The $C$-Spectral Sequence, Lagrangian Formalism, and Conservation Laws. I. The Linear Theory

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INTRODUCTION

The question of finding conservation laws for nonlinear differential equations is an old but actual problem. Up to the present time the only known general method for doing this is the one given by the classical Noether theorem (for a special class of evolution equations—those which can be represented in the form of $L$-$A$ pairs—a different mechanism for constructing conservation laws was found fairly recently).

Since the equations serviced by the Noether theorem are Euler-Lagrange equations, this stimulated the study of the "inverse problem of the calculus of variations," which asks if a given equation can be represented in the form of an Euler-Lagrange equation.

Conservation laws are represented by the appropriate "conserved densities" much in the same way as cohomology classes are represented by cocycles. In this connection the following problem arises: describe the trivial densities, i.e., those to which trivial conservation laws correspond. For this problem, as far as is known to the author, there are also no really general results of any sort. Moreover, one can indicate papers in which "infinite" series of conservation laws are described by their densities, which are actually trivial! A very similar, but simpler question is the following: Is a given Lagrangian density trivial, i.e., does it give rise to a trivial Euler-Lagrange equation?

In this paper we give a solution of the problems mentioned above, as well
as that of some related problems obtained by the author in 1977 and announced in [1; 2, 3, Sect. 6].\(^1\) The starting point of the present study was the desire, motivated by certain "question of principle" of current mathematical and theoretical physics, to understand the Lagrangian formalism with, as well as without, constraints from the point of view of the category \(\text{DE}\) of nonlinear partial differential equations (see Section 6). It turned out that all these questions are different aspects of the \(\mathcal{E}\)-spectral sequence mentioned in the title: this was indeed the fact which made possible their (in certain sense) complete solution.

The \(\mathcal{E}\)-spectral sequence and the rich and interesting homological algebra which surrounds it and contains, for example, the Spencer cohomology systems and their diverse generalizations, is a fragment of a new mathematical theory, which may be called the algebraic topology of differential equations and which contains, as will be shown elsewhere, as its "zero-dimensional" particular case, the algebraic topology of smooth manifolds. The term \(E_1\) of the \(\mathcal{E}\)-spectral sequence is the analogue of the de Rham complex in the category \(\text{DE}\) and this predetermines its importance. In particular, it has now become clear that such branches of mathematics as the theory of characteristic classes (including secondary ones), topological Bott obstructions, the Gelfand–Fuks cohomology, characteristic classes of deformations, etc., are merely different aspects of the theory of \(\mathcal{E}\)-spectral sequences.

The topics considered in this article were studied from various points of view by numerous authors during a period of at least a hundred years and involve a rather large amount of literature. For this reason, our references are limited to relatively recent works which contain ideas and results close to ours. In them the reader may find fairly complete indications concerning earlier work.

All the current work on the "questions of principle" of the Lagrangian formalism are based on the theory of jet spaces. Dedeker was the first to understand the importance of algebraic topology methods and Eresmann’s theory of jets for the calculus of variations. In the early fifties, he carried out an interesting cycle of studies whose results are summarized in [5] (also see [6]) and whose aim was to construct the Hamiltonian formalism for the multi-dimensional calculus of variations. This approach is developed, for instance, in the recent paper [7]. Our point of view, however, is purely Lagrangian, so that the results of these papers are only indirectly related to ours. It is nevertheless interesting to note that Dedeker constructed a certain spectral sequence for the so-called phase spaces of variational problems by using the same general construction which is applied below for developing our \(\mathcal{E}\)-spectral sequence. An approach nearer to ours is carried out in

\(^1\) See reference list at end of Part II, this issue.
Sniatycki's paper [8] and, especially, in the work of Krupka [9, 10]. The one-dimensional calculus of variations from the modern point of view is developed in Hermann's book [11].

In these papers, as well as in the majority of other current work, which is concerned with special questions of the Lagrangian formalism, the study is based on finite order jet spaces in the spirit of modern local differential geometry. An important new step, made by Tulczyjew [24], Kuperschmidt [12], and Takens [18] in 1976, was the passage to jets of infinite order. Among other things, this step turned out to be useful in connection with the theory of higher infinitesimal symmetries constructed in 1975–1976 (see [4, 13, 14]); these symmetries are vector fields precisely on the “manifold” of infinite jets and can be used in the role of “higher” symmetries in the Noether theorem. A new exposition of Kuperschmidt’s work, also using the language of local differential geometry, but in the more algebraic style of differential algebra, is given in Manin's review article [15].

An important advantage of the papers mentioned above is their invariance which, as could be expected, clarifies the logical structure of many aspects of Lagrangian theory. But, on the other hand, it has become clear that invariance in itself is not a guarantee of the necessary functoriality of the theory and thus cannot be used as a method for finding all the essential elements of the structures involved in this theory. Moreover, the invariance obtained in the language of local differential geometry often turns out to be so complicated and tedious that it hampers rather than helps effective computation in concrete cases. Also note that, up to now, there has been no satisfactory formulation of the Lagrangian formalism with constraints.

The introduction of the Lagrangian formalism into the framework of the category DE in the sense of [4] removes, in an appropriate way, the defects noted above; the language of differential calculus in commutative algebras turns out to be extremely convenient in this connection (see [4, 13, 16]).

The detailed contents of this article are the following: In Section 0 we give an exposition of the necessary facts from the theory of differential operators in commutative algebras and fix the notations. Sections 1–5 are purely algebraic and contain a theory which is valid for any smooth algebra of zero characteristic. In Section 1 we construct the algebraic theory of adjoint operators and the corresponding transformation theory. Spencer complexes and the \( \mathcal{S} \)-complexes intimately related to them are defined in Section 2. The general algebraic Green formula turns out to be one of the statements of the relationship between these complexes. The polylinear analogues of \( \mathcal{S} \)-complexes are constructed in Section 3. There we also prove that they are acyclic and use this to construct the linear Lagrangian formalism. The theory of linear conservation laws for the equation \( A = 0 \), where \( A \) is a linear differential operator, is developed in Section 4, where we show that the group \( \mathcal{Z}(A) \) of linear conservation laws of this equation, under certain natural
assumptions, coincides with coker $\Delta^*$. In the general case the group $3(\Delta)$ is isomorphic to the first Spencer homology group of the operator $\Delta$. The section also contains the general construction of nonlinear conservation laws, which, apparently, are rather useful in practice. The linear Noether theorem is developed in Section 5. There, in particular, we show that it is a particular case of the nonlinear construction given in Section 4.

Sections 6–12 (in Part II) are devoted to the nonlinear theory. The main facts and constructions of jet space theory necessary for the sequel are summarised in Section 6. There we also introduce the category $\mathcal{DE}$ and the extremely important class of $\mathcal{E}$-differential operators on objects of the category $\mathcal{DE}$. It is shown in Section 7 that the theory of $\mathcal{A}$-adjoint operators, Spencer complexes, Green’s formula, etc., can be carried over in a natural way to the theory of $\mathcal{E}$-differential operators, which enables us to apply the algebra developed in Sections 1–5 to nonlinear theory. In Section 8 we construct a “naive” Lagrangian formalism in the situation without constraints. Here, in particular, the functorial significance of the notions of Lagrangian and Lagrangian density is clarified. An important part of this section deals with the theory of “transversality conditions,” which, apparently, has not yet been developed in such a general framework. Finally, Section 8 contains the main definitions concerning conservation laws of nonlinear differential equations, the general Noether theorem and its comparison with its classical version. Note that Lagrangians and conservation laws are respectively $n$-dimensional and $(n-1)$-dimensional de Rham $\mathcal{D}$-cohomology classes of objects of the category $\mathcal{DE}$ (see Section 6); this is what actually includes the Lagrangians formalism and the theory of conservation laws into the algebraic topology of the category $\mathcal{DE}$, as we mentioned above. Our version of general Noether theorem resembles a similar result due to Kuperschmidt [12].

The main construction of this paper—the $\mathcal{E}$-spectral sequence $(E_r^{p,q}, d_r^{p,q})$—is described in Section 9. If $\mathcal{A}^*{\mathcal{C}}$ is the appropriately defined de Rham complex for $\mathcal{C} \in \mathcal{E}b\mathcal{DE}$ and $\mathcal{E}\mathcal{A}^*{\mathcal{C}} \subset \mathcal{A}^*{\mathcal{C}}$ is the ideal constituted by the forms which vanish on integral manifolds of the object $\mathcal{C}$, then the $\mathcal{E}$-spectral sequence $E_r^{p,q}(\mathcal{C})$ is the spectral sequence generated by the filtration of the complex $\mathcal{A}^*{\mathcal{C}}$ by powers of the ideal $\mathcal{E}\mathcal{A}^*{\mathcal{C}}$. The significance of the spectral sequence for the group of problems described above is determined by the fact that $E_1^{0,*}(\mathcal{C})$ is the group of all Lagrangians for the variational problems related to $\mathcal{C}$, while the differential $d_1^{0,*}$ is the “Euler operator,” i.e., the operator which assigns to each Lagrangian the corresponding Euler–Lagrange equation ($n$ is the number of independent variables). If $\mathcal{C} = \mathcal{Y}_\infty$ is the infinite prolongation of the differential equation $\mathcal{Y}$, then $E_1^{0,*}(\mathcal{Y}_\infty)$ is the group of Lagrangians for the variational problems in which the constraints are given by the equation $\mathcal{Y}$, while $E_1^{0,n-1}(\mathcal{Y}_\infty)$ is the group of conservation laws for the equation $\mathcal{Y}$. 
The first group of results obtained in Section 9 is based on the explicit description of the $\mathcal{C}$-spectral sequence for "free" objects $\mathcal{O}$ of the category $\mathcal{DE}$, for example, when $\mathcal{O} = J^\infty(\pi)$, where $\pi: E_\pi \to M$ is a submersion. This description yields the global solution of the trivial Lagrangian problem and the inverse problem of the calculus of variations. For example, if $\mathcal{O} = J^\infty(\pi)$, then some operator $V$ is an Euler–Lagrange operator if and only if $d^1_n(V) = 0$ and its cohomology class $\langle V \rangle \in E^1_2 \pi(\mathcal{O}) = H^{n+1}(E_\pi_\ast; R)$ is trivial.

Locally, the problem of trivial Lagrangians was considered by many authors. Its solution for Lagrangians of the first order was obtained, for example, by Krupka [17] and (exhaustively) by Kuperschmidt [12]. Of course, it follows from certain later works. Its global solution was proposed by Takens [18] and the author [2]. Note in this connection that the results of our paper [2] developed here give an effective approach to this problem in the situation with constraint as well (see Section 11). For instance, they yield a complete solution in the case when the constraint equations are linear or more generally, homogeneous with respect to the derivatives which they involve. Apparently these are the first results of this type.

The inverse problem of the calculus of variations is an old one. Numerous particular results concerning it are spread out in oceans of papers (making difficult to review), many of which duplicate each other. Let us mention the works of Douglas [19], Havas [20], Tonti [21, 22], Horndeski [23], Tulczyjew [24], and Kuperschmidt [12] which give its local solution and, on the example of which, the reader may form an opinion on the history of the question and on the manner of quoting “accepted” in this field. An historical review of the topic is given by Santilli [25]. A global solution of this problem was given by Takens [18] and the author [2] (see above). Recently, certain detailisations of these results were made by Anderson and Duchamp [26]. Actually they follow from the methods of the present work.

The inverse problem in the situation with constraints, as far as the author knows, has not even been stated. Amalgamating results of papers [2, 3], we shall indicate a method for its solution which yields, in particular, a definitive answer in the case when the constraint equation is homogeneous with respect to the derivatives (see Section 11). The second group of results obtained in Section 9 is related to the Stokes infinitesimal formula, with which the term $E_1$ of the $\mathcal{C}$-spectral sequence may be supplied. This is a very important fact, since it enables us to compute the term $E_2$ by homotopy methods and, moreover, yields the solutions of specific inverse problems of the calculus of variations and the "potentials" of trivial Lagrangians. In Section 9 we explicitly write out the "resolvent" of the Euler operator, a complicated construction of which was given in [12, 25].

Section 10 is devoted to the study of the $\mathcal{C}$-spectral sequence for $\mathcal{O} = \mathcal{Y}_\infty$, where $\mathcal{Y}$ is a differential equation. To do this, we develop a homological
technique which enables us to compute $E_1(\mathcal{Y}_\infty)$. Using it, we establish the main result of this paper—the “two line theorem” which states that for non-overdetermined equations $\mathcal{Y}$, under certain weak regularity assumptions, the nonzero terms $E^{p,q}_n(\mathcal{Y}_\infty)$ are concentrated in the “lines” $q = n - 1$ and $q = n$ and the “segment” $p = 0$, $0 \leq q \leq n$, where $n$ is the number of independent variables. All the principal results of this paper are obtained by bringing together this theorem and homotopy techniques based on the infinitesimal Stokes formula mentioned above.

Section 11 contains all the applications. Besides the results concerning the trivial Lagrangian problem and the inverse problem of the calculus of variations with constraints, which we mentioned above, it contains the theory of conservation laws, which yields effective computational procedures in many concrete cases. We shall show that for non-overdetermined “regular” equations the differential $d_1^{1,n-1}$ maps the group of conservation laws $E_1^{1,n-1}$ injectively, modulo a certain unessential kernel of purely topological nature, into $E_1^{1,n} = \ker l^*_\circ$, where $\mathcal{Y} = \{ \phi = 0 \}$ and $l_\circ$ is the appropriate universal linearization operator. Since the algebra of higher infinitesimal symmetries of the equation $\mathcal{Y}$ coincides with $\ker l_\circ$, while the Euler–Lagrange equations are self-adjoint, i.e., $l_\circ^* = l_\circ$, the differential sends conservation laws into symmetries in this case. This is the most complete result of the inverse Noether theorem type. Note that it is valid for anti-adjoint equations also.

Knowing the kernel of the operator $l_\circ^*$, we can find the conservation laws contained in it by means of the operator $d_1^{1,n-1}$, which is also effectively described in the section. Note that in this manner we compute the conservation laws directly, and not the conserved densities, thus avoiding the difficult problem of checking the nontriviality of conserved densities from the very start. The above transforms the problem of finding conservation laws for many equations (e.g., for KdV) into a routine procedure.

Another application considered in Section 10 concerns the “generalised Schwartz formula” (see [27–29]), i.e., an $(n - 1)$ form $\theta$ on the space of jets such that for any extremal of the Lagrangian $\mathcal{L} = \int \omega$, we have the relation $\int_{\bar{\gamma}} \omega = \int_{\bar{\partial} \gamma} \theta$, where $\bar{\gamma}$ is the graph of $j_k(\gamma)$ for an appropriate $k$. A formula of this type was found in the Nineteenth century by Schwartz for minimal surfaces; apparently, it considerably simplifies a number of questions in this theory. Dedekke [27–29] believes that the existence of such a formula is a natural generalization of the notion of complete integrability in the case of several independent variables and states a series of interesting results which use his spectral sequence (mentioned above). We notice that the only obstruction to the existence of a “Schwartz formula” lies in $E_2^{0,n}(\mathcal{Y}_\infty)$, where $\mathcal{Y}$ is the corresponding Euler–Lagrange equation, which, for example, enables us to find this formula for Euler–Lagrange equations which are homogeneous with respect to the derivatives.

In Section 12 we briefly indicate certain variants and generalizations of
the $E$-spectral sequence. Here we mention some very interesting remarks
from Tsujishita's preprint [30] which relates the $E$-spectral sequence with
Gelfand-Fuks cohomology, characteristic classes of $G$-structures, Bott
topological obstructions, etc. In the same preprint Tsujishita also gives a new
exposition of some of our results in the so called language of "formal
differential geometry."

This paper is "theoretical" in character. It is mainly concerned with the
description of an apparatus which services the $E$-spectral sequence, and only
obtains its immediate consequences. Its applications (known to us) to the
case of specific equations, as well as the study of certain generalizations,
mentioned in Section 12, will be published separately. Let us note, however,
that the techniques for finding conservation laws described here may be
applied directly to specific equations which arise in geometry and
mathematical physics.

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0. NOTATIONS AND FACTS FROM DIFFERENTIAL CALCULUS IN
COMMUTATIVE ALGEBRAS

0.1. Suppose $M$ is a smooth ($\cong \mathcal{C}^\infty$) manifold, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The $\mathbb{F}$-
algebra of differential $\mathbb{F}$-valued functions on $M$ is denoted by $\mathcal{C}_\mathbb{F}^\infty(M)$
or simply $\mathcal{C}_\mathbb{F}^\infty(M)$. The $\mathcal{C}_\mathbb{F}^\infty(M)$-module of smooth differential forms of degree $i$
on $M$ is denoted by $\mathcal{A}_\mathbb{F}^i(M) = \mathcal{A}_\mathbb{F}^i(M)$, $\mathcal{A}_\mathbb{F}^+(M) = \sum_i \mathcal{A}_\mathbb{F}^i(M)$, $\mathcal{D}(M) = \mathcal{D}_{\mathbb{R}}(M)$
denotes the $\mathcal{C}_\mathbb{F}^\infty(M)$-module of smooth vector fields on $M$ which are identified
with the differentiation of the algebra $\mathcal{C}_\mathbb{F}^\infty(M)$. Further without special
mention we use the standard notation of the calculus of differential forms
and vector fields on manifolds (see, e.g., [31, 32]). One exception however, is
the Lie derivative. Namely, the Lie derivative of some object $\omega$ along the
field $\chi \in \mathcal{D}(M)$ is denoted by $\chi(\omega)$. Throughout the entire paper all
constructions and operations related to smooth manifolds are assumed
smooth, i.e., of class $\mathcal{C}^\infty$.

0.2. Suppose $K$ is a commutative ring with unit and $A$ is a unitary
commutative $K$-algebra. The pair $(a, \Delta)$, $a \in A$, $\Delta \in \text{Hom}_K(P, Q)$, where $P$
and $Q$ are $A$-modules, generates an element $\delta_a(\Delta) \in \text{Hom}_K(P, Q)$: $\delta_a(\Delta)(p) = 
\Delta(ap) - a \Delta(p)$, $p \in P$. The operator $\delta_a : \text{Hom}_K(P, Q) \to \text{Hom}_K(P, Q)$,
$\Delta \mapsto \delta_a(\Delta)$, is a $K$-homomorphism and $\delta_a \circ \delta_b = \delta_b \circ \delta_a$. Put

$$\delta_{a_0, a_1, \ldots, a_s} = \delta_{a_0} \circ \delta_{a_1} \circ \cdots \circ \delta_{a_s}.$$
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DEFINITION. The element $\Delta \in \text{Hom}_K(P, Q)$ is said to be a $K$-differential operator (d.o.) of order $\leq s$ over $A$, if for all $a_0, \ldots, a_s \in A$ we have $\delta_{a_0, \ldots, a_s}(\Delta) = 0$.

