2 \geq d_7 \geq 3 \geq \cdots \geq d_n - 2 \geq 1$. In addition, $\sigma(\pi^*)$ is even. So $\pi^*$ is graphic. By Lemma 3.1, $\pi$ is potentially $K_{3,3}$-graphic. If $d_2 \geq n-2$, then $d_1=4$. Since $d_1 + d_2 \geq 5n - 4 - 4(n-2) = n+4$, the type 1 criteria sequence $\pi^*_1$ of $\pi$ satisfies $d_3^1 = d_3, d_4^1 = d_4, \ldots, d_5^1 = d_5, d_6^1 = d_6, l=d_7^1 = \cdots = d_n^1 = l-1$, where $2 \leq l \leq 3$. Moreover, $\sigma(\pi^*_1)$ is even. Hence $\pi^*_1$ is graphic. Thus by Proposition 2.4, $\pi$ is potentially $K_{3,3}$-graphic. □
Theorem 3.5. \( \sigma(K_{3,3}, 6) = 26, \ \sigma(K_{3,3}, 7) = 34, \ \sigma(K_{3,3}, 8) = 40, \ \sigma(K_{3,3}, 9) = 44 \) and \( \sigma(K_{3,3}, 10) = 48. \)

Proof. By Theorem 3.2, \( \sigma(K_{3,3}, 6) \geq 26. \) Clearly, the sequences \( \pi_1 = (6^2, 4^5), \ \pi_2 = (7^2, 4^6), \ \pi_3 = (8^3, 4^5, 2^1) \) and \( \pi_4 = (9^2, 4^6, 2^3) \) are graphic, but not potentially \( K_{3,3} \)-graphic. Thus \( \sigma(K_{3,3}, 7) \geq \sigma(\pi_1) + 2 = 34, \ \sigma(K_{3,3}, 8) \geq \sigma(\pi_2) + 2 = 40, \ \sigma(K_{3,3}, 9) \geq \sigma(\pi_3) + 2 = 44 \) and \( \sigma(K_{3,3}, 10) \geq \sigma(\pi_4) + 2 = 48. \)

Suppose that \( \pi = (d_1, d_2, \ldots, d_n) \) is graphic and \( \sigma(\pi) \geq 26. \) If \( G \) realizes \( \pi, \) then \( 13 \leq e(G) \leq 15. \) Hence \( G \) is obtained from \( K_6 \) by deleting at most two edges. Clearly, \( G \) contains \( K_{3,3}. \) Hence \( \sigma(K_{3,3}, 6) \leq 26. \) Similarly, we can prove that \( \sigma(K_{3,3}, 7) \leq 34, \ \sigma(K_{3,3}, 8) \leq 40, \ \sigma(K_{3,3}, 9) \leq 44 \) and \( \sigma(K_{3,3}, 10) \leq 48. \) The proof is completed. \( \Box \)

Theorem 3.6. If \( n \geq 11, \) then \( \sigma(K_{3,3}, n) = \begin{cases} 5n - 3 & \text{if } n \text{ is odd}, \\ 5n - 4 & \text{if } n \text{ is even}. \end{cases} \)

Proof. By Theorem 3.2, it is enough to prove that, if \( n \geq 11, \) and a graphic sequence \( \pi = (d_1, d_2, \ldots, d_n) \) satisfies

\[
\sigma(\pi) \geq \begin{cases} 5n - 3 & \text{if } n = 2m + 1, \\ 5n - 4 & \text{if } n = 2m + 2, \end{cases}
\]

