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# $O(p^6)$ extension of the large-$N_C$ partial wave dispersion relations

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**Abstract**

Continuing our previous work [Z.H. Guo, J.J. Sanz-Cillero, H.Q. Zheng, JHEP 0706 (2007) 030], large-$N_C$ techniques and partial wave dispersion relations are used to discuss $\pi\pi$ scattering amplitudes. We get a set of predictions for $O(p^6)$ low-energy chiral perturbation theory couplings. They are provided in terms of the masses and decay widths of scalar and vector mesons.



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## 1. Introduction

Chiral perturbation theory ($\chi$PT) is a powerful tool in the study of low energy hadron physics. An important issue in $\chi$PT is the determination of the values of low energy constants (LECs), which are crucial to make predictions. In addition to an exhaustive phenomenological discussions about the LECs, Refs. [2] and [3] provided a deeper theoretical understanding. In these papers, the authors constructed a phenomenological Lagrangian including the heavy resonances, which were then integrated out to predict the LECs at tree level in terms of the resonance couplings.

In a previous paper [1], we obtained a generalization of the $KSRF$ relation [4], a new relation between resonance couplings and a prediction for the chiral constants $L_2$ and $L_3$ [5]:

$$\frac{144\pi f^2\bar{\Gamma}_V}{\bar{M}_V^3}+\frac{32\pi f^2\bar{\Gamma}_S}{\bar{M}_S^3}=1,$$

$$\frac{9\bar{\Gamma}_V}{\bar{M}_V^5}[\alpha_V+6]+\frac{2\bar{\Gamma}_S}{3\bar{M}_S^5}[\alpha_S+6]=0,$$

$$L_2=12\pi f^4\frac{\bar{\Gamma}_V}{\bar{M}_V^5},$$

$$L_3=4\pi f^4\left(\frac{2\bar{\Gamma}_S}{3\bar{M}_S^5}-\frac{9\bar{\Gamma}_V}{\bar{M}_V^5}\right), \tag{1}$$

where $\bar{\Gamma}_R$ and $\bar{M}_R$ stand, respectively, for the value of the $R$ resonance width and mass in the chiral limit. The parameter $\alpha_R$ is given by their $\mathcal{O}(m_\pi^2)$ correction in the ratio $\frac{\Gamma_R}{M_R^3}=\frac{\bar{\Gamma}_R}{\bar{M}_R^3}[1+\alpha_R\frac{m_\pi^2}{\bar{M}_R^2}+\mathcal{O}(m_\pi^4)]$.

No particular realization of the resonance Lagrangian was considered in Ref. [1]. While in the Lagrangian approach one has to pay attention to different realizations of the vector fields [3], all our analyses only rely on general properties like crossing symmetry and analyticity. Chiral symmetry was incorporated by matching chiral perturbation theory ($\chi$PT) at low energies [6–8]. In Ref. [1], we found that the minimal resonance chiral theory Lagrangian [2] was unable to fulfill the high-energy constraints for the partial wave $\pi\pi$-scattering amplitudes once the matching was taken up to order $p^4$. Another interesting finding is that in large $N_C$ limit the [1,1] Padé ap-

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proximation in $SU(3)$ $\chi$PT for $\pi\pi$ scatterings means to neglect the left-hand cuts contribution completely [9], but the understanding to the latter is very important to accept the $\sigma$ meson even in the non-linear realization of chiral symmetry [10]. However, in Ref. [1] the $\pi\pi$ scattering was only matched up to $\mathcal{O}(p^4)$. This Letter is devoted to extending the discussion up to $O(p^6)$.

## 2. Dispersive analysis

The $\pi\pi$ scattering amplitude $T(s,t,u)$ admits a decomposition into partial waves of definite angular momentum $J$ [11],

$$T(s,t,u)=\sum_J 32\pi(2J+1)P_J(\cos\theta)T_J(s), \tag{2}$$

where every $T_J(s)$ accepts a once-subtracted dispersion relation of the form,

$$T_J(s)-T_J(0)=\frac{s}{\pi}\int_{-\infty}^{0}\frac{ds'\,\mathrm{Im}\,T_J(s')}{s'(s'-s)}+\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\mathrm{Im}\,T_J(s')}{s'(s'-s)}. \tag{3}$$

In general, we will work with amplitudes and partial-waves with definite isospin, $T(s,t,u)^I$ and $T_J^I(s)$, respectively. We however quite often in the following omit the indices $I$, $J$ for simplicity when no confusion is caused.