Suppose $K = F$, $A = C^c_\xi(M)$, $P = \Gamma(\xi)$, and $Q = \Gamma(\eta)$, where $\xi$, $\eta$ are smooth vector bundles over $M$, and $\Gamma(\alpha)$ denotes the set of all smooth sections of the bundle $\alpha$. This situation will be referred to as the classical one.

PROPOSITION. In the classical situation the "ordinary" notion of differential operator coincides with the one introduced above.

The proof of this proposition as well as more detailed considerations concerning the facts listed in this section may be found in [13, 14].

0.3. The set of all d.o.'s of order $\leq s$, from $P$ to $Q$ may be supplied with a left (resp. right) $A$-module structure, by putting $a\Delta = a \circ \Delta$, where $(a\Delta)(p) = a\Delta(p)$ (resp. $\Delta a = \Delta \circ a$, i.e., $(\Delta a)(p) = \Delta(ap)$). The module which arises in this way is denoted by $\text{Diff}_s(P, Q)$ (resp. $\text{Diff}_s^+(P, Q)$). By $\text{Diff}_s^{++}(P, Q)$ we denote the corresponding bimodule. Obviously $\text{Diff}_s^{++}(P, Q) \subset \text{Diff}_s^{++}(P, Q)$, $s \leq t$, so that we may consider the union

$$\text{Diff}^{++}(P, Q) = \bigcup_{s \geq 0} \text{Diff}_s^{++}(P, Q).$$

If $P$, $Q$, and $R$ are $A$-modules and $\Delta_1 \in \text{Hom}_K(P, Q)$, $\Delta_2 \in \text{Hom}_K(Q, R)$ then $\delta_0(\Delta_2 \circ \Delta_1) = \delta_0(\Delta_2) \circ \Delta_1 + \Delta_2 \circ \delta_0(\Delta_1)$ which implies $\Delta_2 \circ \Delta_1 \in \text{Diff}_s^{++}(P, Q)$, if $\Delta_1 \in \text{Diff}_s(P, Q)$, $\Delta_2 \in \text{Diff}_s(Q, R)$. The operation composition of d.o. transforms $\text{Diff}(P, P)$ into an associative $A$-algebra and allows us to view $\text{Diff}(P, Q)$ as a left $\text{Diff}(Q, Q)$-module and a right $\text{Diff}(P, P)$ module.

Suppose $\text{Diff}_s^{++}(Q) = \text{Diff}_s^{+}(A, Q)$, $\text{Diff}^{++}(Q) = \text{Diff}^{++}(A, Q)$, $\Delta^{(+)} = \Delta^{(++)}(Q) : \text{Diff}^{++}(Q) \to Q$, where $\Delta^{(+)}(\Delta) = \Delta(1)$ and $\Delta^{(++)} = \Delta^{(++)}\text{Diff}^{++}(Q)$. The operator $\Delta_s$ is a homomorphism of $A$-modules, while $\Delta^+_s$ is a d.o. of order $\leq s$ as well as the operators $i_+: \text{Diff}_s(P, Q) \to \text{Diff}_s^+(P, Q)$, $i^+ : \text{Diff}_s^{++}(P, Q) \to \text{Diff}_s(P, Q)$ generated by the identity maps of the corresponding sets.

To every $\Delta \in \text{Diff}_s(P, Q)$ corresponds the homomorphism

$$h_\Delta \in \text{Hom}_A(P, \text{Diff}_s^+(Q)) : h_\Delta(p)(a) = \Delta(ap).$$

PROPOSITION. The correspondences $\Delta \to h_\Delta$ and $h \to \Delta_s \circ h$ are isomorphisms inverse to each other of the $A$-modules $\text{Diff}_s(P, Q)$ and $\text{Hom}_A(P, \text{Diff}_s^+(Q))$, where the superscribed point means that the set of all $A$-homomorphisms from $P$ to $\text{Diff}_s^+(Q)$ is supplied with the $A$-module structure induced by the left $A$-module structure in $\text{Diff}_s^{++}(Q)$. 
Suppose $A \in \text{Diff}_+(P, Q)$, $A^{(s)} = \Delta \circ \Delta^+(P)$ and $h_{A(s)}^s : \text{Diff}_+^s P \to \text{Diff}_+^{s+1} Q$. Passing to the limit when $s \to \infty$ we obtain a homomorphism of $A$-modules $h_A^\infty : \text{Diff}_+^\infty P \to \text{Diff}_+^\infty Q$. The module $\text{Diff}_+(P, Q)$ is a homomorphism of $\text{Diff}_+^\infty P$ which is also a homomorphism of $\text{Diff}_+^\infty A$-modules. If $A = \Delta^+(P)$ the homomorphism $h_A^\infty : \text{Diff}_+^\infty (\text{Diff}_+^\infty P) \to \text{Diff}_+^{\infty+1} P$ is said to be the glueing operator and denoted by $c_{\Delta}$.

Note that $\text{Diff}_+^\infty (P, Q) = \text{Hom}_A(P, Q)$ and $\text{Diff}_+^\infty P = \text{Hom}_A(A, P) = P$. In view of this last representation $A = (A - A(1)) + A(1)$ determines a decomposition of the $A$-module $\text{Diff}_+ P$ into a direct sum $\text{Diff}_+ P = P \oplus \text{Diff}_+^\infty P$, where $\text{Diff}_+ P = \{ A \in \text{Diff}_+ P \mid A(1) = 0 \}$. Similarly $\text{Diff}_+ P = P \oplus \text{Diff}_+^\infty P$.

0.4. The module $D(P) = \text{Diff}_+^1(P)$ consists of all $K$-derivations of the $K$-algebra $A$ into the $A$-module $P$, so that in the classical situation $D(C^\infty(M)) = D(M)$. The function $(a_1, \ldots, a_s) \mapsto \varphi(a_1, \ldots, a_s) \in P$, $a_i \in A$ will be referred to as a skew multiderivation if it is skew symmetric and is a $K$-derivation with respect to each variable. The set of all such multiderivations is an $A$-module denoted by $D_s(P)$. Obviously $D_1(P) = D(P)$. We also put $D_0(P) = P$.

The functional definition of the modules $D_i(P)$ is the following. Suppose $(\text{Diff}_+^i)^{(s)}(P) = \text{Diff}_+^i((\text{Diff}_+^i)^{(s-1)}(P))$ and for any subset $Q_1$ of the $A$-module $Q$ define

$$D(Q_1 \subset Q) = \{ \nabla \in D(P) \mid \text{im } \nabla \subset Q_1 \}.$$  

Then the module $D_i(P)$ are defined by induction by the relation

$$D_i(P) = D(D_{i-1}(P) \subset (\text{Diff}_+^i)^{(i-1)}(P)).$$

Here the inclusion $D_i(P) \subset (\text{Diff}_+^i)^{(i)}(P)$ is not a homomorphism of $A$-modules and is the composition of the following natural inclusion: $D_i(P) = D(D_{i-1}(P)) \subset (\text{Diff}_+^{i-1}(P)) \subset D((\text{Diff}_+^{i-1})^{(i-1)}(P)) \subset (\text{Diff}_+^i)^{(i)}(P)$. In the sequel the symbol $D_i(\text{Diff}_+^i Q)$ will denote the same set of multiderivations as $D_i(\text{Diff}_+^i Q)$, but possessing the $A$-module structure induced by the left module structure in $\text{Diff}_+^i Q$, i.e., if $A \in D_s(\text{Diff}_+^i Q)$, then for $a \in A$ we have $(aA)(a_1, \ldots, a_s) = a \circ A(a_1, \ldots, a_s)$.

0.5. Suppose $\mathcal{M}$ is the category of all $A$-modules and $\Phi = D$, $\text{Diff}_s$, $\text{Diff}_s^+$, $\text{Diff}_s^{(+)}$ etc., or is the composition of such symbols, e.g., $D_i(\text{Diff}_+^i)$. Then the correspondence $P \to \Psi(P)$ defines a functor on the category $\mathcal{M}$ with values in the same category or in the category $\mathcal{M}_A$ of $A$-polymodules (e.g., $\text{Diff}_+^{(+)}P \in \mathcal{C}b.\mathcal{M}_A$). If $\Phi$ is such an $A$-module, that the functor $P \to \text{Hom}_A(\Phi, P)$ is isomorphic to the functor $P \to \Phi(P)$ then it is said to be a representative object for $\Phi$.

Functors $\Phi$ of the type described above are examples of functors of the differential calculus. They all possess representative objects. The represen-
tative object corresponding to the functor $\Phi = D_i$ (resp. $\Phi = \text{Diff}_j$) will be called the module of $i$-dimensional differential form (resp. $s$-jets) over $A$ and denote by $A^i = A^i(A)$ (resp. $\mathcal{F}^s = \mathcal{F}^s(A)$). The representative object for the functor $Q \rightarrow \text{Diff}_j(P, Q)$ will be denoted by $\mathcal{F}^j(P)$ and called the module of $s$-jets of the module $P$.

The operator of order $\leq s$ $j_s = j_s(P) : P \rightarrow \mathcal{F}^s(P)$, corresponding, by the isomorphism $\text{Diff}_j(P, \mathcal{F}^s(P)) = \text{Hom}_A(\mathcal{F}^s(P), \mathcal{F}^s(P))$ to the identical homomorphism $\mathbf{1}_{\mathcal{F}^s(P)}$ will be referred to as the operator of taking $s$-jets. If under the natural isomorphism $\text{Diff}_j(P, Q) = \text{Hom}(\mathcal{F}^s(P), Q)$ the operator $\square$ is mapped into the homomorphism $\varphi : \mathcal{F}^s(P) \rightarrow Q$ (denoted further by $\varphi_\square$) we have $\square = \varphi \circ j_s$. Note that as usual $\mathcal{F}^i = \mathcal{F}^i A \ldots \mathcal{F}^i (i \text{ times})$.

0.6. The natural transformation of functors of the differential calculus by duality defines operators between the corresponding representative objects. Here are three examples needed in the sequel.

1. To the natural inclusion $\text{Diff}_{k-1} \hookrightarrow \text{Diff}_k$ correspond the homomorphisms $v_k, r_{k-1} : \mathcal{F}^{k-1}(P) \rightarrow \mathcal{F}^k(P)$ restricting the order of $k$-jets.

2. The representative object for the functor $D_i D_j$ is $A^j \otimes_A A^i$. Then the natural inclusion $D_{i+j} \hookrightarrow D_i D_j$ induces the homomorphism $A^j \otimes A^i \rightarrow A^{i+j}$ which defines the exterior multiplication of differential forms over $A$.

3. The representative object to the functor $D_j(\text{Diff}_j^+)$ is $\mathcal{F}^1(A^i)$. The natural inclusion of functors $D_j \hookrightarrow D_{j-1}(\text{Diff}_j^+)$ constructed inductively from the inductive definition of $D_j$ given above, beginning with the inclusion $D \hookrightarrow \text{Diff}_j$, yields the homomorphism $\mathcal{F}^1(A^{s-1}) \rightarrow A^s$. The operator $d : A^{s-1} \rightarrow A^s$ which is the composition of $A^{s-1} \rightarrow J^1(A^{s-1}) \rightarrow A^s$ is a d.o. of order $\leq 1$ and is said to be the exterior differentiation operator.

The sequence $0 \rightarrow A^0 = A \rightarrow^d A^1 \rightarrow^d A^2 \rightarrow^d \ldots$, is a complex and is said to be the de Rham complex of the algebra $A$.

0.7. Suppose $\mathcal{F}$ is a subcategory of the category $\mathcal{M}_A$, possessing the following property: it contains, together with each of the object $P$, all objects of the form $\Phi(P)$, where $\Phi$ is one of the above-mentioned functors of the differential calculus, as well as the homomorphisms $\Phi_1(P) \rightarrow \Phi_2(P)$ corresponding to the transformations $\Phi_1 \rightarrow \Phi_2$ of these functors. In the situation we may ask if the representative objects for the functors of the differential calculus, considered only on $\mathcal{F}$, actually exist. If such an object exists, we denote it in the same way as in 0.5, adding the subscript $\mathcal{F}$. For example, $\mathcal{F}^i_{\mathcal{F}}$ or $d_{\mathcal{F}} : A^i_{\mathcal{F}} \rightarrow A^{i+1}_{\mathcal{F}}$.

The most important example of such a category is that of geometrical modules $\mathcal{M}_{\mathcal{G}_A}$. A module $Q \in \mathcal{G}_A$ is called geometrical if $\bar{Q} = \bigcap pQ = 0, p \in \text{Spec}_A$. There is a geometrization functor $\mathcal{G} : \mathcal{M}_A \rightarrow \mathcal{M}_{\mathcal{G}_A}, \ Q \mapsto Q/\bar{Q}$.
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**Proposition.** The functors $\Phi$ of the differential calculus are representative in $\mathcal{M}\mathcal{G}_A$, and the geometrization operation $\mathcal{G}$ transforms representative objects in $\mathcal{M}_A$ into the corresponding representative objects in $\mathcal{M}\mathcal{G}_A$.

Consider the classical case (see 0.2). Suppose $\mathcal{H} = \mathcal{M}\mathcal{G}_A$. Then $A_\mathcal{G} = A'(M)$ and $J^k_\mathcal{G}(P) = I(\mathcal{E}_k)$, where $\varepsilon_k : J^k(\mathcal{E}) \to M$ is the bundle of $k$ jets corresponding to the bundle $\mathcal{E}$. Similar statements may be made about other standard constructions related to the differential calculus on the manifold.

The category $\mathcal{H}$ will be called smooth, if the module $A_\mathcal{G}$ is of finite type and projective. The category $\mathcal{M}\mathcal{G}_A$ is smooth in the classical case.

0.8. Consider the quotient modules $\text{smbl}_k(P, Q) = \text{Diff}_k(P, Q)/\text{Diff}_{k-1}(P, Q)$ and put $\text{smbl}(P, Q) = \sum_{k \geq 0} \text{smbl}_k(P, Q)$. If $P = A$ we write $\text{smbl}_k Q$ and $\text{smbl} Q$ instead of $\text{smbl}_k(A, Q)$ and $\text{smbl}(A, Q)$, respectively. For the operator $A \in \text{Diff}_k(P, Q)$ the corresponding element in $\text{smbl}_k(P, Q)$ is denoted by $\text{smbl}_k A$.

The composition of d.o. generates an associative algebra structure in $\text{smbl}(P, P)$. This algebra is commutative, if $P = A$. For the same reason $\text{smbl}(P, Q)$ is a left $\text{smbl}(Q, Q)$-module and a right $\text{smbl}(P, P)$-module. In particular, $\text{smbl} P$ is a $\text{smbl} A$-module.

The homomorphism $h_\mathcal{A}$ (see 0.2), when we pass to quotients, induces the homomorphisms $\text{smbl}_s P \to \text{smbl}_{s+1} Q$, $s \geq 0$, whose direct sum is denotes by $\text{smbl} P \to \text{smbl} Q$. Here we have $\text{smbl}(\nabla \circ A) = \text{smbl} \nabla \circ \text{smbl} A$.

Denote the $A$-module of all symmetric multiderivations in $k$ variables with values in the module $P$ by $D^k_{\text{sym}}(P)$. Every $A \in \text{Diff}_k(P, Q)$ defines a multiderivation $\text{sm}(A) \in D^k_{\text{sym}}(\text{Hom}_A(P, Q))$, $\text{sm}(A) = \delta_{a_1, \ldots, a_k}(A)$.

If $A \in \text{Diff}_k(P, Q)$ then $\text{sm}(A) = 0$ and the map $A \mapsto \text{sm}(A)$ generates a homomorphism of $A$ modules

$$\chi_k : \text{smbl}_k(P, Q) \to D^k_{\text{sym}}(\text{Hom}_A(P, Q)).$$

Suppose the multiderivation $[\nabla_1, \ldots, \nabla_k] \in D^k_{\text{sym}}(A)$, $\nabla_i \in D(A)$, defined by the rule

$$(a_1, \ldots, a_k) \mapsto \frac{1}{k!} \sum_{\sigma} \nabla_1(a_{\sigma(1)}) \times \cdots \times \nabla_k(a_{\sigma(k)})$$

(the sum is taken over all permutations $\sigma$ of the numbers $1, \ldots, k$). We than have the homomorphism

$$\gamma_k : S^k(D) \otimes \text{Hom}_A(P, Q) \to D^k_{\text{sym}}(\text{Hom}_A(P, Q)).$$
where \( S^k(D) \) is the \( k \)th symmetric power of the \( A \) module \( D = D(A) \) and

\[
\gamma_k(V_1 \times \cdots \times V_k \otimes h) = [V_1, \ldots, V_k] \otimes h, \\
V_i \in D(A), \quad h \in \text{Hom}_A(P, Q).
\]

**PROPOSITION.** If the category \( \mathcal{H} \) is smooth \( A \in C^b \mathcal{H} \) and \( P \) is a projective \( A \) module of finite type, then \( \chi_k \) and \( \gamma_k \) are isomorphisms.

In this connection in the classical situation we may identify the modules

\[
\text{smbl}(P, Q), \quad D^k_{\text{sym}}(\text{Hom}_A(P, Q)) \quad \text{and} \quad S^k(D) \otimes \text{Hom}_A(P, Q).
\]

Under these assumptions we may identify \( \ker v_{k,k-1} \) with \( S^k(A^1) \otimes P \), where \( v_{k,k-1} : \mathcal{F}^k(\mathcal{H}) \rightarrow \mathcal{F}^{k-1}(\mathcal{H}) \) and \( S^k(A^1) \) is the \( k \)th symmetric power of the \( A \)-module \( A^1 \).

0.9. Further the theory of differential operators on smooth manifolds is understood as differential calculus in the sense described above over \( A - C^\infty(M) \) in the category \( \mathcal{H} = \mathcal{G}_A \).