then \( \pi \) is potentially \( K_{3,3} \)-graphic. Apply induction on \( m \geq 5. \) Assume that \( m = 5 \) and \( n = 2m + 1 = 11, \) then \( \pi = (d_1, d_2, \ldots, d_{11}) \) satisfies \( \sigma(\pi) \geq 5 \times 11 - 3 = 52. \) If \( d_{11} \leq 2, \) then the sequence \( \pi' \) of \( \pi \) obtained by laying off \( d_{11} \) from \( \pi \) satisfies \( \sigma(\pi') = \sigma(\pi) - 2d_{11} \geq 52 - 4 = 48. \) Since \( \sigma(K_{3,3}, 10) = 48, \) \( \pi' \) and \( \pi \) both are potentially \( K_{3,3} \)-graphic. So we can assume that \( d_{11} \geq 3. \) Then by Lemmas 3.3 and 3.4, \( \pi \) is potentially \( K_{3,3} \)-graphic. Now assume that \( m = 5 \) and \( n = 2m + 2 = 12. \) Similarly, we can prove that \( \pi \) is also potentially \( K_{3,3} \)-graphic. Thus the claim holds for \( m = 5. \) Now assume that the claim holds for \( m - 1 \geq 5. \) We will prove that the claim holds for \( m. \) Assume that \( n = 2m + 1 \) and \( \sigma(\pi) \geq 5n - 3. \) If \( d_n \leq 3, \) then for the sequence \( \pi' \) obtained by laying off \( d_n \) from \( \pi, \) we have \( \sigma(\pi') \geq \sigma(\pi) - 2d_n \geq 5n - 3 - 6 = 5(n - 1) - 4, \) where \( n - 1 = 2(m - 1) + 2. \) By the induction hypothesis, \( \pi' \) and \( \pi \) are potentially \( K_{3,3} \)-graphic. If \( d_n \geq 4, \) then by Lemmas 3.3 and 3.4, \( \pi \) is potentially \( K_{3,3} \)-graphic. Now assume that \( n = 2m + 2 \) and \( \sigma(\pi) \geq 5n - 4. \) Similarly, we can prove that \( \pi \) is also potentially \( K_{3,3} \)-graphic. This shows that the claim holds for \( m \geq 5. \) \( \Box \)

Theorem 3.7. Let \( n \geq 8. \) Then \( \sigma(K_{4,4}, n) \geq 8n - 16. \)

Proof. It is easy to verify that \( \pi = ((n - 1)^3, 6^1, 5^{n-5}, 4^1) \) is graphic, but not potentially \( K_{4,4} \)-graphic. Thus \( \sigma(K_{4,4}, n) \geq \sigma(\pi) + 2 = 8n - 16. \) \( \Box \)

Lemma 3.8. Let \( n \geq 8 \) and \( \pi = (d_1, d_2, \ldots, d_n) \in G_n, \) where \( d_4 \geq 7 \) and \( d_n \geq 5. \) Then \( \pi \) is potentially \( K_{4,4} \)-graphic.

Proof. If \( d_1 = d_2 = d_3 = n - 1, \) clearly \( \pi \) is potentially \( K_{4,4} \)-graphic. So we may assume that \( d_3 \leq n - 2. \) Use induction on \( n. \) If \( n = 8, \) then \( d_1 = d_2 = d_3 = d_4 = 7. \) Obviously, \( \pi \) is
potentially $K_{4,4}$-graphic. Now suppose that the lemma holds for $n-1 (n \geq 9)$, and that $\pi = (d_1, \ldots, d_n)$ is graphic, where $d_4 \geq 7$ and $d_n \geq 5$.

If $d_4 \geq 8$ and $d_5 \geq 6$, then the sequence $\pi'$ obtained by laying off $d_4$ from $\pi$ satisfies $d'_4 \geq 7$ and $d'_n \geq 5$. By the induction hypothesis, $\pi'$ and $\pi$ are potentially $K_{4,4}$-graphic.

If $d_4 \geq 8$ and $d_5 = \cdots = d_n = 5$, then the type 1 criteria sequence $\pi''_1$ of $\pi$ satisfies $d''_5 = \cdots = d''_8 = 1$ and $l = d''_9 \geq \cdots \geq d''_n \geq l-1$, where $1 \leq l \leq 5$. If $1 \leq l \leq 2$, clearly $\pi''_1$ is graphic. If $3 \leq l \leq 5$, then the rearrangement sequence $\pi''_1^{\overline{l}}$ of $\pi''_1$ satisfies $d''_5^{\overline{l}} = l$ and $d''_n^{\overline{l}} = 1$.

It is easy to see that, if $l = 5$, then $n - 4 \geq 20 \geq (5 + 1 + 1)^2/4$; if $l = 4$, then $n - 4 \geq 9 \geq (4 + 1 + 1)^2/4$; if $l = 3$, then $n - 4 \geq 7 \geq (3 + 1 + 1)^2/4$.

By Lemma 2.5, $\pi''_4$ is also graphic. Thus $\pi$ is potentially $K_{4,4}$-graphic. Now suppose $d_4 = 7$. If $d_5 = 7$, then $\pi''_4$ satisfies $d''_5 = \cdots = d''_8 = 3$ and $l = d''_9 \geq \cdots \geq d''_n \geq l-2$, where $3 \leq l \leq 7$.