At large-$N_C$, the resonances become narrow-width states, allowing the recovering of the right-hand cut contribution in Eq. (3). In the previous paper [1], we have demonstrated that the PKU parametrization of $S$ matrix [12] will give the same results in large $N_C$ limit as Eq. (3). The $s$-channel exchange of a resonance $R$ with proper quantum numbers $IJ$ provides for $s>0$ the absorptive contribution,

$$\mathrm{Im}\,T_J^{I,R}(s)=\pi\frac{M_R\Gamma_R}{\rho_R}\delta\left(s-M_R^2\right), \tag{4}$$

where $\rho_R=\sqrt{\frac{M_R^2-4m_\pi^2}{M_R^2}}$ and the subscript $R$ denote the different resonances.

Crossing symmetry relates the right to the left-hand cut through the expression [11],

$$\mathrm{Im}_L\,T_J^I(s)=\frac{1+(-1)^{I+J}}{s-4m_\pi^2}\sum_{J'}\sum_{I'}(2J'+1)C_{II'}^{st}\times\int_{4m_\pi^2}^{4m_\pi^2-s}dt\,P_J\left(1+\frac{2t}{s-4m_\pi^2}\right)\times P_{J'}\left(1+\frac{2s}{t-4m_\pi^2}\right)\mathrm{Im}_R\,T_{J'}^{I'}(t), \tag{5}$$

with $P_n(x)$ the Legendre polynomials. The crossing matrix is also given by [11]

$$C_{II'}^{(st)}=\begin{pmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{pmatrix}. \tag{6}$$

Hence, the imaginary part of $T_J^I(s)$ for $s<0$ produced by the crossed-channel resonance ($R$) exchange is given by

$$\mathrm{Im}\,T_J^{I,L}(s)=\theta\left(-s-M_R^2+4m_\pi^2\right)\times\frac{1+(-1)^{I+J}}{s-4m_\pi^2}(2J'+1)C_{II'}^{st}\times P_J\left(1+\frac{2M_R^2}{s-4m_\pi^2}\right)P_{J'}\left(1+\frac{2s}{M_R^2-4m_\pi^2}\right)\frac{\pi M_R\Gamma_R}{\rho_R}. \tag{7}$$

Putting the different imaginary parts together, it is then possible to calculate the right- and left-hand cut integrals:

$$T^{sR}(s)=\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\mathrm{Im}\,T^R(s')}{s'(s'-s)}, \tag{8}$$

$$T^{tR}(s)=\frac{s}{\pi}\int_{-\infty}^{0}\frac{ds'\,\mathrm{Im}\,T^R(s')}{s'(s'-s)}, \tag{9}$$

where these expressions only depend on the mass and width of the resonances. The precise results for $T^{sR}$ and $t^{tR}$, with $R=S,V$, are given in Ref. [1].

We consider now the low energy limit where the $\pi\pi$ scattering is described by $\chi$PT which determines the left-hand side of Eq. (3). For convenience, the dispersion relation is rewritten in the way,

$$T^{\chi\mathrm{PT}}(s)-T^{\chi\mathrm{PT}}(0)=T^{tR}(s)+T^{sR}(s), \tag{10}$$

where the l.h.s. only contains $\chi$PT couplings and the r.h.s. only contains resonances parameters. Comparing the different terms of the chiral expansion on both sides, one gets the low-energy constants (LECs) in terms of parameters of resonances and some other useful relations.

The $\pi\pi$ scattering amplitude is determined by the function $A(s,t,u)$,

$$A\left[\pi^a(p_1)+\pi^b(p_2)\to\pi^c(p_3)+\pi^d(p_4)\right]=\delta^{ab}\delta^{cd}A(s,t,u)+\delta^{ac}\delta^{bd}A(t,u,s)+\delta^{ad}\delta^{bc}A(u,t,s), \tag{11}$$

which is given up to $\mathcal{O}(p^4)$ in Refs. [7,13], and up to $\mathcal{O}(p^6)$ in Refs. [14,15]. Since we are interested in the $m_\pi$ dependence of the amplitude, we express the amplitude explicitly in terms of LECs, momenta and masses:

$$A(s,t,u)=\frac{s-m_\pi^2}{f^2}+\frac{16m_\pi^4}{f^4}\left(L_2+L_3+L_8-\frac{1}{2}L_5\right)-\frac{16m_\pi^2 s}{f^4}(L_2+L_3)+\frac{2s^2}{f^4}(2L_3+3L_2)+\frac{2(t-u)^2}{f^4}L_2\times\frac{16m_\pi^6}{f^6}\left(-8L_5^2+32L_8L_5-32L_8^2\right)+\frac{m_\pi^6}{f^6}(r_1+2r_f)$$

$$+\frac{m_\pi^4 s}{f^6}(r_2-2r_f)+\frac{m_\pi^2 s^2}{f^6}r_3 + \frac{m_\pi^2(t-u)^2}{f^6}r_4+\frac{s^3}{f^6}r_5+\frac{s(t-u)^2}{f^6}r_6, \quad (12)$$

with $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2=4m_\pi^2-s-t$, and where we have used the chiral expansion of the pion decay constant $f_\pi$ up to $\mathcal{O}(p^6)$ [14,16]:

$$f_\pi=f\left[1+\frac{4L_5m_\pi^2}{f^2}+\left(32L_5^2-64L_8L_5+r_f\right)\frac{m_\pi^4}{f^4}+\mathcal{O}\left(m_\pi^6\right)\right]. \quad (13)$$

In both expressions, only the leading terms in the $1/N_C$ expansion are kept. Following the notation in the former work [1], the large-$N_C$ $\mathcal{O}(p^4)$ $SU(2)$ LECs have been expressed in terms of $SU(3)$ constants [8,17]: $l_1=4L_1+2L_3$, $l_2=4L_2$, $l_3=-8L_4-4L_5+18L_6+8L_8$, $l_4=8L_4+4L_5$, together with the large-$N_C$ relations $L_1=L_2/2$, $L_4=L_6=0$.

The isospin amplitudes are given by the combinations

$$T(s,t,u)^{I=0}=3A(s,t,u)+A(t,s,u)+A(u,s,t),$$
$$T(s,t,u)^{I=1}=A(t,s,u)-A(u,s,t),$$
$$T(s,t,u)^{I=2}=A(t,s,u)+A(u,s,t). \quad (14)$$

Finally, in order to get amplitudes with definite angular momentum, one performs the partial wave projection,

$$T(s)_J^I=\frac{1}{32\pi}\frac{1}{s-4m_\pi^2}\int_{4m_\pi^2-s}^{0}P_J\left(1+\frac{2t}{s-4m_\pi^2}\right)T(s,t,u)^I\,dt. \quad (15)$$

This yields the $\chi$PT results for different partial-wave amplitudes up to $\mathcal{O}(p^6)$:

1. $IJ=00$ channel

$$\text{l.h.s.}=\frac{s}{16\pi f^2}-\frac{10L_2+5L_3}{3\pi f^4}m_\pi^2 s-\frac{-3r_2+8r_3+32r_4+36r_5+4r_6+6r_f}{48\pi f^6}m_\pi^4 s+\frac{25L_2+11L_3}{24\pi f^4}s^2+\frac{11r_3+17r_4+18r_5+10r_6}{96\pi f^6}m_\pi^2 s^2+\frac{15r_5-5r_6}{192\pi f^6}s^3, \quad (16)$$

2. $IJ=11$ channel

$$\text{l.h.s.}=\frac{s}{96\pi f^2}+\frac{L_3}{6\pi f^4}m_\pi^2 s+\frac{5r_2+40r_3-80r_4+216r_5-24r_6-10r_f}{480\pi f^6}m_\pi^4 s+\frac{-L_3}{24\pi f^4}s^2-\frac{5r_3-15r_4+54r_5+14r_6}{480\pi f^6}m_\pi^2 s^2+\frac{3r_5+3r_6}{320\pi f^6}s^3, \quad (17)$$

3. $IJ=20$ channel

$$\text{l.h.s.}=-\frac{s}{32\pi f^2}-\frac{8L_2+L_3}{6\pi f^4}m_\pi^2 s-\frac{3r_2+16r_3+40r_4+72r_5+56r_6-6r_f}{96\pi f^6}m_\pi^4 s+\frac{5L_2+L_3}{12\pi f^4}s^2+\frac{r_3+7r_4+9r_5+17r_6}{48\pi f^6}m_\pi^2 s^2-\frac{3r_5+11r_6}{192\pi f^6}s^3, \quad (18)$$

where l.h.s. means the left-hand side of Eq. (10).