1. **ADJOINT OPERATORS**

In this section the theory of adjoint operators is constructed in such a way that it can be carried over to the situations where the usual coordinate approach is unjustified. Here we fix the manifold \( M, \dim M = n \) and write briefly \( \Lambda^\sharp = \Lambda^\sharp(M), \ A = C^\infty(M) \).

1.1. **Action of Operators on \( \Lambda^n \)**

We define the action of the operator \( A \in \text{Diff} A \) on the form \( \omega \in \Lambda^n \), denoted by \( A[\omega] \in \Lambda^n \) by the following axioms

\[
(1) \quad (A_1 + A_2)[\omega] = A_1[\omega] + A_2[\omega], \quad A_1, A_2 \in \text{Diff} A,
\]

\[
(2) \quad f[\omega] = f\omega, \quad f \in \text{Diff} A = A,
\]

\[
(3) \quad X[\omega] = -X(\omega), \quad X \in D(M),
\]

\[
(4) \quad (A_1 \circ A_2)[\omega] = A_2[A_1[\omega]].
\]

**PROPOSITION.** The action satisfying the above axioms exists and is unique.

**Proof.** Any operator \( A \in \text{Diff} A \) may be represented in the form of the sum of a "scalar" \( f = A(1) \) and an expression of the form \( X_1 \circ \cdots \circ X_s; \)

\( X_i \in D(M) \). But \( (X_1 \circ \cdots \circ X_s)[\omega] = (-1)^s X_s(\cdots (X_1(\omega)) \cdots) \) in view of (3) and (4). Thus \( A[\omega] = f\omega + \Sigma(-1)^s X_s(\cdots (X_1(\omega)) \cdots) \) which proves uniqueness.
To prove existence first notice that $(f X)[\omega] = (X \circ f)[\omega] - X(f)[\omega]$ and $(X \circ Y)[\omega] = (Y \circ X)[\omega] + ([X, Y])[\omega]$. To prove the first statement note that $Z(\omega) = Z \perp d\omega + d(Z \perp \omega) = d(Z \perp \omega)$, since $d\omega = 0$ and $0 = Y \perp (df \wedge \omega) = Y(f)\omega - df \wedge (Y \perp \omega)$, i.e., $Y(f)\omega = df \wedge (Y \perp \omega)$. Therefore $(f X)(\omega) = d(f X \perp \omega) = df(X \perp \omega) + df \wedge (X \perp \omega) = f X(\omega) + X(f)\omega$, so that $(f X)[\omega] = -f X(\omega) - X(f)\omega$. On the other hand $(X \circ f - X(f))[\omega] = f[X(\omega)] - X(f)[\omega] = -f X(\omega) - X(f)\omega$.

The second statement follows directly from the well-known property of the Lie derivative $X(Y(W)) = Y(X(W)) = [X, Y](W)$.

Represent the operator $A$ in the form $A = f + \sum_i X_i \circ \cdots \circ X_i$, where $X_i \in D(M)$ and put $A[\omega] = f\omega + \sum_i (-1)^{i}\cdots X_i(\omega) \cdots$. In order to show that this formula yields the required action we must show that it is well defined. In other words, we must show that $f\omega + \sum_i (-1)^{i}\cdots X_i(\omega) \cdots = 0$, if $f + \sum_i X_i \circ \cdots \circ X_i = 0$. Consider the “monomial” $A = X_i \circ \cdots \circ X_i$. It follows from (1) and (2) that $A[\omega]$ is well defined above with respect to the following elementary operations: (a) putting $X_i \circ X_i+ = fY$ replace the fragment of the monomial $X_i \circ X_i$ by $fX_i \circ Y - X_i(f)Y$; (b) replace $X_i \circ X_i+ \circ X_i$ by $X_i \circ X_i \circ X_i \circ X_i \cdots$. But as we easily see, the expression $f + \sum X_i \circ \cdots \circ X_i = 0$ may be transformed into zero by means of these elementary operations. 1

1.2. Adjoint Operator

First assume that $A \in Diff M^n$ and $M$ is orientable. Suppose $\omega A^n$ is a volume form (i.e., $\omega \neq 0$, $\forall \alpha \in M$). Define the operator $A_\omega \in Diff A$ by putting $A(\omega) = A_\omega(f)\omega$.

**DEFINITION.** The operator $A^* : A \rightarrow A^*$ is called adjoint to $A \in Diff A^n$, if $A^*(f) = A_\omega[f\omega]$.

Suppose $\omega' = g\omega$ is another volume form. Then $A_\omega = gA_\omega$, $A_\omega[f\omega] = (g \circ A_\omega)[f\omega] = A_\omega[f\omega] = A_\omega[f\omega']$ and therefore $A^*$ is well defined.

If the manifold $M$ is non-orientable, by using the above we may construct an adjoint operator $A^*_\beta$ on any oriented domain $U \subset M$. But then, by the uniqueness of the action $A[\cdot], A^*_\beta|_{U \cap V} = A^*_\beta|_{U \cap V} = A^*_\beta|_{U \cap V}$ so that the system of operators $A^*_\beta$ defines a unique operator $A^*$ on $M$.

**Proposition.** (1) if $A \in Diff_s A^n$ then $A^* \in Diff_s A^n$.

(2) $\omega^* = \omega, \omega \in A^n = Diff_s A^n$.

(3) If $X \in D(A^n)$ then $X + X^* \in Diff_0 A^n = A^n$.

(4) $\text{smbl}_s A^* = (-1)^s \text{smbl}_s A, A \in Diff_s A^n$.

(5) $(A^*)^* = A$. 
Proof. (1) \[ \delta_\varepsilon(A^*)(f) = A^*(gf) - gA^*(f) = A\omega[g\varepsilon - gA\omega]\varepsilon[f\varepsilon] = -\delta_\varepsilon(A\omega)[f\varepsilon] = -\delta_\varepsilon(A\omega, f\varepsilon) = -\delta_\varepsilon(A^*)(f) \] i.e., \( \delta_\varepsilon(A^*) = -\delta_\varepsilon(A) \). Thus \[ 0 = [\delta_{\varepsilon_0, \ldots, \varepsilon_1}(A)]^* = (-1)^{\varepsilon_0, \ldots, \varepsilon_1}A^*, \]

(2) Obvious.

(3) Suppose \( X \in D(A^n) \). Then \( X_\omega \in D(M) \) and \( X^*(f) = -X_\omega(f\omega) = -X_\omega(f)\omega - fX_\omega(\omega) \). Hence \( X^* = -X - X_\omega(\omega) \) and \( X + X^* = -X_\omega(\omega) \).

(4) First suppose \( \Delta \) satisfies \( \Delta_\omega = X_1 \cdots \circ X_k, X_i \in D(M), k \leq s \).

Then \( \Delta^*(f) = (-1)^k X_1(\cdots (X_k(f\omega) \cdots ) = (-1)^k (X_k \circ \cdots \circ X_1)(f) \omega + \Delta'(f), \)

where \( \Delta' \in \text{Diff}_{k-1} A^n \). Further \( X_1 \cdots \circ X_i = X_1 \circ \cdots \circ X_k + \Delta'' \).

\( \Delta'' \in \text{Diff}_{k-1} A^n \), so that \( \text{smbl}_s \Delta^* = \text{smbl}_s \Delta = 0 \), if \( k < s \) and \( \text{smbl}_s \Delta^* = \Delta^* \mod \text{Diff}_{s-1} A^n \).

The general case follows from the above in view of the decomposition \( \Delta_\omega = \Delta_\omega(1) + \sum_i X_i^i \circ \cdots \circ X_i^i, \)

where \( s(i) \leq s, X_i^i \in D(M) \).

(5) First let us show that \( (\Delta \circ X)^* = -\bar{X} \circ \bar{A}^* \) and \( (\bar{X} \circ \bar{A})^* = -\Delta \circ \Delta \), where \( \Delta, \bar{X} \in \text{Diff} A^n, X \in D(M), \bar{X} \in \text{Diff}_{1}(A^n, A^n), \)

and \( \bar{X}(\rho) = X(\rho), \rho \in \mathcal{A}^* \). First note that \( (\Delta \circ X)_\omega = \Delta_\omega \circ X \). Therefore \( (\Delta \circ X)^*(f) = (\Delta \circ X)_\omega[f\varepsilon] = -X(\Delta_\omega[f\varepsilon]) = -X(\Delta^*(f)) = (\bar{X} \circ \bar{A}^*)(f) \).

Further \( (\bar{X} \circ \bar{A})^*(f) = X(\bar{A}(f) \omega) = (X \circ \bar{A} + g\bar{A})(f) \omega \), where \( g\omega = X(\omega) \). Therefore \( (\bar{X} \circ \bar{A})_\omega = (X + g) \circ \bar{A} \omega \) so that \( (\bar{X} \circ \bar{A})^*(f) = ((X + g) \circ \bar{A})_\omega[f\varepsilon] = \bar{A}_\omega[(X + g)[f\varepsilon]] = -\bar{A}_\omega[X(f) \omega] = -\Delta^*(X(f)) \).

Now suppose \( \Delta = \bar{X} \circ \bar{A}, \bar{X} \in \text{Diff} A^n, X \in D(M) \), then it follows from the relations proved above that \( \Delta^* = \bar{X} \circ \bar{A}^* \). Since \( \forall \Delta \in \text{Diff} A^n \), \( \Delta^* \) may be represented in the form \( \Delta = \sum_i X_i^i \circ \bar{D} + \Delta(1), \bar{D} \in \text{Diff}_{s-1} A^n, X_i^i \in D(M) \), the last relation allows us to prove the required fact by induction over \( \deg \Delta \).

Remark. If \( X \in D(A^n) \) then the form \( \text{div} \ X = X + X^* = -X_\omega(\omega) \in A^n \) is naturally referred to as the divergence of the "field" \( X \).

1.3. Suppose \( \bar{Q} = \text{Hom}_*(Q, A^n) \), where \( Q \) is a \( A \)-module. Now construct the adjoint operator in the general case, namely, when \( \Delta \in \text{Diff}(P, \bar{Q}) \), the \( P \) and \( Q \) being \( A \)-modules. To do this consider the family of operators \( \Delta(p, q) \in \text{Diff} A^n, p \in P, q \in Q, \Delta(p, q)(f) = \Delta(fp)(q), f \in A \). Obviously the operator \( \Delta \) is defined by the family \( \Delta(p, q) \). The following statement is also obvious.

**Lemma.** For the family of operators \( \Delta[p, q] \in \text{Diff} A^n, p \in P, q \in Q, \) we can find an operator \( \Delta \in \text{Diff}(P, \bar{Q}) \) such that \( \Delta[p, q] = \Delta(p, q), \) if and only if \( \Delta[p, q] = \sum_i f_i A[p, q_i], \Delta[\sum_i f_i q_i] = \sum_i f_i A[p, q_i]. \)

In view of this lemma the operators \( \Delta^*(p, q) = \Delta(p, q)^* \), uniquely determines the operator \( \Delta^* \in \text{Diff}(Q, \bar{P}) \), which will be called adjoint to \( \Delta \).
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**Proposition.** (1) For every $\Delta \in \text{Diff}(P, \hat{Q})$, we have $\Delta^{**} = \Delta$.

(2) If the $A$-module $Q$ is projected then for all $\Delta_1 \in \text{Diff}(P, \hat{Q})$, $\Delta_2 \in \text{Diff}(\hat{Q}, \hat{R})$ we have $(\Delta_2 \circ \Delta_1)^* = \Delta_1^* \circ \Delta_2^*$ (here $\hat{Q}$ is identified with $Q$).

**Proof.** (1) $\Delta^{**}(p, q) = (\Delta^*(q, p))^{**} = (\Delta(p, q))^{**}$, but since $\Delta(p, q) \in \text{Diff} A^n$, it follows from Proposition 1.2(5) that $(\Delta(p, q))^{**} = \Delta(p, q)$.

(2) further we always identify $\hat{P}$ with $P$ for a projective $P$. First consider the special case $A^+ A \rightarrow A^2 A^-$ first when $0$ is free and then when it is projective. Finally, the general cases reduces to this one, since $\Delta_2 \circ \Delta_1^* = A^\pi \circ A^\pi$.

If $p \in P$ then the homomorphism $A \rightarrow P, f \mapsto fp$ will also be denoted by $p$.

In this notation $A(p, q) = q^* \circ A \circ p$ and $(q^* \circ A \circ p)^* = p^* \circ A^* \circ q$. It follows from the last relation that the required formula holds if $\Delta_1$ or $\Delta_2$ are homomorphisms. Thus it follows that $\hat{Q} = \oplus \hat{Q}_i$, $\alpha_i : \hat{Q}_i \rightarrow \hat{Q}_j, i \in D(M)$ and using the relation $\square \circ X^* = -X \circ \square^*$, proved above we get $(\Delta_2 \circ \Delta_1)^* = \Delta_1 \circ \Delta_2^*$, where $\Delta_1 : \text{Diff} \rightarrow \text{Diff}$, $\Delta_1(w) = \Delta_1[w]$. It also follows directly from the definition that $\square = \square^*$, $\square \in \text{Diff} A$ so that in the case under consideration $(\Delta_2 \circ \Delta_1)^* = \Delta_1^* \circ \Delta_2^*$.

1.4. Coordinates

Suppose $A \in \text{Diff}_s A$ and $\Delta : A^n \rightarrow A^n$ is defined by $\Delta(\omega) = \Delta[\omega]$. Then $\delta(\hat{\Delta}) = \delta(\Delta)$, showing that $\hat{\Delta} \in \text{Diff}_s(A^n, A^n)$. In particular, the operator $\hat{A}$ is local and therefore $(\hat{\Delta}|_U)(\omega|_U) = \hat{\Delta}(\omega|_U)$, where $U \subset M$ is an open submanifold. In other words $(\Delta|_U)(\omega|_U) = \Delta(\omega)|_U$, i.e., the action of the operator $A \in \text{Diff} A$ on $n$-forms is local. In view of this we also have $(\Delta|_U)^* = \Delta^*|_U$. This enables us to compute the expressions for $A(\omega)|_U$ and $A^*|_U$ as $(\Delta|_U)(\omega|_U)$ and $(\Delta|_U)^*$, respectively, in local coordinates if $U$ is a coordinate neighbourhood.

Suppose $x_1, \ldots, x_n$ are coordinates in $U$, $\omega = dx_1 \wedge \cdots \wedge dx_n$

$$\frac{\partial \omega}{\partial x_\sigma} = \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}},$$

where $\sigma = (i_1, \ldots, i_k)$ is a non-ordered set of integers $1 \leq i_j \leq n$, $|\sigma| = k$. Since $(\partial / \partial x_i)(f \omega) = (\partial f / \partial x_i) \omega$ we have

$$\left( a \frac{\partial^{\sigma}}{\partial x_\sigma} \right) \omega = (1)^{|\sigma|} \frac{\partial^{\sigma}(a)}{\partial x_\sigma} \omega.$$
Therefore for any $\Delta \in \text{Diff} \, A^*$ satisfying $\Delta_\omega = \sum_\sigma a_\sigma (\partial^{[\sigma]} / \partial x_\sigma)$ we have

$$(\Delta^*)_\omega = \sum_\sigma (-1)^{[\sigma]} \frac{\partial^{[\omega]}}{\partial x_\sigma} \circ a_\sigma.$$  

Suppose $\alpha_i : E_{\alpha_i} \to M$ are vector bundles ($i = 1, 2$), $P = r(\alpha_1)$, $Q = r(\alpha_2)$. Assume that the coordinate neighbourhood $U \subset M$ is chosen so that the bundle $\alpha_i$ have a base of smooth sections over $U$, say $e_1, \ldots, e_r$ and $f_1, \ldots, f_s$, respectively. Let $e_i \in P, f_i \in Q$ satisfy $\alpha_i(e_j) = \delta_{ij} \alpha, \alpha_i(f_j) = \delta_{ij} \alpha$. Then for $p \in P, q \in Q$, we have $p = \sum p_i e_i$ and $\Delta(p) = \sum p_i \Delta_i(p_i) f_j = \sum f_j^* \circ \Delta_i(p_i)$, where $\Delta_i = \Delta(e_i, f_j), \Delta_j = (\Delta_j)_{|U}$. Then by definition $\Delta^*(q) = \sum c_{ki}^* \circ \Delta_k^{*}(q_i) = \sum (\Delta_k^{*}(q_i)) \hat{e}_k$, where $q = \sum q_i f_i$. In other words, if the operator $\Delta$ is given by the operator matrix $C = [\Delta_{ij}]$, acting on the column $(p_1, \ldots, p_r)^t$, then the operator $\Delta^*$ is given by the matrix $[c_{ki}]$, where $c_{ki}^* = (\Delta_k^{*})_{ijkl}$, acting on the columns $(q_1, \ldots, q_s)^t$.

1.5. Transformations

We conclude the section by describing the transformational properties of the operations described above in two situations the finite one and the infinitesimal one. By the latter we mean the following: Suppose that in the classical situation $F_r$ is a one-parameter family of transformations. Then it is natural to call the operation $Z \to X(Z) = (d/dt) F_r(Z)|_{t=0}$ an infinitesimal transformation of the object $Z$. In the general algebraic situation by an infinitesimal transformation of the object $Z$ we mean an operation which formally is of the type indicated above. For example, suppose $F:A_1 \to A_2$ is an isomorphism of two commutative $K$-algebras, $F_A$ (or simply $F$) is the corresponding map of differential forms and $\Delta \in \text{Diff} \, A^k(A_1)$. Then we put $F(\Delta) = F_A \circ \Delta \circ F^{-1} \in \text{Diff} \, A^k(A_2)$. If $A_1 \rightarrow A_2 = A$ and $X \in D(A)$ then the expression $X(\Delta)$ should be defined by "taking the derivative" of the expression $(F_A)_A \circ \Delta \circ F^{-1}$ assuming that $X = (dF/\partial t)|_{t=0}$. Thus we come to the definition $X(\Delta) = X_A \circ \Delta - \Delta \circ X$, or simplifying the notation $= |X, \Delta|$. Here $X_A(\omega)$ denotes the Lie derivative of the form $\omega$. Whenever it is necessary to show explicitly what object is being mapped by a morphism induced by an isomorphism $F$, we shall supply the letter $F$ with the appropriate subscript. For example, $F_{\text{Diff}, A^k} : \text{Diff} \, A^k(A_1) \to \text{Diff} \, A^k(A_2)$ denotes the map considered above. A similar system of notations will be used in the infinitesimal situation. For example, $X_{\text{Diff}, A^k}(\Delta) = |X, \Delta|$. Let s finally note that in the classical case the role of the algebra $A_i$ is played by the algebras $C^\infty(M_i)$ while $F = f^*$, where $f : M_1 \to M_2$ is a smooth map.