If $3 \leq l \leq 4$, obviously $\pi''_4$ is graphic. If $5 \leq l \leq 7$, then $\pi''_4^{\overline{l}}$ satisfies $d''_5^{\overline{l}} = l$ and $d''_n^{\overline{l}} = 3$. If $l = 7$, then $n - 4 \geq 16 \geq (7 + 3 + 1)^2/4 \times 3$. If $l = 6$, then $n - 4 \geq 10 \geq (6 + 3 + 1)^2/4 \times 3$. If $l = 5$, then $n - 4 \geq 7 \geq (5 + 3 + 1)^2/4 \times 3$. By Lemma 2.5, $\pi''_4^{\overline{l}}$ is graphic. Hence $\pi$ is potentially $K_{4,4}$-graphic. Similarly, we can prove that $\pi$ is potentially $K_{4,4}$-graphic if $5 \leq d_9 \leq 6$. □

**Lemma 3.9.** Let $n \geq 8$, $\pi = (d_1, \ldots, d_n) \in G_n$, where $d_4 = 6$, $d_n \geq 5$ and $\sigma(\pi) \geq 8n - 16$. If $\pi \neq (9^3, 6^2, 5^3)$ and $(8^3, 6^2, 5^4)$, then $\pi$ is potentially $K_{4,4}$-graphic.

**Proof.** It is easy to determine by $\sigma(\pi) \geq 8n - 16$ that $\pi = (d_1, d_2, d_3, 6^2, d_6, \ldots, d_n)$. If $n = 8$, then $\pi = (7^3, 6^4, 5^1; 7^3, 6^2, 5^3; 6^6; 7^1, 6^6, 5^1; 7^2, 6^4, 5^2)$ or $(7^2, 6^3)$. Since the type 1 criteria sequences of $(6^6; 7^1, 6^6, 5^1; 7^2, 6^4, 5^2)$ are graphic sequences $(2^4), (2^4), (2^2, 1^2)$ and $(2^4)$, respectively, and the type 2 criteria sequences of $\pi = (7^3, 6^4, 5^1)$ and $(7^3, 6^2, 5^3)$ are graphic sequences $(1^2, 0^1, 2^2)$ or $(1^3, 3^1, 2^1)$. Hence by Proposition 2.4 and Lemma 3.1, $\pi$ is potentially $K_{4,4}$-graphic. Now assume $n \geq 9$. Consider the following cases:

**Case 1:** $d_1 = d_2 = d_3 = n - 1$. If $n \geq 11$, then the type 2 criteria sequence $\pi^*$ satisfies $1 \geq d_6 - 5 \geq d_7 - 5 \geq d_8 - 5 \geq 0$ and $d_9 = \cdots = d_n = 3 = 2$ if $d_9 = 5$, or $d_6 - 5 = d_7 - 5 = d_8 - 5 = 1$ and $3 \geq d_3 - 3 \geq \cdots \geq d_n = 3 \geq 2$ if $d_9 = 6$. By $\sigma(\pi^*)$ is even, $\pi^*$ is graphic.

If $n = 10$, then $\pi = (9^3, 6^4, 5^1)$ or $(9^3, 6^6, 5^1)$, and the type 2 criteria sequence of $\pi$ is graphic sequence $(1^2, 0^1, 2^2)$ or $(1^3, 3^1, 2^1)$. Hence by Proposition 2.4, $\pi$ is potentially $K_{4,4}$-graphic. Clearly the type 2 criteria sequences of $\pi$ is also graphic sequences $(1^2, 0^1, 2^1)$ or $(1^3, 3^1)$.

**Case 2:** $d_1 \leq n - 2$. Since $d_1 + d_2 + d_3 + 6 \geq 8n - 16 - 6(n - 4) = 2n + 8$, i.e., $d_1 + d_2 + d_3 + 6 \geq 2n - 8 = (n - 8) + 8$, the type 1 criteria sequence $\pi''_4$ of $\pi$ satisfies $d''_5 = \cdots = d''_8 = 2$ and $4 \geq d''_9 \geq \cdots \geq d''_n \geq 3$ when $d_9 = 6$, or $2 \geq d''_9 \geq \cdots \geq d''_n \geq 1$ and $3 \geq d''_9 \geq \cdots \geq d''_n \geq 2$ when $d_9 = 5$. It is easy to follow from $\sigma(\pi''_4)$ being even that $\pi''_4$ is graphic. Hence by Proposition 2.4, $\pi$ is potentially $K_{4,4}$-graphic. □

**Theorem 3.10.** $\sigma(K_{4,4}, 8) = 50$, $\sigma(K_{4,4}, 9) = 58$, $\sigma(K_{4,4}, 10) = 66$, and $\sigma(K_{4,4}, n) = 8n - 16$ for $n \geq 11$. 