For the r.h.s. of Eq. (10), a similar chiral expansion is performed up to $\mathcal{O}(p^6)$:

1. $IJ=00$ channel

$$T^{sR}=\frac{\Gamma_S}{M_S^3}s+\frac{2\Gamma_S}{M_S^5}m_\pi^2 s+\frac{6\Gamma_S}{M_S^7}m_\pi^4 s+\frac{\Gamma_S}{M_S^5}s^2+\frac{2\Gamma_S}{M_S^7}m_\pi^2 s^2+\frac{\Gamma_S}{M_S^7}s^3+\mathcal{O}\left(p^8\right), \quad (19)$$

$$T^{tR}=\frac{-\Gamma_S}{3M_S^3}s-\frac{22\Gamma_S}{9M_S^5}m_\pi^2 s-\frac{122\Gamma_S}{9M_S^7}m_\pi^4 s+\frac{9\Gamma_V}{M_V^3}s+\frac{74\Gamma_V}{M_V^5}m_\pi^2 s+\frac{446\Gamma_V}{M_V^7}m_\pi^4 s+\frac{2\Gamma_S}{9M_S^5}s^2+\frac{22\Gamma_S}{9M_S^7}m_\pi^2 s^2-\frac{\Gamma_S}{6M_S^7}s^3-\frac{4\Gamma_V}{M_V^5}s^2-\frac{46\Gamma_V}{M_V^7}m_\pi^2 s^2+\frac{5\Gamma_V}{2M_V^7}s^3+\mathcal{O}\left(p^8\right), \quad (20)$$

2. $IJ=11$ channel

$$T^{sR}=\frac{\Gamma_V}{M_V^3}s+\frac{2\Gamma_V}{M_V^5}m_\pi^2 s+\frac{6\Gamma_V}{M_V^7}m_\pi^4 s+\frac{\Gamma_V}{M_V^5}s^2+\frac{2\Gamma_V}{M_V^7}m_\pi^2 s^2+\frac{\Gamma_V}{M_V^7}s^3+\mathcal{O}\left(p^8\right), \quad (21)$$

$$T^{tR}=\frac{\Gamma_S}{9M_S^3}s+\frac{10\Gamma_S}{9M_S^5}m_\pi^2 s+\frac{326\Gamma_S}{45M_S^7}m_\pi^4 s+\frac{\Gamma_V}{2M_V^3}s+\frac{\Gamma_V}{M_V^5}m_\pi^2 s-\frac{37\Gamma_V}{5M_V^7}m_\pi^4 s-\frac{\Gamma_S}{9M_S^5}s^2-\frac{64\Gamma_S}{45M_S^7}m_\pi^2 s^2+\frac{\Gamma_S}{10M_S^7}s^3+\frac{\Gamma_V}{2M_V^5}s^2+\frac{38\Gamma_V}{5M_V^7}m_\pi^2 s^2-\frac{11\Gamma_V}{20M_V^7}s^3+\mathcal{O}\left(p^8\right), \quad (22)$$

3. $IJ=20$ channel

$$T^{sR}=0, \quad (23)$$

$$T^{tR}=-\frac{\Gamma_S}{3M_S^3}s-\frac{22\Gamma_S}{9M_S^5}m_\pi^2 s-\frac{122\Gamma_S}{9M_S^7}m_\pi^4 s-\frac{9\Gamma_V}{2M_V^3}s-\frac{37\Gamma_V}{M_V^5}m_\pi^2 s-\frac{223\Gamma_V}{M_V^7}m_\pi^4 s+\frac{2\Gamma_S}{9M_S^5}s^2+\frac{22\Gamma_S}{9M_S^7}m_\pi^2 s^2$$

$$- \frac{\Gamma_S}{6M_S^7}s^3 + \frac{2\Gamma_V}{M_V^5}s^2 + \frac{23\Gamma_V}{M_V^7}m_\pi^2 s^2 - \frac{5\Gamma_V}{4M_V^7}s^3 + \mathcal{O}(p^8), \tag{24}$$

where only the lightest multiplet of vector and scalar resonances is taken into account, respectively denoted by the subscripts $V$ and $S$.