The transformation of the action $| \cdot |$ given over $A_1$, into an action over $A_2$ denoted by $| \cdot |_F$ is defined by the formula

$$\Delta |\omega|_F = F_A (F^{-1}(\Delta)|F_A^{-1}(\omega)|).$$
while the infinitesimal analogue denoted by $|X|_\lambda$ for \(X \in D(A)\) is defined by the formula

\[
A[\omega]_\lambda = X_A(A[\omega]) - [X, A][\omega] - A[X_A(\omega)].
\]

In view of Proposition 1.1, claiming uniqueness, we have

\[
A[\omega] = F_A(F^{-1}(A)[F^{-1}_A(\omega)]).
\]

The infinitesimal analogue of this formula is

\[
A[\omega]_\lambda = X(A[\omega]) - [X, A][\omega] + A[X_A(\omega)].
\]

For the proof it suffices to write out the commutator $[X, A]$ in detail and use the action axioms.

Let us show that

\[
\circ \circ F_{\text{Diff}\, A} = F_{\text{Diff}\, A} \circ \circ \circ, \quad \text{or} \quad F(A)^* = F(A^*), A \in \text{Diff} A^n. \tag{1.5.3}
\]

We first check that $F(A)_{F(\omega)} = F(A_\omega)$. Indeed, $(F(A)_{F(\omega)}(a)) F(\omega) = F(A)(a) = F(A(F^{-1}(a) \omega)) = (F \circ A \circ F^{-1})(a) F(\omega)$. Further $F(A)^*(a) = F(A)_{F(\omega)} \cdot [a F(\omega)] = F(A_{\omega})[F(F^{-1}(a) \cdot \omega)]$. In view of (1.5.1) for $F^{-1}$ this last expression equals $F(A_{\omega}[F^{-1}(a) \omega]) = F(A^*(F^{-1}a)) = F(A^*)(a)$.

The infinitesimal analogue of (1.5.3) is of the form

\[
\circ \circ X_{\text{Diff}\, A} = X_{\text{Diff}\, A} \circ \circ \circ \quad \text{or} \quad [X, A]^* = [X, A^*], A \in \text{Diff} A^n \tag{1.5.4}
\]

and is obviously true.

We shall say that the $K$-linear map $F_p : P_i \to P_2$, where $P_i$ is a $A_i$ module covers $F : A_1 \to A_2$, if $F_p(ap) = F(a)F_p(p)$, $a \in A$, $p \in P$. Similarly $X_p \in \text{Diff}_1(P, P)$ covers $X \in D(A)$, if $X_p(ap) = X(a)p + aX_p(p)$.

Put $\text{Der} P - \{(X, X) \mid X \in D(A), X_p \in \text{Diff}_1(P, P), X_p \text{ covers } X\}$.

$\text{Der} P$ has an obvious $A$-module structure and the formula $\delta_a(X_p) = X(a)$ shows that $X_p$ determines $X$, i.e., $\text{Der} P \subset \text{Diff}_1(P, P)$. If $P = \Gamma(\alpha_1)$ in the notations of the previous subsection, then $X_p \in \text{Der} P \Leftrightarrow X_p = XI_r + B$, where $I_r$ is the unit matrix and $B = \|b_{ij}\|$ is also a $(r \times r)$ matrix. $b_{ij} \in C^\infty(U), \ X \in D(U)$.

If we are given the bijection $F_p : P_1 \to P_2$, $F_Q : Q_1 \to Q_2$ covering $F$, then we can define $\text{Diff} : \text{Diff}(P_1, Q_1) \to \text{Diff}(P_2, Q_2)$, by putting $\text{Diff}_i(A) = F_Q \circ A \circ F_p^{-1}$. Similarly, if both $X_p \in \text{Der} P$, $X_Q \in \text{Der} Q$ cover $X \in D(A)$ we put $X_{\text{Diff}}(A) = X_Q \circ A - A \circ X_Q = [X, A] \in \text{Diff}(P, Q)$.

In this situation we also have the formulas $\circ \circ F_{\text{Diff}} = F_{\text{Diff}} \circ \circ \circ$, $\circ \circ X_{\text{Diff}} = X_{\text{Diff}} \circ \circ \circ \circ \circ$. To prove the first of them (the second one is obvious again) note that $F(\square)(F_p p_1, \hat{q}_2) = F(\square)(p_1, F_0^{-1}(\hat{q}_2))$, $p \in P_1$, $\hat{q}_2 \in \hat{Q}_2$. Hence
\[ F(\Delta^*)(q_2, p_2) = F(\Delta^*(F^{-1}_\phi q_2, F^{-1}_\psi p_2)) = F(\Delta(F^{-1}_\psi p_2, F^{-1}_\phi q_2)^*) \] but this expression, by (1.5.3) equals \[ [F(\Delta(F^{-1}_\psi p_2, F^{-1}_\phi q_2))]^* = [F(\Delta)(p_2, q_2)]^* = F(\Delta^*)(q_2, p_2), \forall q_2 \in \hat{Q}_2, p_2 \in P_2. \] Thus \( F(\Delta^*) = F(\Delta)^* \).

2. SPENCER COMPLEXES AND GREEN’S FORMULA

The aim of this section is to establish the relationship between the operator * and the Spencer Diff-complex. The classical Green formula is one of the aspects of this relationship.

2.1. Spencer Diff-complexes

Let us recall the construction of the Spencer Diff-complex (for details see [13] as well as [16]). Consider the composition

\[ \text{Diff}: P \rightarrow \text{Diff}:(\text{Diff}: P) \rightarrow \text{Diff}:(\text{Diff}+ P), \]

where \( \alpha \) is generated by the transformation of functors \( D_i \rightarrow D_{i-1}(\text{Diff}^+ P) \) (see 0.5) while \( \beta \) is the glueing homomorphism \( \gamma_{1,s} : \text{Diff}_1(\text{Diff}^+ P) \rightarrow \text{Diff}^+ P \). The operators \( S = S_{i,s} \equiv \beta \circ \alpha : \hat{D}_i(\text{Diff}^+ P) \rightarrow \hat{D}_{i-1}(\text{Diff}^+ P) \), called Spencer Diff-operators generate the Spencer Diff-complex \( S_k P \):

\[ k \text{Diff}^+ P \rightarrow \cdots, \]

or, for \( k = \infty \) its “stable” variant \( S^P \):

\[ k \text{Diff}^+ P \rightarrow \cdots, \]

where \( S^P \) is the direct limit of the chain of complexes \( \cdots \rightarrow S_k P \rightarrow S_{k+1} P \rightarrow \cdots \). Note that the operators \( S_{i,j} \) are \( A \)-homomorphisms.

Further we shall need a more direct description of a Spencer operator based on the interpretation of elements from \( D_i(P) \) as multiderivations. The multiderivation \( (f_1, \ldots, f_i) \mapsto [(X_1 \wedge \cdots \wedge X_i) \cup (df_1 \wedge \cdots \wedge df_i)] p \), where \( X_i \in D(A), f_i \in A, p \in P \), belonging to \( D_i(P) \) is denoted by \( X \cdot f_1 \cdots f_i \).

If \( P = \text{Diff}^+ Q \), then by \( \Delta \circ X_1 \wedge \cdots \wedge X_i \) we denote the multiderivation \( (f_1, \ldots, f_i) \mapsto [X_1 \wedge \cdots \wedge X_i \cup (df_1 \wedge \cdots \wedge df_i)] \) with values in \( D_i(\text{Diff}^+ Q), \Delta \in \text{Diff}^+ Q \). In this notation \( S_{i,j}(\Delta \circ X_1 \wedge \cdots \wedge X_i) = \sum_{j=1}^i (-1)^{i-j} \Delta \circ X^{(j)} \otimes X_j \), where \( X^{(j)} = X_1 \wedge \cdots \wedge X_j \) and the
multiderivation $\Delta \circ X^{(j)} \otimes X_j$ acts according to the rule $(f_1, \ldots, f_{i-1}) \mapsto 
abla \circ X^{(j)}(f_1, \ldots, f_{i-1}) X_j$. Here $X^{(j)}(f_1, \ldots, f_{i-1}) = (X_1 \wedge \cdots \wedge i X_j \wedge \cdots \wedge X_i)_\perp (df_1 \wedge \cdots \wedge df_{i-1})$.

Note that multiderivations of the form $\Delta \circ X_1 \wedge \cdots \wedge X_i$ generate $\hat{D}_i(Diff^*_P)$ additively.

We shall also need the natural isomorphism $\Pi = \Pi_x : D_i(A^n) \to A^{n-i}$ (Poincaré duality) defined in the following way. Suppose $z = x_1 \wedge \cdots \wedge X_i \otimes \omega \in D_i(A^n)$, where $\omega \in A^n$ is a (local) volume form. Then $\Pi(z) = X_1 \wedge (\cdots (X_i \wedge \omega) \cdots )$. This definition obviously does not depend on the choice of $\omega$ and may be extended by linearity to all of $D_i(A^n)$.

2.2.

Consider the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A^n & \overset{\partial}{\longrightarrow} & Diff_k A^n & \overset{S}{\longrightarrow} & \hat{D}(Diff^+_{k-1} A^n) & \overset{S}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^n & \overset{\partial}{\longrightarrow} & Diff^+_k A^n & \overset{\gamma}{\longrightarrow} & Diff^+_{k-1} A^{n-1} & \overset{\gamma}{\longrightarrow} & \cdots \\
\end{array}
\]

whose upper line is the Spencer complex. The $A$-homomorphisms appearing in it are defined as follows: $\gamma(V) = d \circ V \in Diff^+_{k+1} A^{s+1}$, $V \in Diff^+ A^s$; $\mu(A) = A^*(1)$, $A \in Diff_k A^n$; $\psi$ is the composition of the isomorphisms

\[
\hat{D}_s(Diff^+_k A^n) \overset{D_s(A^*)}{\longrightarrow} D_s(Diff^+_{k+s} A^n) \overset{\gamma}{\longrightarrow} Diff^+_{k-s}(D_s(A^n)) \overset{\partial}{\longrightarrow} Diff^+_{k-s} A^{n-s} \overset{\partial}{\longrightarrow} Diff^+_{k-s} D_s.
\]

where the middle homomorphism is generated by the permutation of functors $D_s Diff^+_{k-s} \rightarrow Diff^+_{k-s} D_s$.

**Theorem.** Every square of this diagram is anti-commutative (the case $k = \infty$ is not excluded).

**Proof.** The anti-commutativity of the left square is obvious. Suppose $Z = \Delta \circ X_1 \wedge \cdots \wedge X_s \in D_s(Diff^+_{k-s} A^n)$, then according to 2.1, $Z' = (D_{s-1}(\gamma) \circ S) Z$ is the multi-derivation
\[(f_1, \ldots, f_{s-1}) \mapsto \sum_j (-1)^{j-1} [\Delta \circ X^{(j)}(f_1, \ldots, f_{s-1}) \circ X_j]^{*}\]

\[= \sum_j (-1)^j \vec{X}_j \circ X^{(j)}(f_1, \ldots, f_{s-1}) \circ \Delta^{*}\]

\[= \sum_j (-1)^j \left[ X^{(j)}(f_1, \ldots, f_{s-1}) \vec{X}_j \circ \Delta^{*} + X_j(X^{(j)}(f_1, \ldots, f_{s-1})) \Delta^{*} \right]\]

\[= \sum_j (-1)^j X^{(j)}(f_1, \ldots, f_{s-1}) \vec{X}_j \circ \Delta^{*}\]

\[+ \sum_{i<j} (-1)^{i+j} X^{(i,j)}(f_1, \ldots, f_{s-1}) \Delta^{*} \in D_{s-1}(\text{Diff}_{k-s-1} A^{*}),\]

where \(X^{(i,j)} = [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \vec{X}_i \wedge \cdots \wedge \vec{X}_j \wedge \cdots \wedge X_s\). The functor permutation operation \(D_{s-1} \text{Diff}_{k-s-1} \to \text{Diff}_{k-s+1} D_{s-1}\) transforms \(Z'\) into the differential operator \(f \mapsto \sum_f (-1)^j X^{(j)} \otimes X_j(A^{*}(f)) + \sum_{i<j} (-1)^{i+j} X^{(i,j)} \otimes A^{*}(f)\), so that finally for \(\Box = (\psi \circ S)(Z) \in \text{Diff}_{k-s} A^{*}\) we have

\[\Box(f) = \sum_i (-1)^i X^{(i)} \jmath X_i(A^{*}(f)) + \sum_{i<j} (-1)^{i+j} X^{(i,j)} \jmath A^{*}(f),\]

where \(W \jmath \rho\) if \(W = Y_1 \wedge \cdots \wedge Y_i\) denotes \(Y_1 \jmath (\cdots (Y_i \jmath \rho) \cdots)\).

On the other hand, \(\psi(Z)(f) = X_1 \jmath (\cdots (X_s \jmath A^{*}(f)) \cdots)\), so that \(\Box'(f) = d(X_1 \jmath \cdots \jmath X_s \jmath A^{*}(f) \cdots)\), where \(\Box' = \mathcal{S}'(\psi(Z))\). Hence the required statement follows from the following lemma if we take into consideration the relation \(dA^{*}(f) = 0\) and \(X_j(A^{*}(f)) = d(X_j \jmath A^{*}(f))\).

**Lemma.** If \(\rho \in A^k, X_i \in D(M)\) then

\[d(X_1 \jmath \cdots \jmath X_s \jmath \rho) = \sum_j (-1)^{j-1} X^{(j)} \jmath d(X_j \jmath \rho) + \sum_{i<j} (-1)^{i+j-1} X^{(i,j)} \jmath d \rho + (-1)^{s-1} (s-1) X_1 \jmath \cdots \jmath X_s \jmath d \rho.\]

For \(s = 1\) the statement is a tautology. Using the formula \(i_{[X,Y]} = [L_X, i_Y]\), where \(L_X\) is the Lie derivative and \(i_Y \rho = X \jmath \rho, \rho \in A^*\) (see, e.g., [31]), we get

\[[X, Y] \jmath \rho = X(Y \jmath \rho) - Y \jmath X(\rho)\]

\[= X \jmath d(Y \jmath \rho) + d(X \jmath Y \jmath \rho) - Y \jmath X \jmath d \rho - Y \jmath d(X \jmath \rho)\]
or
\[ d(X \cup Y \cup \rho) = Y \cup X \cup dp + Y \cup d(X \cup \rho) - X \cup d(Y \cup \rho) + [X, Y] \cup \rho, \]
i.e., the statement of the lemma for \( s = 2 \). Further the lemma is proved by an obvious induction using this formula. \[ \]

Since the Spencer Diff-complexes are exact for any geometric \( C^\infty(M) \)-module \( P \) (see [13]) the theorem proved above has the following:

**Corollary.** The complex \( \mathcal{S}_k \) (\( k = \infty \) is admissible)

\[
0 \rightarrow A^n \xleftarrow{\mu} \text{Diff}_k^+ A^n \xrightarrow{\mathcal{F}} \text{Diff}_{k-1}^+ A^{n-1} \xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} \text{Diff}_{k-n}^+ A \rightarrow 0
\]
is exact.

Applying the functor \( \text{Hom}^\_\_ P, \cdot \) to the complex \( \mathcal{S}_k \) and taking into consideration the relation \( \text{Hom}^\_\_ P, \text{Diff}_k^+ Q \) = \( \text{Diff}_k(P, Q) \), we get the complex \( \mathcal{S}_k P : 0 \rightarrow \text{Diff}_{k-n}(P, A) \rightarrow \mathcal{F} \cdots \rightarrow \mathcal{F} \text{Diff}_{k-1}(P, A^{n-1}) \rightarrow \mathcal{F} \text{Diff}_k(P, A^n) = 0 \). Here, for the sake of brevity we write \( \mu \) and \( \mathcal{F} \) instead of \( \text{Hom}^\_\_ P, \mu \) and \( \text{Hom}^\_\_ P, \mathcal{F} \), respectively. If \( P \) is projective, then the complex \( \mathcal{S}_k P \) is obviously exact.

Finally, applying the functor \( \text{Diff}_l(Q, \cdot) \) to \( \mathcal{S}_k P \) (\( l = \infty \) is admissible) we obtain the complex \( \mathcal{S}_{k+l} PQ : 0 \rightarrow \text{Diff}_l(Q, \text{Diff}_{k-n}(P, A)) \rightarrow \mathcal{F} \cdots \rightarrow \mathcal{F} \text{Diff}_l(Q, \text{Diff}_k(P, A^n)) \)

\[
\rightarrow \bar{\mu} \text{Diff}_l(Q, \bar{P}) \rightarrow 0,
\]
where \( \bar{\mu} = \text{Diff}_l(Q, \mu) \), \( \mathcal{F} = \text{Diff}_l(Q, \mathcal{F}) \). This complex is exact if \( l = \infty \) and \( P \) and \( Q \) are projective. This will be proved later.

**2.3.**

In this subsection we describe the complexes dual to the de Rham complex and the complex \( \mathcal{S}_k P \). Suppose \( \cdots \rightarrow P_i \xrightarrow{\Delta_i} P_{i+1} \rightarrow \cdots \), is a complex of projective \( A \)-modules. Then \( P_i = \bar{P}_i \) and we may assume that \( \Delta_i^* \) acts from \( \bar{P}_{i+1} \) to \( \bar{P}_i \). Since \( \Delta_{i-1}^* \circ \Delta_i^* = (\Delta_i \circ \Delta_{i-1})^* = 0 \), we obtain the complex \( \cdots \rightarrow \bar{P}_i \xrightarrow{\Delta_i^*} \bar{P}_{i+1} \rightarrow \cdots \), which is said to be dual to the given one.