Proof. Firstly, it is easy to verify that the sequences $\pi=(7^3, 6^4, 3^1)$, $\pi=(8^3, 6^4, 4^2)$ and $\pi=(9^3, 6^2, 5^3)$ are graphic, but not potentially $K_{4,4}$-graphic. Hence, $\sigma(K_{4,4}, 8) \geq 50$, $\sigma(K_{4,4}, 9) \geq 58$ and $\sigma(K_{4,4}, 10) \geq 66$.

Next, assume that $\pi=(d_1, \ldots, d_n)$ is graphic, $\sigma(\pi) \geq 50$ and $G$ realizes $\pi$. Then $25 \leq e(G) \leq 28$. Hence $G$ is obtained by deleting at most three edges from $K_8$. Clearly $G$ contains $K_{4,4}$ as its subgraph. Hence $\sigma(K_{4,4}, 8) \leq 50$. Similarly, we can prove that $\sigma(K_{4,4}, 9) \leq 58$ and $\sigma(K_{4,4}, 10) \leq 66$.

If $n \geq 11$, then by Theorem 3.7, $\sigma(K_{4,4}, n) \geq 8n-16$. Now assume that $\pi=(d_1, \ldots, d_n)$ is graphic and $\sigma(\pi) \geq 8n-16$. We use induction on $n \geq 11$ to prove that ($\ast$): $\pi$ is potentially $K_{4,4}$-graphic. When $n=11$, $\sigma(\pi) \geq 72$. If $d_{11} \geq 5$, then by Lemmas 3.8 and 3.9, $\pi$ is potentially $K_{4,4}$-graphic. If $d_{11} \leq 3$, or $d_{11}=4$ and $\sigma(\pi) \geq 74$, then the sequence $\pi'$ obtained by laying off $d_{11}$ from $\pi$ satisfies $\sigma(\pi')=\sigma(\pi)-2d_{11} \geq 66=\sigma(K_{4,4}, 10)$. Hence, $\pi'$ and $\pi$ are potentially $K_{4,4}$-graphic. Now suppose $d_{11}=4$ and $\sigma(\pi)=72$. Since $\pi$ is graphic, by Theorem 2.1, we have $d_7 \geq 5$. If $d_{10} \geq 5$, then the sequence $\pi'$ obtained by laying off $d_{11}$ from $\pi$ satisfies $\sigma(\pi') \geq 64$. If $\pi' \neq (9^3, 6^2, 5^3)$, then by Lemmas 3.8 and 3.9, $\pi'$ and $\pi$ are potentially $K_{4,4}$-graphic. If $\pi'=(9^3, 6^2, 5^3)$, then $\pi=(10^3, 7^3, 6^2, 5^3)$ or $(10^3, 6^5, 4^4)$. If $\pi=(10^3, 7^3, 6^2, 5^3)$, clearly $\pi$ is potentially $K_{4,4}$-graphic because $d_1=d_2=d_3=10$ and $d_4=7$. If $\pi=(10^3, 6^5, 4^4)$, then the type 2 criteria sequence of $\pi$ is graphic sequence $(2^2, 1^2, 0^2)$. By Lemma 3.1, $\pi$ is also potentially $K_{4,4}$-graphic. Thus we may further assume that $d_{10}=4$. By a similar way, we can prove that $\pi$ is also potentially $K_{4,4}$-graphic.

Now suppose that ($\ast$) holds for $n-1 \geq 11$. We will prove that ($\ast$) holds for $n$. If $d_n \geq 5$, then by Lemmas 3.8 and 3.9, $\pi$ is potentially $K_{4,4}$-graphic. If $d_n \leq 4$, then the sequence $\pi'$ obtained by laying off $d_n$ from $\pi$ satisfies $\sigma(\pi') \geq \sigma(\pi)-8 \geq 8(n-1)-16$. By the induction hypothesis, $\pi'$ and $\pi$ both are potentially $K_{4,4}$-graphic. \[\square\]

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References

Combinatorics, Graph Theory, and Algorithms, Vol. 1, New Issues Press, Kalamazoo, Michigan, 1999,
pp. 387–400.
[10] D.J. Kleitman, D.L. Wang, Algorithm for constructing graphs and digraphs with given valences and
factors, Discrete Math. 6 (1973) 79–88.
[12] Jiong-Sheng Li, Zi-Xia Song, The smallest degree sum that yields potentially $P_k$-graphic sequences,
[13] Jiong-Sheng Li, Zi-Xia Song, An extremal problem on the potentially $P_k$-graphic sequence, Discrete
[15] Jiong-Sheng Li, Jian-Hua Yin, The threshold for the Erdős, Jacobson and Lehel conjecture being true,
submitted for publication.