The masses $M_R$ and decay widths $\Gamma_R$ in Eqs. (19)–(24) denote the physical ones at large-$N_C$. They carry an implicit $m_\pi^2$ dependence that we parameterize in the form

$$\frac{\Gamma_R}{M_R^5} = \frac{\bar{\Gamma}_R}{\bar{M}_R^5}\left[1 + \beta_R \frac{m_\pi^2}{\bar{M}_R^2} + \mathcal{O}\left(m_\pi^4\right)\right], \tag{25}$$

$$\frac{\Gamma_R}{M_R^3} = \frac{\bar{\Gamma}_R}{\bar{M}_R^3}\left[1 + \alpha_R \frac{m_\pi^2}{\bar{M}_R^2} + \gamma_R \frac{m_\pi^4}{\bar{M}_R^4} + \mathcal{O}\left(m_\pi^6\right)\right], \tag{26}$$

where $\bar{M}_R$ and $\bar{\Gamma}_R$ are the chiral limit of $M_R$ and $\Gamma_R$, respectively. Notice that $\bar{\Gamma}_R$ and $\bar{M}_R$ were denoted as $M_R^{(0)}$ and $\Gamma_R^{(0)}$ in Ref. [1].

After expanding the resonance contributions on the r.h.s. of Eq. (10) in powers of $s$ and $m_\pi^2$, it is possible to perform a matching with $\chi$PT. Ref. [1] was devoted to the analysis of the constraints derived from $\chi$PT at $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$. The present work studies the relations that stem from the matching at $\mathcal{O}(p^6)$

1. $IJ = 00$ channel

$$\frac{3r_2 - 8r_3 - 32r_4 - 36r_5 - 4r_6 - 6r_f}{48\pi f^6} = \frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(-\frac{68}{9} - \frac{4\beta_S}{9} + \frac{2\gamma_S}{3}\right) + \frac{\bar{\Gamma}_V}{\bar{M}_V^7}(446 + 74\beta_V + 9\gamma_V), \tag{27}$$

$$\frac{11r_3 + 17r_4 + 18r_5 + 10r_6}{96\pi f^6} = \frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(\frac{40}{9} + \frac{11\beta_S}{9}\right) + \frac{\bar{\Gamma}_V}{\bar{M}_V^7}(-46 - 4\beta_V), \tag{28}$$

$$\frac{15r_5 - 5r_6}{192\pi f^6} = \frac{5\bar{\Gamma}_S}{6\bar{M}_S^7} + \frac{5\bar{\Gamma}_V}{2\bar{M}_V^7}. \tag{29}$$

2. $IJ = 11$ channel

$$\frac{5r_2 + 40r_3 - 80r_4 + 216r_5 - 24r_6 - 10r_f}{480\pi f^6} = \frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(\frac{326}{45} + \frac{10\beta_S}{9} + \frac{\gamma_S}{9}\right) + \frac{\bar{\Gamma}_V}{\bar{M}_V^7}\left(-\frac{7}{5} + 3\beta_V + \frac{3\gamma_V}{2}\right), \tag{30}$$

$$\frac{-5r_3 + 15r_4 - 54r_5 - 14r_6}{480\pi f^6} = \frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(-\frac{\beta_S}{9} - \frac{64}{45}\right) + \frac{\bar{\Gamma}_V}{\bar{M}_V^7}\left(\frac{48}{5} + \frac{3\beta_V}{2}\right), \tag{31}$$

$$\frac{3r_5 + 3r_6}{320\pi f^6} = \frac{\bar{\Gamma}_S}{10\bar{M}_S^7} + \frac{9\bar{\Gamma}_V}{20\bar{M}_V^7}. \tag{32}$$

3. $IJ = 20$ channel

$$\frac{-3r_2 - 16r_3 - 40r_4 - 72r_5 - 56r_6 + 6r_f}{96\pi f^2} = -\frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(\frac{122}{9} + \frac{22\beta_S}{9} + \frac{\gamma_S}{3}\right) - \frac{\bar{\Gamma}_V}{\bar{M}_V^7}\left(223 + 37\beta_V + \frac{9\gamma_V}{2}\right), \tag{33}$$

$$\frac{r_3 + 8r_4 + 9r_5 + 17r_6}{96\pi f^6} = \frac{\bar{\Gamma}_S}{\bar{M}_S^7}\left(\frac{22}{9} + \frac{2\beta_S}{9}\right) + \frac{\bar{\Gamma}_V}{\bar{M}_V^7}(23 + 2\beta_V), \tag{34}$$

$$\frac{-3r_5 - 11r_6}{192\pi f^6} = -\frac{\bar{\Gamma}_S}{6\bar{M}_S^7} - \frac{5\bar{\Gamma}_V}{4\bar{M}_V^7}. \tag{35}$$

Eqs. (27), (30) and (33) refer to the matching of the terms $\mathcal{O}(m_\pi^4 s)$. Eqs. (28), (31) and (34) correspond to the $\mathcal{O}(m_\pi^2 s^2)$ terms. Eqs. (27), (30) and (33) provide the matching at $\mathcal{O}(s^3)$.