First consider the operator \( d = d_i : A^i \rightarrow A^{i+1} \) in order to compute \( d_i^* : \tilde{A}^{i+1} \rightarrow \tilde{A}^i \). But \( \tilde{A}^i = \text{Hom}_A(A^s, A^n) = D_s(A^n) \), which, by Poincare duality, is \( A^n \rightarrow s \). Therefore we may assume that \( d_i^* \) acts from \( A^{n-i} \) to \( A^n \).

Let us describe in more detail how \( \theta \in A^{n-s} \) should be understood as an element of \( \tilde{A}^s \). If \( \theta = X_1 \cup \cdots \cup X_s \cup \omega \), \( X_i \in D(M) \), \( \omega \in A^n \), then by
Poincaré duality it corresponds to the multidervation $X_1 \wedge \cdots \wedge X_s \otimes \omega \in D_s(A^n)$ and therefore is a homomorphism $h_\theta' : A^s \to A^n$, such that $h_\theta'(df_1 \wedge \cdots \wedge df_s) = |(X_1 \wedge \cdots \wedge X_s) \mapsto (df_1 \wedge \cdots \wedge df_s)| \omega$. Since the forms $df_1 \wedge \cdots \wedge df_s$ generate the $A$-module $A^s$, we have

$$h_\theta'(\rho) = (X_1 \wedge \cdots \wedge X_s \mapsto \rho) \omega = \rho \wedge (X_1 \wedge \cdots \wedge X_s \mapsto \omega) = \rho \wedge \theta, \quad \forall \rho \in A.$$  

Further we shall put $h_\theta = (-1)^{(h-s)}h_\theta' : A^s \to A^n$, $h_\theta(\rho) = \theta \wedge \rho$.  

**Lemma.** Suppose $\square(f) = \partial f \wedge \theta, \theta \in A^{n-1}$, then $\square^*(f) = -df \wedge \theta - f d\theta$.

**Proof.** Suppose (locally) $\theta = X \wedge \omega, \omega$ is the volume form. Then $\square(f) = \partial f \wedge (X \wedge \omega) = X(f) \omega$, i.e., $\square, \omega = X$. Hence $\square^*(f) = \square, [f \omega] = -X(f \omega) = -X(f) \omega - f X(\omega)$. But $X(f) \omega = df \wedge (X \wedge \omega) = df \wedge \theta, X(\omega) = d(X \wedge \omega) = d\theta$.

Now let us compute the operator $d^+_i(\theta, \rho) = d_i(\theta, \rho)^*, \rho \in A^i, \theta \in A^{n-i-1}$ (see 1.3). We have $d_i(\theta, \rho)(f) = (\theta^* \circ d_i \circ \rho)(f) = d(fp) \wedge \theta = df \wedge (\rho \wedge \theta) + f dp \wedge \theta$.

Hence by lemma $d_i(\theta, \rho)^*(f) = -df \wedge (\rho \wedge \theta) = f d(\rho \wedge \theta) + f dp \wedge \theta = (-1)^{n-i+1} d(f \theta) \wedge \rho$. But since in a similar way $d_{n-i-1}(\theta, \rho)(f) = d(f \theta) \wedge \rho$, we get $d_i^* = (-1)^{n+1} d_{n-i-1}$. In particular, for an even $n$ the de Rham complex is skew symmetric.

Now consider the Spencer jet complex $\mathcal{J}^k P$ of the $A$-module $P: 0 \to P \to \mathcal{J}^k P$. We do not exclude $k = \infty$. Let us compute the dual complex. Note first of all that $\mathcal{J}^\ast(P) \otimes A^i = \text{Hom}_A(\mathcal{J}^\ast(P) \otimes A^i, A^n) = \text{Hom}_A(\mathcal{J}^\ast(P), \text{Hom}_A(\mathcal{J}^\ast(P), A^n)) = \text{Hom}_A(\mathcal{J}^\ast(P), D_\ast(A^n)) = \text{Diff}_\ast(P, A^{n-i}).$ Therefore assuming that the module $P$ is finitely generated and projective we may suppose that the operator $\mathcal{J}^\ast_{s,i}(j_i(p) \otimes \omega) = j_{s-i}(p) \otimes \omega, \rho \in P, \omega \in A^i$. We do not exclude the case $k = \infty$. Let us compute the dual complex. Note first of all that $\mathcal{J}^\ast(P) \otimes A^i = \text{Hom}_A(\mathcal{J}^\ast(P) \otimes A^i, A^n) = \text{Hom}_A(\mathcal{J}^\ast(P), \text{Hom}_A(\mathcal{J}^\ast(P), A^n)) = \text{Hom}_A(\mathcal{J}^\ast(P), D_\ast(A^n)) = \text{Diff}_\ast(P, A^{n-i}).$ Therefore assuming that the module $P$ is finitely generated and projective we may suppose that the operator $\mathcal{J}^\ast_{s,i}(j_i(p) \otimes \omega) = j_{s-i}(p) \otimes \omega, \rho \in P, \omega \in A^i$. Therefore $\Sigma_{s,i}(j_i(p) \otimes \theta, \Delta(f) = (\Delta, \Sigma_{s,i}(j_i(p) \otimes f \theta)) = d(f \theta) \wedge \Delta(p), \rho \in P, \theta \in A^i, \Delta \in \text{Diff}_{s-i}(P, A^{n-i-1}).$ Above in our computation of the operator $d^*$ we noted that the operator dual to the operator $f \mapsto d(f \theta) \wedge \omega$, is $f \mapsto (-1)^{1+ns} d(f \omega) \wedge \theta$. Therefore

$\Sigma^*_{s,i}(\Delta) = \Sigma^*_{s,i}(\Delta) = \Sigma^*_{s,i}(\Delta)(1) = \Sigma_{s,i}(j_i(p) \otimes \theta, \Delta)(1) = (-1)^{ns+1} d\Delta(p) \wedge \theta = (-1)^{s+1} \theta \wedge d\Delta(p)$.
and identifying $\Sigma_{s,k}^*(\Delta) \in \text{Hom}_\mathcal{C}(\mathcal{F}^s(P) \otimes \Lambda^s, \Lambda^n)$ with the corresponding element of $\text{Diff}_\mathcal{C}(P, \Lambda^{n-s})$ via isomorphism described above we have

$$\Sigma_{s,k}^*(\Delta) = (-1)^{s+1} d \circ \Delta = (-1)^{s+1} \Sigma_{s,k-s}.$$ 

Computing the operator $j_k^*$, where $j_k : P \to \mathcal{F}^k(P)$, note that $j_k(p, \Delta)(\phi) = \Delta(\phi), \quad p \in P, \quad \Delta \in \text{Diff}_\mathcal{C}(P, \Lambda^n)$. Therefore $j_k^*(\Delta)(p) = j_k^*(\Delta)(P) = j_k(\Delta)^*(1) = (\Delta \circ P)^*(1) = \Delta^*(1)(p) \Rightarrow j_k^*(\Delta) = A^*(1) = \mu(\Delta) \Rightarrow j_k^* = \mu$.

Putting all this together we obtain the following:

**Proposition.**

1. If $n$ is even then the de Rham complex is skew symmetric.
2. If $n$ is odd then the de Rham complex is dual to the complex $\{A^s, (-1)^{s+1} d^s\}$.
3. The complex $\mathcal{J}_k P$ is dual to the "twisted" Spencer jet complex $\{\mathcal{J}^{k-1}(P) \otimes \Lambda^s, (-1)^{s+1} \Sigma_{s,k-s}\}$.

2.4. Green's formula

Consider the "square" which is anti-commutative by Theorem 2.2 ($k = \infty$ is possible):

$$\text{Diff}_k \Lambda^n \leftarrow S \downarrow \mathit{D} \downarrow \text{Diff}^{k-1}_k \Lambda^n$$

Note that $\text{im} \ S = \text{Diff}_k \Lambda^n = \{ \Delta \in \text{Diff}_k \Lambda^n | \Delta(1) = 0 \}$. Since the module $\text{Diff}_k \Lambda^n$ is projective there exists a homomorphism $\lambda : \text{Diff}_k \Lambda^n \to \mathit{D}(\text{Diff}^{k-1}_k \Lambda^n)$, such that $S \circ \lambda = 1$.

The relation $\star \circ S + A \circ \psi = 0$ applied to the element $\lambda(\square), \square \in \text{Diff}_k \Lambda^n$ yields $\square^* + d \circ \lambda(\square) = 0$, where $\lambda(\square) = \psi \circ \lambda$. In particular, $\square^*(1) + d(\lambda(\square)(1)) = 0$. Putting $\lambda(\square) = \Pi_{k-1} \circ \lambda \circ v, \square = v(\Delta)$, where $v(\Delta) = \Delta - \Delta(1)$, and $\Delta \in \text{Diff}_k \Lambda^n$, and then taking into consideration the fact that $d \circ \Pi_{k-1}^{\lambda*}(\Lambda^{n-1}) = \Pi_k^*(\Lambda^n) \circ d$ we see that the last equality may be rewritten in the form (the Green formula for the operator $A$)

$$\Delta(1) - \Delta^*(1) = d^\lambda \lambda(A).$$

Note that since $\lambda(\square)$ is a homomorphism $\lambda(\square) : \text{Diff}_k \Lambda^n \to \Lambda^{n-1}$ is an operator of order $\leq k - 1$. If $\Delta \in \text{Diff}_k(P, \mathcal{Q})$, then $\Delta(p, q) \in \text{Diff}_k \Lambda^n, \quad p \in P, \quad q \in Q$ and $\Delta(p, q)(1) = \langle \Delta(p), q \rangle$, where the brackets $\langle \cdot, \cdot \rangle$ denote the natural pairing $\mathcal{Q} \otimes Q \to \Lambda^n$. Hence Green's formula for the operator $A(p, q)$ may be written as (the general Green's formula)

$$\langle \Delta(p), q \rangle - \langle p, A^*(q) \rangle = d^\lambda \lambda \langle \Delta(p), q \rangle.$$
Consider the expression \( a(p, q) = X_k(A(p, q)) \). Let us show that it is a bidifferential operator, or to be more precise that we always have \( c_p \in \text{Diff}_{k-1}(Q, A^{n-1}) \), and \( b_q \in \text{Diff}_{k-1}(P, A^{n-1}) \) under certain conditions imposed on \( \lambda \), where \( b_q(p) = a(p, q) = c_p(q) \). Indeed, the operator \( c_p \) is the composition \( h_{k+1} \circ X_k \circ h_{k-1} \), where \( h(q) = q^* \circ A \circ p \), and \( h, v, X_k \) are homomorphisms while \( h_{k+1} \) is an operator of order \( k - 1 \).

The splitting \( \lambda : \text{Diff}_k A^n \to \text{Diff}_{k-1} A^n \) will be called homogeneous, if \( \lambda(\text{Diff}_i A^n) \subseteq D(\text{Diff}_{i-1} A^n) \), \( i = 1, \ldots, k \). In view of the projectivity of the modules \( \text{Diff}_i A^n / \text{Diff}_{i-1} A^n \) homogeneous splittings exist and are easily constructed by induction over \( k \). It turns out that \( b_q \in \text{Diff}_{k-1}(P, A^{n-1}) \), if \( \lambda \) is homogeneous. For every homomorphism \( \gamma : \text{Diff}_k A^n \to D(\text{Diff}_{k-1} A^n) \) denote by \( \gamma^+ : \text{Diff}_k A^n \to D(\text{Diff}_{k-1} A^n) \) the map which coincides with \( \gamma \) as a map of sets. Then \( \delta(\gamma^+)(\Delta) = \gamma(\delta(\Delta)) - \tilde{f} \cdot \gamma(\Delta) \), where \( \tilde{f} \) is the difference between the operator of right-hand and left-hand multiplication in \( D(\text{Diff}_{k-1} A^n) \). Hence by induction we immediately obtain the relation
\[
\delta_{f_1, \ldots, f_k}(\gamma^+)(\Delta) = \sum_{\sigma} (-1)^{\lvert \sigma \rvert} \tilde{f}_\sigma \gamma(\delta(\Delta)),
\]
where \( \sigma = \{i_1, \ldots, i_m\} \) is a subset of \( \{1, \ldots, s\} \), \( \lvert \sigma \rvert = m \), \( \tilde{f}_\sigma = f_{i_1} \cdots f_{i_m} \), \( \delta = \{1, \ldots, s\} \setminus \sigma = \{j_1, \ldots, j_{s-m}\} \), \( \delta_\sigma = \delta_{j_1} \cdots \delta_{j_{s-m}} \). Using this formula for \( s = k \) and \( \gamma = \lambda \circ v \) we see that \( \delta_{f_1, \ldots, f_k}(\gamma^+) = 0 \). Indeed, since \( \lambda \) is homogeneous we have \( \gamma(\delta(\Delta)) \in D(\text{Diff}_{m-1} A^n) \) and \( g_1 \circ \cdots \circ g_m D(\text{Diff}_{m-1} A^n) - i \) for all \( g_1, \ldots, g_m \in A \), so that all the terms in the right-hand side vanish. Thus \( \gamma^+ \) is an operator of order \( \leq k - 1 \). This proves the required statement, since the operator \( b_q \) is the composition of the homomorphism \( p \mapsto q^* \circ A \circ p \) from \( P \to \text{Diff}_1 A^n \), the operator \( \lambda^+ \) and the homomorphism \( h_{k-1} \circ (A^{n-1}) \circ \psi^+ \), where \( \psi^+ : D(\text{Diff}_{k-1} A^n) \to \text{Diff}_{k-1} A^n \) coincides with \( \psi \) as a map of sets.

If we do not require the homogeneity of \( \lambda \) we can only claim that \( b_q \in \text{Diff}_{2k-2}(P, A^{n-1}) \) since in this case \( \gamma^+ \) is an operator of order \( \leq 2k - 2 \).

Remark. In the following two cases the splitting \( \lambda \) is uniquely determined: \( k = 1 \) and \( \dim M = 1 \), since in this case the operator \( S_{1, k-1} \) is an isomorphism of \( D(\text{Diff}_{k-1} A^n) \) onto \( \text{Diff}_k A^n \).

2.6. Coordinates

In order to write out Green's formula in coordinates it suffices, by 1.4 describe the operator \( X_k \). In the notation of 1.4 one of the possible homogeneous splittings \( \lambda \) may be described by the formula
\[
\lambda(\Delta) = \sum_{\sigma} a_\sigma \left( \frac{\partial}{\partial x_\sigma} \otimes \frac{\partial}{\partial x_{i(\sigma)}} \right),
\]
where \( \Delta = \sum_{\lvert \sigma \rvert > 0} a_\sigma \left( \frac{\partial}{\partial x_\sigma} / \partial x_{i(\sigma)} \right) \), \( \sigma = \{i_1, \ldots, i_s\} \), \( \sigma = \{i_1, \ldots, i_{s-1}\} \), and \( i(\sigma) = i_s \).
under the condition that the indices $i_j$ are written in increasing order. Otherwise,

$$\lambda(A)(f) = \sum_{\sigma} \left( a_{\sigma} \frac{\partial |\sigma|}{\partial x_{\sigma}} \circ \frac{\partial f}{\partial x_{i(\sigma)}} \right) \omega.$$ 

Since the Spencer complex is exact and the module $\widetilde{\text{Diff}}_k A^n$ is projective we can find, for any two splittings $\lambda, \lambda'$, a homomorphism $h : \widetilde{\text{Diff}}_k A^n \to \hat{D}(\text{Diff}_{k-1} A^n)$ such that $A' - A = S_{r+1} 0 h$ and conversely. Hence any splitting may be obtained from the one above by adding the summand $S_{r+1} 0 h$, where $h$ is arbitrary.

Putting $\omega_i = (\partial / \partial x_i) \omega = (-1)^{i-1} dx_1 \wedge \ldots \wedge dx_i \wedge \ldots dx_n$ we obtain directly from the definition:

$$\mathcal{X}_A(\Delta) = \sum_{|\sigma| > 0} (-1)^{|\sigma|} \frac{\partial |\sigma|}{\partial x_{\sigma}} a_{\sigma} \omega_{i(\sigma)}.$$ 

In view of the above $\mathcal{X}_A(\Delta) = \mathcal{X}_A(\Delta) + dh'(\Delta)$, where $h' = \bar{\Delta}_{k-2} \circ \psi \circ h \circ v : \text{Diff}_k A^n \to A^{n-2}$ is an arbitrary homomorphism.

2.7. Transformations

In the notations of 1.5 put

$$F(\mathcal{X}_A) = F_A \circ \mathcal{X}_{\Delta} \circ F_{\text{Diff}}^{-1}, \quad F(\mu) = F_A \circ \mu \circ F_{\text{Diff}}^{-1},$$

$$X(\mathcal{X}_A) = X_A \circ \mathcal{X}_{\Delta} - \mathcal{X}_{\Delta} \circ X_{\text{Diff}}$$

or

$$X(\mathcal{X}_A)(\Delta) = X(\mathcal{X}_A(\Delta)) - \mathcal{X}_A([X, \Delta]),$$

$$X(\mu) = X_A \circ \mu - \mu \circ X_{\text{Diff}}.$$ 

Note that $F(\mu) = \mu$ and $X(\mu) = 0$. Indeed $F(\mu)(\Delta) = F(\mu(F^{-1}(\Delta))) = F(F^{-1}(\Delta^*)(F(1))) = \Delta^*(1) = \mu(\Delta)$, $X(\mu)(\Delta) = X(\mu(\Delta)) - \mu([X, \Delta]) = X(\Delta^*(1)) - [X, \Delta]^*(1) = 0$. A consequence of the above are the equalities $F(D_i(\mu)) = D_i(\mu)$, $X(D_i(\mu)) = 0$, where $D_i(\mu) : D_i(\text{Diff}^+ A^n) \to D_i(A^n) = A^{n-i}$, since $|F, D_i| = |X, D_i| = 0$ from functional considerations.