It is remarkable that the system of nine equations for six unknowns ($r_i$, with $i = f, 2, \ldots, 6$) is actually compatible. The $\mathcal{O}(s^3)$ relations determine $r_5$ and $r_6$. After that, it is then possible to extract $r_3$ and $r_4$ from the $\mathcal{O}(m_\pi^2 s^2)$ equations. Finally, using these values, one can extract the combination $r_2 - 2r_f$ from the $\mathcal{O}(m_\pi^4 s)$ constraints. The LECs always appear in this particular combination, avoiding an independent determination of $r_2$ and $r_f$. This yields the predictions:

$$r_2 - 2r_f = \frac{64\pi f^6 \bar{\Gamma}_S}{\bar{M}_S^7}\left(1 + \frac{\beta_S}{3} + \frac{\gamma_S}{6}\right) + \frac{\pi f^6 \bar{\Gamma}_V}{\bar{M}_V^7}(7584 + 1248\beta_V + 144\gamma_V), \tag{36}$$

$$r_3 = \frac{64\pi f^6 \bar{\Gamma}_S}{3\bar{M}_S^7}\left(1 + \frac{\beta_S}{2}\right) - \frac{768\pi f^6 \bar{\Gamma}_V}{\bar{M}_V^7}\left(1 + \frac{3\beta_V}{32}\right), \tag{37}$$

$$r_4 = \frac{192\pi f^6 \bar{\Gamma}_V}{\bar{M}_V^7}\left(1 + \frac{\beta_V}{8}\right), \tag{38}$$

$$r_5 = \frac{32\pi f^6 \bar{\Gamma}_S}{3\bar{M}_S^7} + \frac{36\pi f^6 \bar{\Gamma}_V}{\bar{M}_V^7}, \tag{39}$$

$$r_6 = \frac{12\pi f^6 \bar{\Gamma}_V}{\bar{M}_V^7}. \tag{40}$$

## 3. An example of $\mathcal{O}(p^6)$ coupling determination

The authors of Ref. [14] provide an estimate of the $\mathcal{O}(p^6)$ LECs $r_i$ in terms of resonances couplings, where they consider a phenomenological Lagrangian including one multiplet of vector and scalar resonances. The vector interaction is given by

$$\mathcal{L}_V = -i\frac{g_V}{2\sqrt{2}}\langle \hat{V}_{\mu\nu}[u^\mu, u^\nu]\rangle + f_\chi \langle \hat{V}_\mu [u^\mu, \chi_-]\rangle, \tag{41}$$

and for the scalar,

$$\mathcal{L}_S = c_d\langle Su^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle + \tilde{c}_d S_1\langle u^\mu u_\mu\rangle + \tilde{c}_m S_1\langle \chi_+\rangle, \quad (42)$$

where $\langle\cdots\rangle$ is short for trace in flavour space and the tensors $u^\mu$, $\chi_\pm$ introduce the chiral Goldstones. For further details on the notations, see Ref. [14] and references therein. At large-$N_C$, the $SU(3)$ singlet and octet states become degenerate and one has $\tilde{c}_d = c_d/\sqrt{3}$, $\tilde{c}_m = c_m/\sqrt{3}$, $M_{S_1} = M_S$ [2]. Using this Lagrangian, the authors computed the contributions to the $\pi\pi$ scattering from resonance exchanges and provided a set of values for the LECs $r_i$ [14].

As an example of our method, we will rederive their result. In order to do that, in a first step, we will neglect the wave-function renormalizations $Z_R$ and $Z_\pi$, and only the resonance exchange contribution will be considered, as it was done in Ref. [14]. At large-$N_C$, the meson wave functions get renormalized if there are tree-level tadpole diagrams that connect the scalar meson field to the vacuum [18,19]. After recovering the results in Ref. [14], we will compute the LECs including also the effect of $Z_\pi$ and $Z_R$ and their impact on the numerical estimates will be analyzed.