In the sequel we shall need the following:

**Proposition.** $X(\mathcal{X}_A) = d \circ v$, where $v \in \text{Diff}(\text{Diff}_k A^n, A^{n-2})$. 

In the notations of \( 2.4 \) \( \mathcal{K}_1 = \mathcal{D}^+ \circ \psi \circ \lambda \circ \nu. \) Putting \( \lambda' = \lambda \circ \nu, \)
\( \psi = \mathcal{D}^+ \circ \psi \) we get \( X(\mathcal{K}_1) = X(\psi) \circ \lambda' + \psi \circ X(\lambda'). \) Since \( \psi = \Pi \circ D(\mu) \) and \( X(\Pi) = 0, \) we have \( X(\psi) = X(\Pi) \circ D(\mu) + \Pi \circ X(D(\mu)) = 0 \) and therefore \( X(\mathcal{K}_1) = \psi \circ X(\lambda'). \) Now let us show that \( \text{im} \ X(\lambda') \subset \text{im} \ S_{2,k-2}. \) In view of the exactness of the Spencer complex it suffices to show that \( S_{1,k-1} \circ X(\lambda') = 0. \) But \( S_{1,k-1} \circ \lambda' = \nu, \) and \( X(S_{1,k-1}) = 0, \) \( X(\nu) = 0, \) since the operators \( S_{1,k-1} \) and \( \nu \) are natural. Hence \( 0 = X(\nu) = X(S_{1,k-1} \circ \lambda') = X(S_{1,k-1}) \circ \lambda' + S_{1,k-1} \circ X(\lambda') = S_{1,k-1} \circ X(\lambda). \) Further \( \text{im} \ S_{2,k-1} \oplus \text{im} \lambda \approx \text{Diff}_k \mathcal{A}^n \) and the module \( \text{im} \ S_{2,k-2} \) is projective. Therefore there is a splitting \( \alpha : \text{im} S_{2,k-2} \rightarrow \text{Diff}_{k-2}^+ \mathcal{A}^n, \) \( S_{2,k-2} \circ \alpha = 1. \) In view of Theorem 2.2 we have \( d \circ \nu = \psi \circ \alpha \circ X(\lambda'), \) we have \( d \circ \nu = -d \circ \nu \circ \alpha \circ X(\lambda') = \psi \circ S_{2,k-2} \circ \alpha \circ X(\lambda') = \psi \circ X(\lambda'). \)

### 3. QUADRATIC LAGRANGIANS AND THE EULER OPERATOR

Here we consider the linear Lagrangian formalism which yields the linear Euler–Lagrange equations. The complex \( \mathcal{S}_k \mathcal{P}Q = \mathcal{S}_k \mathcal{Q} \mathcal{P}Q \) (see 2.2) plays the main role in it.

#### 3.1. The complex \( \mathcal{S}_k \mathcal{P}Q \)

Let us show that this complex is acyclic. To do this we shall need the following elementary lemma.

**Lemma.** Suppose the complex of the form

\[
\text{im} a_i \rightarrow K_i \rightarrow a_{i-1} K_{i-1} \rightarrow \ldots \rightarrow a_0 K_0 \rightarrow 0
\]

consisting of projective \( A \)-modules, the differentials \( a_i \) being \( A \)-homomorphisms is acyclic. Then the modules \( \text{im} a_i \) are projective.

**Proof.** If \( \text{im} a_i \) is projective then there exists a splitting \( \alpha : \text{im} a_i \rightarrow K_{i+1} \) and \( K_{i+1} = a_i(\text{im} a_i) \oplus \text{im} a_{i+1}, \) so that the projectivity of \( \text{im} a_{i+1} \) follows. But since \( \text{im} a_i = K_i, \) the lemma is proved by induction. \( \square \)

Consider the complex \( \mathcal{S}_k \mathcal{P}^+ \), obtained from the complex \( \mathcal{S}_k \) by applying the functor \( \text{Hom}(P, \cdot) \):

\[
0 \leftarrow \mathcal{P} \leftarrow \text{Diff}_k^+(P, \mathcal{A}^n) \leftarrow \mathcal{S}_k^+ \mathcal{P} \mathcal{Q} \mathcal{P} \mathcal{Q} \leftarrow \text{Diff}_k^+(P, \mathcal{A}^{n-1}) \leftarrow \mathcal{S}_k^+ \mathcal{P} \mathcal{Q} \mathcal{P} \mathcal{Q} \leftarrow \ldots.
\]

If the module \( P \) is projective then all the terms of this complex are also projective. Moreover, \( \mu^+ \) and \( \mathcal{S}_k^+ \mathcal{P} \mathcal{Q} \mathcal{P} \mathcal{Q} \) are homomorphisms. We therefore have
COROLLARY. If \( P \) is projective then so is \( \mathcal{S}_{k-i,i}^+ \).

Suppose \( e: \mathcal{S}_{k}^+P \to \mathcal{S}_{k}^+P^+ \) is a chain map which is the identity as a map of sets and \( \alpha_i: \operatorname{im} \mathcal{S}_{k-i,i}^+ \to \operatorname{Diff}_{k-i-1}(P, A^{n-i-1}) \) is a splitting of the epimorphism \( \operatorname{Diff}_{k-i-1}(P, A^{n-i-1}) \to \operatorname{im} \mathcal{S}_{k-i,i}^+ \), which exists since \( \operatorname{im} \mathcal{S}_{k-i,i}^+ \) is projective. But \( e_i = e | \operatorname{Diff}_{k}(P, A^{n-i}) \) is an operator of order \( \leq k - i \) as well as \( e_i \), the map \( \beta_i = e_i^{-1} \circ \alpha_i \circ e_i | \operatorname{im} \mathcal{S}_{k-i,i}^+ \) is a differential operator of order \( < 2(k - i) \) satisfying \( e_i \circ \beta = 1 \). Thus if \( 0 \in \operatorname{Diff}(Q, \operatorname{Diff}_{k-i-1}(P, A^{n-i})) \) and \( e_i(0) = 0 \), then \( \operatorname{im} 0 \subset \operatorname{im} \mathcal{S}_{k-i,i}^+ \) since \( \mathcal{S}_{k}^+P \) is acyclic and therefore \( 0 = e_i^{-1}(0) = \beta_i(0) \), where \( \beta_i \circ 0 \in \operatorname{Diff}(Q, \operatorname{Diff}_{k-i-1}(P, A^{n-i})) \). Thus we have proved the following:

**Theorem.** The complex \( \mathcal{S}_{P}PQ \) is acyclic if the module \( P \) is projective.

3.2. The Involution \( w \)

There is a natural identification \( w = w_{P,Q} : \operatorname{Diff}_{k}(P, \operatorname{Diff}_{k}(P, Q)) \to \operatorname{Diff}_{k}(P, \operatorname{Diff}_{k}(P, Q)) \). Namely if \( A \in \operatorname{Diff}_{k}(P, \operatorname{Diff}_{k}(P, Q)) \), then the operator \( V = w(A) \) is defined by the formula \( V(p)(p') = A(p')(p) \). Obviously, \( w^2 = 1 \).

The operation \( w \) induces an involution (also denoted by \( w \)) in the complex \( \mathcal{S}_{P}PP \), which is the direct limit of the complex \( \mathcal{S}_{k}PP \) when \( k \to \infty \). This involution acts on elements \( A \) of \( \operatorname{Diff}_{k}(P, \operatorname{Diff}_{k}(P, A^*)) \), in the manner described above. If \( 0 \in \operatorname{Diff}(P, \hat{P}) \), we put \( w(0) = 0 \).

**Proposition.** The operation \( w \) is an automorphism of the complex \( \mathcal{S}_{P}PP \).

**Proof.** The fact that \( w \circ \mathcal{F} = \mathcal{F} \circ w \) is obvious. Let us show that \( w \circ \bar{w} = \bar{w} \circ w \). If \( A \in \operatorname{Diff}_{k}(P, \operatorname{Diff}_{k}(P, A^*)) \), we shall write \( A_p \) instead of \( A(p) \), \( p \in P \). Suppose also that \( 0 = \bar{w}(A) \), \( V = w(A) \). Then \( \langle 0^*(p), p' \rangle = \langle A^*(p'), p' \rangle = \langle \bar{A}(p'), p' \rangle = \langle j_k(p), \bar{A}_{p} \rangle = d\mathcal{X}_{k} \bar{A}(p', j_k(p)) \). On the other hand, \( \bar{A}(V)(p)(p') = \langle \bar{A}(p)(p'), p' \rangle = \langle 1, \bar{A}(p') \rangle = d\mathcal{X}_{k} \bar{A}(p'). \) Since \( \langle \bar{A}(p), \bar{A}_{p} \rangle = \bar{A}_{p} = \langle \bar{A}(p), \bar{A}_{p} \rangle = \langle 1, \bar{A}(p') \rangle = d\mathcal{X}_{k} \bar{A}(p, p') \), it suffices to show that \( \bar{A}(p', j_k(p)) = \bar{A}(p', 1) \). But \( \bar{A}(p', j_k(p))(f) = \langle j_k(p), \bar{A}(f) \rangle = \bar{A}(f, p) = \langle \bar{A}(f, p'), 1 \rangle = \bar{A}(f, p') = \bar{A}(p')(f) \).

Consider the complexes \( \mathcal{S}_{P}P = \ker(1 - w) \) and \( \mathcal{S}_{P}P = \ker(1 + w) \). Then the Theorem 3.1 has the following:

**Corollary.** The complexes \( \mathcal{S}_{P}P \) and \( \mathcal{S}_{P}P \) are acyclic.

By a polydifferential operator from the \( A \)-module \( P \) into the \( A \)-module \( Q \) we mean the \( \mathbb{F} \)-polylinear function \( (p_1, \ldots, p_s) \mapsto \Delta(p_1, \ldots, p_s) \in Q, p_i \in P \), possessing the following properties: for any \( i, 1 \leq i \leq s \), and any family \( p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_s \in P \), the correspondence \( p_i \mapsto \Delta(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_s) \) determines a differential operator whose order is bounded from above when \( p_1, \ldots, p_{i-1} \) vary. The upper limit of these orders is called the order with
respect to the $i$th variable. The module of polydifferential operators whose order with respect to the $i$th variable is $\leq k_i$ shall be identified with the module

$$\text{Diff}_{k_1}(P, \text{Diff}_{k_2}(P, \cdots \text{Diff}_{k_s}(P, Q) \cdots ) = \text{Diff}_{(k_1,\ldots,k_s)}(P, Q)$$

by identifying $\Delta \in \text{Diff}_{(k_1,\ldots,k_s)}(P; Q)$ with the polydifferential operator

$$\nabla : (p_1,\ldots,p_s) \mapsto (\cdots (\Delta(p_1))(p_2) \cdots (p_s) \cdots ) = \nabla(p_1,\ldots,p_s).$$

We may consider either all the symmetric or all the anti-symmetric $s$-differential $Q$-valued operators of order $\leq k$ which constitute $A$-modules denoted, respectively, $\text{Diff}_{\leq k_s}(P, Q)$, $\text{Diff}_{\leq k_s}^\text{sym}(P, Q)$, $\text{Diff}_{\leq k_s}^\text{alt}(P, Q)$. In particular, $i$-dimensional chains of the complex $\mathcal{S}_{\text{sym}}(\mathcal{S}_{\text{alt}})$ may be identified with $\text{Diff}_{\leq k_s}(P, A^{n-i})$ $(\text{Diff}_{\leq k_s}^\text{alt}(P, A^{n-i}), i \geq 0$, while $(-1)$-dimensional chains are identified with the module $\text{Diff}_{\text{sym}}(P, \hat{P})(\text{Diff}_{\text{alt}}(P, \hat{P}))$ of self-adjoint (skewadjoint) operators from $\text{Diff}(P, \hat{P})$. The symbols $\text{Diff}_{\ast,s}$, $\text{Diff}_{\ast,s}^\text{sym}$ and $\text{Diff}_{\ast,s}^\text{alt}$ denote the "direct limits" of the symbols $\text{Diff}_{k_s}^\ast$, $\text{Diff}_{k_s}^\text{sym}$, $\text{Diff}_{k_s}^\text{alt}$ when $k \to \infty$. Note that any symmetric polydifferential operator $\Delta$ is uniquely determined from its values on the diagonal, i.e., by a nonlinear differential operator of the form $p \mapsto \Delta(p,\ldots,p)$. When $s = 2$ such operators will be called quadratic. Thus, the chains of the complex $\mathcal{S}_{\text{sym}} P$, except the $(-1)$-dimensional ones, may be identified with quadratic operators from $P$ into $A^i$, $0 \leq i \leq n$.

3.3. Linear Lagrangian Formalism

**DEFINITION.** The operator $L \in \text{Diff}_{k_1}^\text{sym}(P, A^n)$ will be called the density of the quadratic Lagrangian of order $\leq k$, while the coset of the operator $L$ with respect to $\mathcal{S}_{\text{sym}}(\text{Diff}_{k_1}^\text{sym}(P, A^{n-1}))$ in $\text{Diff}_{k_1}^\text{sym}(P; A^n)$ is the quadratic Lagrangian corresponding to the density $L$.

Denote the set of all quadratic Lagrangians on $P$ by $\mathcal{L}(P)$; we see that, in view of the fact that $\mathcal{S}_{\text{sym}} P$ is acyclic, the operator $\tilde{\mu}$ induces an isomorphism of the $A$-module $\mathcal{L}(P)$ on a submodule of the module $\text{Diff}(P, \hat{P})$, consisting of all self-adjoint operators. This isomorphism will be denoted by $\mathcal{F}$ and called the Euler operator.

**DEFINITION.** The Euler–Lagrangian equation corresponding to the Lagrangian $\mathcal{L} \in \mathcal{L}(P)$ is the equation $\mathcal{F}(\mathcal{L}) = 0$.

If $L$ is a density of the Lagrangian $\mathcal{L}$, then obviously $\mathcal{F}(\mathcal{L}) = \tilde{\mu}(L)$.

The fact that the quadratic densities introduced above coincide with the "ordinary" ones follows, for example, from their expressions in coordinates. The motivation of the definition of Lagrangians given above will become
clear when we consider the nonlinear Lagrangian formalism. Finally, the fact
that the Euler–Lagrange equations introduced above coincide, within a given
local coordinate system, with the "ordinary" ones can be shown by
expressing them in coordinates. Instead of doing this we shall show directly
that the solutions of these equations are the extremal values of the
corresponding variational problem.

Thus suppose \( L \in \text{Diff}^{\text{sym}}(P, A^n) \) and \( L(p, q) = L_\rho(q) \), \( p, q \in P \). Consider
a variation of the form \( I.(p, p) \) (equal to the value of the density at \( p \in P \))
corresponding to the change of the variable \( p \mapsto p + \varepsilon h, \varepsilon \in \mathbb{F}, h \in P \). Then
\( L(p + \varepsilon h, p + \varepsilon h) = L(p, p) + 2\varepsilon L(p, h) + \varepsilon^2 L(h, h) \). The fact that \( p \) is an
extremum means that \( \int_V L(p, h) = 0 \) for all \( h \in P \), \( h \mid \partial V = 0 \), where \( V \subset M \)
is the domain in which we are solving the variational problem. Green’s
formula for the operator \( L_\rho(h, 1) \), in view of the relation \( L_\rho(h, 1)(1) = L_\rho(h) = L(p, h) \)
can be written in the form
\[
L(p, h) - \langle L_\rho^*(1), h \rangle = d\mathcal{K}_A(L_\rho(h, 1)).
\]
Therefore
\[
\int_V L(p, h) = \int_V \{ \langle L_\rho^*(1), h \rangle + d\mathcal{K}_A(L_\rho(h, 1)) \}.
\]
Suppose the variation \( h \) has the carrier \( W \). Then the carrier of the form
\( \mathcal{K}_A(L_\rho(h, 1)) \) is contained in \( W \) since \( h \mapsto \mathcal{K}_A(L_\rho(h, 1)) \) is a differential
operator (see 2.5). If \( \partial V \cap W = \emptyset \), then \( \int_V L(p, h) = \int_V \langle L_\rho^*(1), h \rangle \). Since the
expression \( \langle L_\rho^*(1), h \rangle \) is \( A \)-linear with respect to \( h \) in accordance to the
Dubois–Raymond lemma, the equality \( \int_V \langle L_\rho^*(1), h \rangle = 0 \) for all \( h \) with
compact carrier contained in \( V \) implies that \( L_\rho^*(1) = 0 \). It remains to note
that \( L_\rho^*(1) = \bar{\mu}(L)(p) = \mathcal{L}(L)(p) \).

Introducing the notation \( \bar{\mu}(L) = \mathcal{L} \), we may rewrite the previous formula
as
\[
L(p, h) - \langle \mathcal{L}(p), h \rangle = d\mathcal{K}_A(L_\rho(h, 1)).
\]
Further it shall be referred to as the Green \( \mathcal{L} \)-formula. Note that \( \mathcal{L} \), being
the composition \( P \rightarrow L \text{Diff}_k(P, A^n) \rightarrow u \tilde{P} \) of operators of order \( \leq k \) is an
operator of order \( \leq 2k \).

3.4. To Describe the Symbol of \( \mathcal{L} \)

Let us consider the following commutative diagram
\[
\begin{array}{ccccccccc}
P & \xrightarrow{j_k} & \mathcal{F}^k(P) & \xrightarrow{\alpha} & \ker v_{k,k-1} & \xleftarrow{i} & S^k(A^1) \otimes P \\
\downarrow{\bar{\tau}} & & \downarrow{I} & & \downarrow{\bar{\sigma}} & & \\
\tilde{P} & \xrightarrow{u} & \text{Diff}_k(P, A^n) & \longrightarrow & \text{smbl}(P, A^n) & \xrightarrow{\chi_k} & \text{Hom}(S^k(A^1, \tilde{P})),
\end{array}
\]
\( \alpha \) (resp. \( \beta \)) being the natural inclusion (resp., projection).
\[ i(df_1 \otimes \cdots \otimes df_n) = 1/k! \left[ \delta_{f_1, \ldots, f_n}(j_k)(p) \right], \quad \bar{L} \] is the corresponding \( L \) homomorphism and \( \bar{\sigma}_L = \beta \circ \bar{L} \circ \alpha \). Consider the map \( \chi_k \circ \bar{\sigma}_L \circ i \in \text{Hom}(S^k(A^1) \otimes P, \text{Hom}(S^k(A^1), \tilde{P})) = B \). The module \( B \) may naturally be identified with \( \text{Hom}(S^k(A^1) \otimes S^k(A^1), \text{Hom}(P, \tilde{P})) \). Suppose \( \sigma_L \) is the element of this module corresponding to \( \chi_k \circ \bar{\sigma}_L \circ i \).