We need first to calculate the $R\to\pi\pi$ decay widths corresponding to this Lagrangian. Ignoring the wave-function renormalizations, one gets

$$\Gamma_V = \frac{g_V^2 M_V^5 \rho_V^3}{48\pi f^4}\left[1 + \frac{4\sqrt{2}f_\chi}{g_V}\frac{m_\pi^2}{M_V^2}\right]^2, \quad (43)$$

$$\Gamma_S = \frac{3c_d^2 M_S^3 \rho_S}{16\pi f^4}\left[1 + \frac{2(c_m - c_d)}{c_d}\frac{m_\pi^2}{M_S^2}\right]^2, \quad (44)$$

where the subscript $S$ denote the $SU(2)$ singlet $\sigma = \sqrt{\frac{2}{3}}S_0 - \sqrt{\frac{1}{3}}S_8 \sim \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$. The large-$N_C$ resonances masses are $m_\pi$-independent within this model, i.e., $M_R = \bar{M}_R$.

With the above expressions of $\Gamma_V$ and $\Gamma_S$, we can get the parameters $\alpha_R$, $\beta_R$ and $\gamma_R$ defined in Eqs. (25) and (26)

$$\alpha_V = \beta_V = \frac{8\sqrt{2}f_\chi}{g_V} - 6, \quad (45)$$

$$\gamma_V = \frac{32 f_\chi^2}{g_V^2} - \frac{48\sqrt{2}f_\chi}{g_V} + 6, \quad (46)$$

$$\alpha_S = \beta_S = \frac{4c_m}{c_d} - 6, \quad (47)$$

$$\gamma_S = 10 - \frac{16 c_m}{c_d} + \frac{4c_m^2}{c_d^2}. \quad (48)$$

Using Eqs. (36)–(40), one gets the predictions on $\mathcal{O}(p^6)$ LECs in terms of the resonance large-$N_C$ parameters $g_V$, $f_\chi$, $c_d$ and $c_m$:

$$r_2 - 2r_f = 20a_V + 16b_V + 3c_V + \frac{8f^2(c_m - c_d)^2}{M_S^4}, \quad (49)$$

$$r_3 = -7a_V - 3b_V + \frac{8f^2 c_d(c_m - c_d)}{M_S^4}, \quad (50)$$

$$r_4 = a_V + b_V, \quad (51)$$

$$r_5 = \frac{3}{4}a_V + \frac{2f^2 c_d^2}{M_S^4}, \quad (52)$$

$$r_6 = \frac{1}{4}a_V, \quad (53)$$

with $a_V \equiv g_V^2 f^2/M_V^2$, $b_V \equiv 4\sqrt{2}f_\chi g_V f^2/M_V^2$, $c_V \equiv 32 f_\chi^2 f^2/M_V^2$. If one neglects the wave-function renormalization and the tadpole effects then the pion decay constant is given by $f_\pi = f$ and therefore $r_f = 0$. Taking this into account, we get a set of predictions for LECs $r_2, \ldots, r_5$, in complete agreement with the results in Ref. [14].

However, all the former results ignored the effects of the scalar tadpole [18,19]. The term $c_m\langle S\chi_+\rangle$ connects the scalar field to the vacuum, inducing a pion wave-function renormalization and a more complicate relation between $m_\pi$ and the quark mass [18]. Thus, one has the large-$N_C$ relations,

$$Z_\pi = 1 - \frac{8c_d c_m}{f^2}\frac{m_\pi^2}{M_S^2} + \frac{64 c_d c_m^3}{f^4}\frac{m_\pi^4}{M_S^4} + \mathcal{O}(m_\pi^6), \quad (54)$$

$$2B_0\hat{m} = m_\pi^2 + \frac{8c_m(c_d - c_m)}{f^2}\frac{m_\pi^4}{M_S^2} + \mathcal{O}(m_\pi^6), \quad (55)$$

with $\hat{m}$ the $u$ and $d$ quark masses in the isospin limit. The expressions for $r_i$ provided in Ref. [14] did not take this effect into account. Our results in Eqs. (36)–(40) are fully general and allow a simple implementation of this correction. Thus, one gets the corrected widths,

$$\Gamma_V = \frac{g_V^2 M_V^5 \rho_V^3}{48\pi f_\pi^4}\left[1 + \frac{4\sqrt{2}f_\chi}{g_V}\frac{2B_0\hat{m}}{M_V^2}\right]^2, \quad (56)$$