**Proposition.** The composition \( S^{2k}(A^1) \to S^k(A^1) \otimes S^k(A^1) \to \text{Hom}(P, \tilde{P}), \) were the first map is the natural inclusion, equals \((-1)^k \text{smbl}_{2k}(\mathcal{S}_L)\).

**Proof.** It suffices to prove that for all \( f \in A, p \in P \) the value of the map

\[ (-1)^k \text{smbl}_{2k}(\mathcal{S}_L) : S^{2k}(A^1) \otimes P \to \tilde{P} \]

on the element \((df)^k \otimes p\) coincides with the value of the compositions

\[ S^{2k}(A^1) \to S^k(A^1) \otimes S^k(A^1) \to \text{Hom}(P, \tilde{P}) \]

on the element \((df)^{2k}\) applied to \( p \). The latter equals

\[ (\chi_k \circ \beta \circ \bar{L} \circ \alpha \circ i)((df)^k \otimes p) \]

\[ = \frac{1}{k!} \left[ (\chi_k \circ \beta)(\bar{L}(\delta^k_f(j_k)(p))) \right] ((df)^k) \]

\[ = \frac{1}{k!} \left[ (\chi_k \circ \beta)(\delta^k_f(L)(p)) \right] ((df)^k) = \frac{1}{(k!)^2} \delta^k_f(\delta^k_f(L)(p)). \]

On the other hand, since \( \mathcal{S}_L = \mu \circ L \) while \( \mu \) and \( L \) are d.o. of order \( \leq k \) we have

\[ \text{smbl}_{2k}(\mathcal{S}_L)((df)^k \otimes p) \]

\[ = \frac{1}{(2k!)} \delta^k_f(\mu \circ L)(p) \]

\[ = \frac{1}{(2k)!} C^k_{\alpha}(\delta^k_f(\mu))(\delta^k_f(L)(p)) = \frac{1}{(k!)^2} \left[ (\delta^k_f(L)(p))^* \right] (1) \]

\[ = \frac{(-1)^k}{(k!)^2} (\delta^k_f(\delta^k_f(L)(p)))^* (1) = \frac{(-1)^k}{(k!)^2} \delta^k_f(\delta^k_f(L)(p)). \]

In this calculation we used the fact that \( \delta^k_\alpha(\mu)(A) = \delta^k_\alpha(A^*)(1), \delta^k_\alpha(A^*) = -\delta^k_\alpha(A)^* \) and also the fact that \( {}^*|_{\text{Hom}(P, \Lambda^n)} : \text{Hom}(P, \Lambda^n) \to \text{Hom}(A, \tilde{P}) = \tilde{P} \) is the identity. \( \blacksquare \)
4. CONSERVATION LAWS IN LINEAR THEORY

Using the apparatus of Spencer sequences once again, we shall consider here the theory of linear conservation laws, and latter establish its relationship to the linear Noether theorem.

4.1. Suppose $A \in \text{Diff}_s(P, Q)$ and $E = \{ A(u) = 0 \}$ is the corresponding equation.

**Definition.** ("naive" version). The operator $V \in \text{Diff}(P, A^{n-1})$ will be called the density of the linear conservation law or c-density, if the form $V(p) \in A^{n-1}$ is closed as soon as $p \in P|_U$ is a solution of the equation $E$ on the open set $U \subset M$.

The integral of the form $V(p)$ on a certain $(n-1)$-dimensional submanifold $\mathcal{Y} \subset M$ is an "invariant expression" in the sense that it does not change when $\mathcal{Y}$ is replaced by a submanifold, homologic to it. In the classic theory submanifolds of the form $t = \text{const.}$ in $\mathbb{R}^{n+1} = \{(x, t)\}$ play the role of $\mathcal{Y}$ and in this case $\int_{t=t_0} V(p)$ does not depend on the time $t_0$, which justifies the terminology.

Now assume that the equation $E$ is such that any of its formal solutions of order $k$ at each point $x \in M$ can be extended to a real one in some neighbourhood $U \ni x$. Such equations will be called FP-equations; note that practically interesting equations are usually of this type.

**Proposition.** If $E$ is a FP-equation then for some operator $\Box \in \text{Diff}(P, P')$ satisfying $\Box(p) = 0$ as soon as $A(p) = 0$, there exists an operator $\Box' \in \text{Diff}(Q, P')$ satisfying $\Box = \Box' \circ A$.

**Proof.** Suppose $E_k \subset \mathcal{F}^{k+s}(P)$ is the submodule generated by the formal solutions of equations $E$ of order $k+s$. It coincides with the submodule of $\mathcal{F}^{k+s}(P)$ generated by elements of the form $j_{k+s}(p)$, where $p$ is a local solution, since $E$ is a FP-equation. If $\Box \in \text{Diff}_{k+s}(P, P')$ and $\varphi_\Box : \mathcal{F}^{k+s}(P) \to P'$ is the corresponding homomorphism, then $\ker \varphi_\Box \supset E_k$. Since $E_k = \ker \varphi_{A_k}$, where $A_k = j_k \circ \Delta : P \to \mathcal{F}^k(Q)$, it follows that $\varphi_{\Box}$ may be represented as the composition $\mathcal{F}^{k+s}(P) \to^{\varphi_{A_k}} \mathcal{F}^k(Q) \to P'$. It remains to put $\Box' = \varphi \circ j_k$.

**Corollary.** If $E$ is a FP-equation while $V$ is its linear c-density, then $d \circ V = \Box \circ A$, where $\Box \in \text{Diff}(Q, A^n)$, and conversely.

4.2. Corollary 4.1 enables us, in the case of FP-equations to give the following homological meaning to c-densities. Consider the complex $\mathcal{F}_k P$ ($k = \infty$ is admissible)
obtained from $\mathcal{P}$ by deleting the fragment $\to \mathcal{P} \to 0$. The homology of this complex, since $\mathcal{P}$ is acyclic, is trivial in all dimensions different from $n$, while the $n$-dimensional homology coincides with $\mathcal{P}$. The operator $\Delta \in \text{Diff}(P, Q)$ generates the chain map $\mathcal{P}^\Delta : \mathcal{P} \to \mathcal{P}^\Delta$. Namely if $\mathcal{P} \in \text{Diff}_k(Q, \Lambda^{n-k})$, then $\mathcal{P}^\Delta(\mathcal{P}) = \mathcal{P} \circ \Delta \in \text{Diff}_{k+1}(P, \Lambda^{n-1})$. Obviously we may assume that the complex $\mathcal{P} = \mathcal{P}^\Delta$ is filtered by the subcomplexes $\mathcal{P}$, $k \geq 0$, while the map $\mathcal{P} = \mathcal{P}^\Delta$ raises the filtration by $s$.

Consider the complex $\text{coker } \mathcal{P}$ and denote the natural projection $\mathcal{P} \to \text{coker } \mathcal{P}$ by $\alpha = \alpha_\Delta$. Then Corollary 4.1 is obviously equivalent to the fact that the following two statements hold simultaneously: $\nabla$ is a c-density of the FP equation $\Delta = 0$, $\alpha(\nabla)$ is a cocycle of the complex $\text{coker } \mathcal{P}$. The Definition 4.1 is unsatisfactory since it is not constructive and requires knowing the solution of the equation $\Delta = 0$. In particular, for this reason it is impossible in general to describe the set of c-densities of a given equation. Therefore we shall set down the following definition, motivated by the previous arguments.

**Definition.** The operator $\nabla \in \text{Diff}(P, \Lambda^{n-1})$ is called the c-density of the equation $\Delta = 0$, if $\alpha_\Delta(\nabla)$ is a cocycle of the complex $\text{coker } \mathcal{P}$.

**Remark.** We can also say that Definition 4.1 becomes Definition 4.2 if the notion of solution is extended to include formal solutions.

4.3. If $p \in \ker \Delta$, then $\nabla(p) = \nabla'(p)$, whenever $\nabla$ and $\nabla'$ are c-densities such that $\nabla' = \nabla + \Delta \circ \Delta$. Moreover, $\int_M \nabla(p) = \int_M \nabla'(p)$ for all $\Lambda^{n-1} \subset M$ and $p \in \ker \Delta$, whenever $\nabla' = \nabla + d \circ \Delta \in \text{Diff}(P, \Lambda^{n-1})$. Thus the integral $\int_M \nabla(p)$ depends only on the cohomology class of the cocycle $\alpha(\nabla)$ of the complex $\text{coker } \mathcal{P}$. This motivates the following:

**Definition.** A linear conservation law for the equation $\Delta = 0$ is any class of $(n-1)$-dimensional cohomology of the complex $\text{coker } \mathcal{P}$. Let us compute the cohomology of the complex $\text{coker } \mathcal{P}$, assuming that $\ker \mathcal{P} = 0$. In view of the fact that $H^i(\mathcal{P}) = H^i(\Delta \mathcal{P}) = 0$, $i \neq n$, the exact cohomology sequence corresponding to the short exact sequence of complexes $0 \to \mathcal{Q} \to \mathcal{P} \to \ker \mathcal{P} \to 0$, is of the form $0 \to H^{n-1}(\ker \mathcal{P}) \to H^n(\mathcal{Q}) \to H^n(\mathcal{P}) \to H^n(\ker \mathcal{P}) \to 0$. As we already noticed $H^n(\mathcal{Q}) = \mathcal{Q}$, $H^n(\mathcal{P}) = \mathcal{P}$. Here the map $H^n(\mathcal{Q}) \to H^n(\mathcal{P})$ corresponds to the map $\mathcal{Q} \to \Delta \mathcal{P}$. This follows from the commutativity of the diagram.
which may be checked directly. Hence \( H^n(\text{coker } J_\Delta) = \text{coker } A^*, H^{n-1}(\text{coker } J_\Delta) = \ker A^* \).

Now let us try to find out the significance of the condition \( \ker J_\Delta = 0 \). To do this, consider the graded complexes \( J_P_{\text{gr}} \) and \( J_Q_{\text{gr}} \), associated with the filtered complexes \( J_P \) and \( J_Q \), and the map \( J_{\text{gr}}^A : J_Q_{\text{gr}} \to J_P_{\text{gr}} \), generated by \( J_\Delta \). If \( \ker J_{\text{gr}}^A = 0 \) then obviously \( \ker J_\Delta = 0 \).

Note further that for projective module \( S \) smbl(A(S, \tau) = \text{Diff}(S, A)/\text{Diff}(S, T) = \text{Hom}(S, \text{smbl} A T) \) and that \( J_{\text{gr}}^A(\text{smbl} k\square) = \text{smbl} k+3\square \circ \text{smbl} A \). Therefore \( \ker J_{\text{gr}}^A = 0 \), if \( \text{smbl} A \)-homomorphism \( \beta : \text{Hom}(\text{smbl} Q, \text{smbl} A^{n-1}) \to \text{Hom}(\text{smbl} P, \text{smbl} A^{n-1}) \), sending \( \text{smbl} \square \) into \( \text{smbl} \square \circ \text{smbl} A \), is injective. This is necessarily so if \( X \subseteq Y = X \), where \( X \) and \( Y \) are the carriers of the \( \text{smbl} A \)-modules \( \text{smbl} Q \) and \( \text{coker}(\text{smbl} A) \), respectively, while the modules \( P \) and \( Q \) are geometric. In the case when \( \xi, \eta \) are vector bundles over \( M \) and \( P = \Gamma(\xi), Q = \Gamma(\eta) \), the map \( \text{smbl} A \) may be realized as an element \( \tilde{\beta} \in \text{Hom}(\pi^*(\xi), \pi^*(\eta)) \), where \( \pi : T^*(M) \to M \) is the natural projection. Then if \( \dim \eta > 0 \), \( X = T^*(M) \) and \( Y = \text{char} A^* \) is the closure of the set \( \{ \theta \in T^*(M) \mid \text{coker } \tilde{\beta}_\theta \neq 0 \} \), consisting of characteristic covectors of \( A^* \). Putting all this together we can state the following:

**Theorem.** If the set \( \text{char} A^* \) is nowhere dense in \( T^*(M) \), then \( \ker J_\Delta = 0 \) so that the group \( 3(A) \) of linear conservation laws is isomorphic to \( \ker A^* \).

**Remark.** The assumption of this theorem is "almost" equivalent to the equation being non-overdetermined.

4.4. Green's formula for the operator \( A \) enables us to indicate the isomorphism \( A^* \approx 3(A) \) explicitly. Indeed, suppose \( A^*(\hat{q}) = 0 \), \( \hat{q} \in \hat{Q} \), \( A \in \text{Diff}(P, Q) \). Then Green's formula for \( p \in P \), \( \hat{q} \in \hat{Q} \) may be written in the form \( \langle A(p), \hat{q} \rangle = d\text{K}_A(A(p, \hat{q})) \). Therefore if \( A(p) = 0 \) then \( d\text{K}(A(p, \hat{q})) = 0 \), i.e., the operator \( p \mapsto \text{K}_A(A(p, \hat{q})) \) is a \( c \)-density. Thus we obtain the map \( \zeta : \ker A^* \to 3(A) \), where \( \zeta(\hat{q}) \) is the conservation law corresponding to the \( c \)-density \( p \mapsto \text{K}_A(A(p, \hat{q})) \). Let us show that \( \zeta \) is an isomorphism in Theorem 4.3.

If \( \nabla \) is the \( c \)-density and \( [\nabla] \) the corresponding conservation law, the element \( \hat{q} \) from \( \ker A^* \) corresponding to it in accordance to the construction given in 4.3 may be found as \( \hat{q} = \mu(\square) \), where the operator \( \square \in \text{Diff}(Q, A^\kappa) \) is defined from the relation \( d \circ \nabla = \square \circ A \). If we understand the element \( \hat{q} \) as
an operator from $Q$ to $A^n$ the equality $\langle A(p), \hat{q} \rangle = \mathcal{K}_A(A(p, q))$ may be rewritten in operator form as $\hat{q} \circ A = d \circ V$, where $V(p) = \mathcal{K}_A A(p, \hat{q})$. Hence $\mathcal{I} = \hat{q}$ and $\mu(\hat{q}) = \hat{q}$.

Now we no longer assume that $\ker \mathcal{S}_\Delta = 0$. Then the exact cohomology sequences corresponding to the exact sequences of complexes $0 \to \ker \mathcal{S}_\Delta \to \mathcal{S}_\Delta Q \to \im \mathcal{S}_\Delta \to 0$ and $0 \to \im \mathcal{S}_\Delta \to \mathcal{S}_\Delta P \to \ker \mathcal{S}_\Delta \to 0$ give us

$$0 \to H^{n-1}(\im \mathcal{S}_\Delta) \xrightarrow{i_0} H^n(\ker \mathcal{S}_\Delta) \xrightarrow{i_1} \hat{Q} \xrightarrow{i_2} H^n(\im \mathcal{S}_\Delta) \to 0,$$

$$0 \to H^{n-1}(\coker \mathcal{S}_\Delta) \xrightarrow{j_1} H^n(\im \mathcal{S}_\Delta) \xrightarrow{j_2} P \to H^n(\coker \mathcal{S}_\Delta) \to 0,$$

where $j_2 \circ i_2 = A^*$. Hence we see that $i_2$ induces an isomorphism of $\ker A^*/\im i_1$ onto $H^{n-1}(\coker \mathcal{S}_\Delta) = \mathcal{H}(A)$ the composition $\ker A^* \to \ker A^*/\im i_1 \to H^{n-1}(\coker \mathcal{S}_\Delta) = \mathcal{H}(A)$ coinciding with $\zeta$. Thus we have obtained the following:

**Proposition.** The sequence

$$0 \to H^{n-1}(\im \mathcal{S}_\Delta) \xrightarrow{i_0} H^n(\ker \mathcal{S}_\Delta) \xrightarrow{i_1} \ker A^* \xrightarrow{\zeta} \mathcal{H}(A) \to 0$$

is exact.

4.5. In connection with the Noether theorem another expression of the map $\zeta$ for the Euler–Lagrange equations will be useful. Suppose $L \in \text{Diff}_{\text{cv}}(P, A^n)$ and $\mathcal{A} = \mathcal{E}_1$. Taking the difference of the Green $\mathcal{E}_\cdot$-formulas (see 3.3) for the pairs $(p, q)$ and $(q, p)$, $p, q \in P$, and then using the fact that $L(p, q) = L(q, p)$ we get $\langle A(q), p \rangle - \langle A(p), q \rangle = dL(p, \hat{q}) - L_q(p, 1)$, which implies that the form $c(p, q) = c_{\mathcal{A}_1}(p, q) = \mathcal{K}_A(L(p, q, 1) - L_q(p, 1))$ is closed whenever $A(p) = A(q) = 0$. In other words, the operator $\nabla_P : P \to A^{n-1}$, $\nabla_P(q) = c(p, q)$ is the linear $c$-density for the equation $A = 0$. If $A(p) = 0$, then $(p^* \circ A)(q) = d\nabla_P(q)$. Together with the argument in 4.3 this shows that the conservation law corresponding to the $c$-density $\nabla_P$ coincides with $\zeta(p)$.

4.6. The “linear algebra” developed above enables us to construct nonlinear conservation laws for linear equations as well.

**Definition (“naive” version).** The nonlinear operator $\nabla : P \to A^{n-1}$ is called a “nonlinear” $c$-density for the equation $A = 0$, if $d\nabla(p) = 0$ for all local solutions $p$ of this equation.

It is natural to call two $c$-densities $\nabla_1$ and $\nabla_2$ equivalent, if there is an operator $\Box : P \to A^{n-2}$ such that $\nabla_2 = \nabla_1 + d \circ \Box$ on the equation $A = 0$, and to call an equivalence class of densities a conservation law.
Remark. A non-naive variant of this definition will be given in the sequel when we construct the nonlinear Lagrangian formalism.

Suppose $A_i : P_i \to Q_i$, $i = 1, 2$. The nonlinear differential operator $\Box_i : P_i \to P_i$ will be called a morphism of the equation $A_1 = 0$ into the equation $A_2 = 0$ if it sends formal solutions of the first equation into formal solutions of the second. In particular, $\Box_i(\ker A_1) \subseteq \ker A_2$. If $P_1 = P_2$ we shall say we are considering an endomorphism. The following important fact immediately follows from subsections 1.6 and 4.4 and definitions.

**Theorem.** (1) Suppose $A \in \text{Diff}(P, Q)$ and $\Box_1$ is an endomorphism of the equation $A = 0$, while $\Box_2$ is a morphism of the equation $A = 0$ into $A^* = 0$. Then the operator $p \mapsto \mathcal{K}_A(A(\Box_1 p, \Box_2 p))$ is the $c$-density of the equation $A = 0$ and its equivalence class does not depend on $\lambda$.

(2) If $A = \mathcal{E}_L$ while $\Box_1$ is an endomorphism of the equation $A = 0$ then the operator $p \mapsto c_{l_A}(\Box_1 p, \Box_2 p)$ is the $c$-density of the equation $A = 0$ whose equivalence class does not depend on $\lambda$.

(3) The $c$-densities described in (1) and (2) are equivalent when $A = \mathcal{E}_L$.

Remark 2. If we do not exclude multivalued conservation laws we can take, in the role of $\Box_i$, $i = 1, 2$, the corresponding Backlund transformation.

Remark 3. If $[A, \Box] = 0$ then $\Box$ is an endomorphism of the equation $A = 0$ and therefore determines the $c$-density $p \mapsto \mathcal{K}_A(A(\Box p, p))$, when $A = 0$ is the Euler-Lagrange equation.

Note that an infinitesimal symmetry of the Lagrangian is at the same time a symmetry of the equation $\mathcal{E}_L = 0$, i.e., its automorphism (see below). Thus, the theorem stated above, indicates a method for constructing conservation laws which is more general (and as we feel more transparent) than the classical Noether theorem. Further the relationship between the theorem and Noether theorem will be specified.

Let us stress in conclusion that we have obtained a method for constructing conservation laws from symmetries for arbitrary linear equations and not only from the Euler-Lagrange equations.

### 5. Automorphisms and the Linear Noether Theorem

In this section we shall consider the linear Noether theorem in algebraic form in order to compare it with Theorem 4.5. To do this it is necessary to give an algebraic description of the transformation laws of the objects which this theorem involves. This is done in the first half of this section.

5.1. Suppose $F : A_1 \to A_2$ and $F_p : P_1 \to P_2$ are as in 1.5. If
where $F_{\text{Diff}}(\mathcal{A}) = F_{\mathcal{A}^n} \circ \mathcal{A} \circ F_{\mathcal{A}^n}^{-1}$, $\mathcal{A} \in \text{Diff}_k(P_1, \mathcal{A}^n(A_1))$, i.e., $F_{\mathcal{A}}(L) = F_{\text{Diff}} \circ L \circ F_{\mathcal{A}}^{-1}$. For this reason for $(X_{\mathcal{A}}, X) \in \text{Der} P$ we should put $X_{\mathcal{A}}(L) = X_{\text{Diff}} \circ L - L \circ X_{\mathcal{A}}$, where $X_{\text{Diff}}(\mathcal{A}) = X_{\mathcal{A}} \circ X_{\mathcal{A}}$. In particular, if $L$ is the density of the Lagrangian, i.e., $L \in \text{Diff}_k^\text{sym}(P_1, \mathcal{A}^n(A_1))$, then $F_{\mathcal{A}}(L) \in \text{Diff}_k^\text{sym}(P_2, \mathcal{A}^n(A_2))$ and is called a transformation of the density $L$ while $X_{\mathcal{A}}(L) \in \text{Diff}_k^\text{sym}(P_2, \mathcal{A}^n(A_2))$ is its variation under the infinitesimal automorphism $X_{\mathcal{A}} \in \text{Der} P$. Since, for obvious reasons, $F_{\mathcal{A}} \circ \mathcal{F} = \mathcal{F} \circ F_{\mathcal{A}}$ and $X_{\mathcal{A}} \circ \mathcal{F} = \mathcal{F} \circ X_{\mathcal{A}}$ and also $F(\mu) = \mu$ and $X(\mu) = 0$ (see 2.7) we see that

$$F_{\mathcal{A}}: \mathcal{F}_\text{sym} P_1 \to \mathcal{F}_\text{sym} P_2, \quad X_{\mathcal{A}}: \mathcal{F}_\text{sym} P \to \mathcal{F}_\text{sym} P$$

are homomorphisms of chain complexes and generate the maps $\mathcal{J}(P_1) \to \mathcal{J}(P_2)$, $\mathcal{J}(P) \to \mathcal{J}(P)$, respectively, which we shall continue to denote by $F_{\mathcal{A}}$ and $X_{\mathcal{A}}$.

Setting $\bar{L} = X_{\mathcal{A}}(L)$ for brevity we get $\bar{L}_{\mathcal{A}} = X_{\mathcal{A}} \circ L_{\mathcal{A}} - L_{\mathcal{A}} \circ X_{\mathcal{A}} - L_{\mathcal{X}_{\mathcal{A}}(p)}$ so that

$$\bar{L}(p, q) = X(L(p, q)) - L(p, X(p)) - L(X(p, q), q), \quad p, q \in P. \quad (5.1.1)$$

Further $\bar{L}_{\mathcal{A}}(q, 1) = \bar{L}_{\mathcal{A}} \circ q = X \circ L_{\mathcal{A}}(q, 1) - L_{\mathcal{A}} \circ X(p, q) - L_{X_{\mathcal{A}}(p)}(q, 1)$. Noticing that $L_{\mathcal{A}} \circ X(p, q) = L_{\mathcal{A}} \circ q \circ X + L_{\mathcal{A}} \circ X_{\mathcal{A}}(q, 1)$, we can rewrite the last expression in the form $X \circ L_{\mathcal{A}}(q, 1) - L_{\mathcal{A}}(q, 1) \circ X - L_{\mathcal{A}}(X(p, q), 1) - L_{X_{\mathcal{A}}(p)}(q, 1)$. Thus

$$\bar{L}_{\mathcal{A}}(q, 1) = [X, L_{\mathcal{A}}(q, 1)] - L_{\mathcal{A}}(X_{\mathcal{A}}(q), 1) - L_{X_{\mathcal{A}}(p)}(q, 1). \quad (5.1.2)$$

Formulas (5.1.1) and (5.1.2) will be used later to derive Noether's theorem. Using Green's $\mathcal{L}$-formula let us transform the second one of these relations: $X(L(p, q)) = d(X \mathcal{A} L(p, q))$, $L(p, X_{\mathcal{A}}(q)) = d \mathcal{J}_{\mathcal{A}}(p, X_{\mathcal{A}}(q)) + \langle \mathcal{F}_{\mathcal{A}}(p), X_{\mathcal{A}}(q) \rangle$, $L(X_{\mathcal{A}}(p, q)) = L(q, X_{\mathcal{A}}(p)) = d \mathcal{J}_{\mathcal{A}}(q, X_{\mathcal{A}}(p)) + \langle \mathcal{F}_{\mathcal{A}}(q), X_{\mathcal{A}}(p) \rangle$, where $\mathcal{J}_{\mathcal{A}}(p, q) = \mathcal{J}_{\mathcal{A}}(L_{\mathcal{A}}(p, q))$. Thus we obtain the following formula for the first variation in linear theory

$$\bar{L}(p, q) = d(X \mathcal{A} L(p, q) - \mathcal{J}_{\mathcal{A}}(p, X_{\mathcal{A}}(q)) - \mathcal{J}_{\mathcal{A}}(q, X_{\mathcal{A}}(p))) \quad - \langle \mathcal{F}_{\mathcal{A}}(p), X_{\mathcal{A}}(q) \rangle - \langle \mathcal{F}(q), X_{\mathcal{A}}(p) \rangle. \quad (5.1.3)$$
5.2. Noether’s Linear Theorem

In the classical calculus of variations the Lagrangian \( \mathcal{L} = \int L \) is said to be invariant with respect to some transformation if the “action” \( \int_\mathcal{V} L \) is not changed for an arbitrary compact domain \( \mathcal{V} \). In view of the Dubois-Raymond lemma this is equivalent to the invariance of the density of the Lagrangian \( L \) with respect to the transformation. For this reason \( F_p \) (resp. \( X_p \)) will be called a symmetry (resp. an infinitesimal symmetry) of the density of the Lagrangian \( L \), if \( F_p(L) = L \) (resp. \( X_p(L) = 0 \)). Since the map \( F_p \) (for \( P_1 = P_2 \)) and \( X_p \) are endomorphisms of the complex \( \mathcal{S}_{\text{sym}} P \), it follows that \( F_p(g_L) = g_L \), and \( X_p(\mathcal{S}_L) = 0 \), whenever \( F_p \) and \( X_p \) are symmetries of the density \( L \). In particular, \( X_p(p) \in \ker \mathcal{S}_L \), when \( p \in \ker \mathcal{S}_L \). This remark, together with the variation formula, yields the following result.

**NOETHER’S THEOREM.** If \( X_p \) is a symmetry of the density \( L \) of the Lagrangian \( \mathcal{L} \) then the \((n - 1)\)-form \( n_\lambda(p, q) = X_p L(p, q) - X_p X_q(p) - \mathcal{J}(X_q X_p(p)) \) is closed whenever \( p, q \in \ker \mathcal{S}_L \), and its homology class does not depend on the choice of \( \lambda \).

The form \( n_\lambda(p) = n_\lambda(p, p) \) is the classical Noether current.

**Remark.** As in the classical proof of the Noether theorem the assumptions may be weakened by requiring the invariant of the action only on the extremums, i.e., requiring \( \int_\mathcal{V} L(p, q) = 0 \) or, equivalently, \( \int_\mathcal{V} \tilde{L}(p, q) = 0 \) whenever \( p, q \in \ker \mathcal{S}_L \) for any integration domain \( \mathcal{V} \). Practically, this modification is only apparent. Suppose, for example, \( \Delta = \mathcal{S}_L \), \( L \in \text{Diff}_{k,2}^\text{sym}(P, \Lambda^n) \). Then \( \ker \varphi_\Delta \subset \mathcal{S}_2^k(P) \), where \( \varphi_\Delta : \mathcal{S}_2^k(P) \to \mathcal{B} \) is the homomorphism corresponding to \( \Delta \). In this case, if \( v_{2k,d}(\ker \varphi_\Delta) = \mathcal{S}_2^k(P) \), where \( v_{a,b} : \mathcal{S}_2^a(P) \to \mathcal{S}_2^b(P) \) is a natural projection, it is easy to see that the weakened requirement implies the invariance of the density \( L \). The requirement \( v_{2k,k}(\ker \varphi_\Delta) = \mathcal{S}_2^k(P) \) essentially means that equation \( \mathcal{S}_L = 0 \) does not degenerate into an equation of order \( \leq k \).

5.3. A significant strengthening of Noether’s theorem may be obtained by widening the class of operators \( X_p \). Indeed, suppose

\[
X_p(L) = \mathcal{J}(L'), \quad L' \in \text{Diff}_{k,2}^\text{sym}(P, \Lambda^{n-1}).
\]

Then \( \tilde{L}(p, q) = dL'(p, q) \), where \( L'(p, q) = L'(p)(q), \ p, q \in P, \) and the first variation formula implies that the form

\[
n_\lambda(p, q) = n_\lambda(p, q) - L'(p, q), \quad p, q \in \ker \mathcal{S}_L
\]

is closed. Just as above we can show that its cohomology class does not depend on the choice of \( \lambda \). In view of the exactness of the complex \( \mathcal{S}_{\text{sym}} P \), it does not depend on the choice of \( L' \) satisfying the condition \( X_p(L) = \mathcal{J}(L') \).
This condition again in view of the exactness of the complex $\mathcal{S}_{\text{sym}} P$ is equivalent to $\mu(X_p(L)) = 0$. But $0 = \mu(X_p(L)) = X_p(\mu(L)) = X_p(\mathcal{F}_\lambda)$. Therefore it implies that $X_p$ is an infinitesimal symmetry of the operator $\mathcal{F}_\lambda$. Obviously, the converse is also true. Thus we have proved the following:

**Theorem.** If $X_p$ is a symmetry of the operator $\mathcal{F}_\lambda$, i.e., $[X_p, \mathcal{F}_\lambda] = 0$ and the module $P$ is projective, then the form $\tilde{n}_A(p, q)$ is closed if $p, q \in \ker \mathcal{F}_\lambda$ and its cohomology class does not depend on the choice of the splitting $\lambda$ and the choice of the solution $L'$ of the equation $X_p(L) = \mathcal{F}(L')$. In particular the map $p \mapsto \tilde{n}_A(p, p)$ is a $c$-density for the equation $\mathcal{F}_\lambda = 0$ and the corresponding conservation law does not depend on the choice of $\lambda$ and $L'$.

**Remark 1.** Let us stress that $\tilde{n}_A$ can be effectively computed from $X_p$. To do this, consider the homomorphism

$$\lambda_1 : \ker \mu = \text{im } \mathcal{S} \to \text{Diff}_{k-1}^+(A_n^{-1}), \quad \lambda_1 = -\psi \circ \lambda \circ \ast,$$

where $\text{im } \mathcal{S}$ is understood as a submodule of $\text{Diff}_{k}^+ A^n$. Then $\text{Hom}(P, \text{im } \mathcal{S})$ coincides with the image of the homomorphism

$$\mathcal{S}_{k,0} = \mathcal{S} : \text{Diff}_{k-1}^+(P, A_n^{-1}) \to \text{Diff}_{k}^+(P, A^n)$$

and therefore

$$\lambda_2 = \text{Hom}(P, \lambda_1) : \text{Hom}(P, \text{im } \mathcal{S}) \to \text{Diff}_{k-1}^+(P, A_n^{-1})$$

is the right inverse for $\mathcal{S}_{k,0}$, just as

$$\tilde{\lambda} = \text{Diff}(P, \lambda_2) : \text{im } \mathcal{S} \to \text{Diff}(P, \text{Diff}_{k-1}^+(P, A_n^{-1}))$$

is the right inverse for $\mathcal{S}$. Therefore we can put

$$L' = \tilde{\lambda}(X_p(L)).$$

**Remark 2.** A weaker requirement which can be imposed on $X_p$, namely, that $L(p, q) = dL'(p, q)$ only on the extremals, in fact gives nothing new since under the assumptions of Remark 5.2 it turns out to be equivalent to the condition $X_p(L) = \mathcal{F}(L')$.

5.4. We shall now show that the linear Noether theorem is an extremely particular case of Theorem 4.6.

**Proposition.** Closed forms $n_\lambda(p, q)$ and $c_\lambda(X_p(p), q)$, where $X_p \in \text{Der } P$ is a symmetry of the density of the Lagrangian $L$ are cohomologic to each other for all $p, q \in \ker \mathcal{F}_\lambda$. In particular, the conservation law which
corresponds to the Noether c-density \( n_\lambda(p) \) coincides with the conservation law corresponding to the c-density \( c_\lambda(X_p(p), p) \).

**Proof.** Suppose \( w(p, q) = n_\lambda(p, q) - c_\lambda(X_p(p), q) = X \circ L(p, q) - \mathcal{K}_\lambda(p, X_p(q)) - \mathcal{K}_\lambda(X_p(p), q) \). If \( \mathcal{E}_\lambda(p) = 0 \) then by Gren's \( \mathcal{E}^\prime \)-formula we have

\[
X \circ L(p, q) = X \circ d\mathcal{K}_\lambda(p, q) = X(\mathcal{K}_\lambda(p, q)) - d(X \circ \mathcal{K}(p, q)).
\]

On the other hand in view of (2.7.1) we have

\[
X(\mathcal{K}_\lambda(p, q)) = X(\mathcal{K}_\lambda(L_p(q, 1))) = X(\mathcal{K}_\lambda)(L_p(q, 1)) + \mathcal{K}_\lambda([X, L_p(q, 1)]).
\]

Using Proposition 2.7 we now obtain

\[
w(p, q) = dv(p, q) - X \circ \mathcal{K}_\lambda(p, q),
\]

where \( v(p, q) = v(L_p(q, 1)) \) (see 2.7). Thus we have

\[
w(p, q) = d\{v(p, q) - X \circ \mathcal{K}_\lambda(p, q)\}
+ \mathcal{K}_\lambda([X, L_p(q, 1)]) - \mathcal{K}_\lambda(L_p(X_p(q), 1)) - \mathcal{K}_\lambda(L_{(p,X_p)}(q, 1))
= du(p, q) + \mathcal{K}_\lambda([X, L_p(q, 1)]) - L_p(X_p(q, 1) - L_{(p,X_p)}(q, 1)),
\]

where \( u(p, q) = v(p, q) - X \circ \mathcal{K}_\lambda(p, q) \). The expression which is the argument of the function \( \mathcal{K}_\lambda \) in the last equality equals, according to (5.1.2) to \( \tilde{L}_p(q, 1) \) and therefore vanishes since \( X_p \) is a symmetry of \( L \).

Now let us note the following relation which we obtained in the process of the proof

\[
n_\lambda(p, q) - c_\lambda(X_p(p), q) = du(p, q) + \mathcal{K}_\lambda(\tilde{L}_p(q, 1)), \quad p \in \ker \mathcal{E}_L. \tag{5.4.1}
\]

This relation implies the following:

**Corollary.** If the symmetry \( X_p \) of the equation \( \mathcal{E}_L = 0 \) is such that the form \( n_\lambda(p, q) \) is closed for all \( p, q \in \ker \mathcal{E}_L \) (i.e., the Noether current is "conserved") then under the assumptions of Remark 5.2 \( X_p \) is actually a symmetry of the density \( L \).

**Proof.** Since \( X_p \) is a symmetry of the equation \( \mathcal{E}_L = 0 \), it follows that \( X_p(p) \in \ker \mathcal{E}_L \), whenever \( p \in \ker \mathcal{E}_L \) and therefore \( dc_\lambda(X_p(p), q) = 0 \) if \( p, q \in \ker \mathcal{E}_L \). Therefore (5.4.1) implies in this case

\[
dn_\lambda(p, q) = d\mathcal{K}_\lambda(\tilde{L}_p(q, 1)) = \tilde{L}(p, q) - \langle \mathcal{E}_L(p), q \rangle
= \tilde{L}(p, q) - \langle X_p(\mathcal{E}_L)(p), q \rangle = \tilde{L}(p, q).
\]
Thus we have $\tilde{L}(p, q) = 0$ if $p, q \in \ker \mathcal{E}_L$. Therefore, under the assumptions of Remark 5.2 it follows that $\tilde{L} = X_p(L) = 0$. ■

In this way, a Noether current fails to be an invariant expression if we carry out a natural generalization of the class of infinitesimal transformations.