$$\Gamma_S = \frac{3c_d^2 M_S^3 \rho_S}{16\pi f_\pi^4}\left[1 - \frac{2m_\pi^2}{M_S^2} + \frac{2c_m}{c_d}\frac{2B_0\hat{m}}{M_S^2}\right]^2, \quad (57)$$

with $f_\pi = f Z_\pi^{-\frac{1}{2}}$ [18]. The resonance masses remain $m_\pi$ independent. From this, one is able to recover the real parameters that provide the LECs:

$$\alpha_V = \beta_V = \frac{8\sqrt{2}f_\chi}{g_V} - 6 - \frac{16 c_d c_m M_V^2}{f^2 M_S^2}, \quad (58)$$

$$\gamma_V = \frac{32 f_\chi^2}{g_V^2} - \frac{48\sqrt{2}f_\chi}{g_V}\left[1 + \frac{4c_m(c_d + c_m)M_V^2}{3f^2 M_S^2}\right] + 6\left[1 + \frac{16 c_d c_m M_V^2}{f^2 M_S^2} + \frac{32 c_m^2 c_d(c_d + 2c_m)M_V^4}{3f^4 M_S^4}\right], \quad (59)$$

$$\alpha_S = \beta_S = \frac{4c_m}{c_d} - 6 - \frac{16 c_d c_m}{f^2}, \quad (60)$$

$$\gamma_S = 10\left[1 + \frac{48 c_d c_m}{5f^2} + \frac{32 c_d^2 c_m^2}{5f^4}\right] - \frac{16c_m}{c_d}\left[1 + \frac{2c_d c_m}{f^2} - \frac{8c_d^2 c_m^2}{f^4}\right] + \frac{4c_m^2}{c_d^2}\left[1 - \frac{8c_d c_m}{f^2}\right]. \quad (61)$$

Substituting these values in Eqs. (36)–(40), one recovers the proper values for $(r_2 - 2r_f), r_3, \ldots, r_6$. Notice that now the original parameters in Eq. (45)–(48) has gained extra terms proportional to $c_m$ due to the scalar tadpole originated by the operators $c_m \langle S\chi_+\rangle$. However, $r_5$ and $r_6$ remain unchanged and only the couplings $r_f, r_2, r_3, r_4$ gets modified.

In order to get the value of $r_f$ (allowing the separate extraction of $r_2$), we need the value of the $\mathcal{O}(p^4)$ LECs [2]

$$L_5 = \frac{c_d c_m}{M_S^2}, \qquad L_8 = \frac{c_m^2}{2M_S^2}. \tag{62}$$

The wave-function renormalization in Eq. (54) provides the value of $f_\pi$ in the resonance theory under consideration. Comparing this to the $f_\pi$ expression in $\chi$PT from Eq. (13) and using the values of $L_5$ and $L_8$ from Eq. (62), one can extract the corresponding $\mathcal{O}(p^6)$ LEC in terms of the resonance couplings:

$$r_f = -\frac{8c_d^2 c_m^2}{M_S^4}. \tag{63}$$

We proceed now to a numerical comparison of our new calculation and the original results in Ref. [14], where one had

$$r_2 = 1.3\times 10^{-4}, \qquad r_3 = -1.7\times 10^{-4},$$
$$r_4 = -1.0\times 10^{-4}. \tag{64}$$

This can be compared to our determinations

$$r_2 = 18\times 10^{-4}, \qquad r_3 = 0.9\times 10^{-4},$$
$$r_4 = -1.9\times 10^{-4}, \tag{65}$$

where we took the same inputs used in Ref. [14] to extract the values of the LECs in Eq. (64), $f = 93.2$ MeV, $g_V = 0.09$, $f_\chi = -0.03$, $M_V = M_\rho = 770$ MeV, $c_d = 32$ MeV, $c_m = 42$ MeV, $M_S = 983$ MeV. The kaon and eta contributions [14] have also been added in Eq. (65) in order to compare with Eq. (64). The impact of this modifications on the whole amplitude is not large since it is an $\mathcal{O}(p^6)$ effect.

Observing the scattering-lengths derived from Ref. [14], we get slight shifts on the values:

$$\delta a_0^0 = 0.004, \qquad \delta b_0^0 = 0.004,$$
$$10\cdot\delta a_0^2 = -0.003, \qquad 10\cdot\delta b_0^2 = -0.017,$$
$$10\cdot\delta a_1^1 = 0.001, \qquad 10\cdot\delta b_1^1 = -0.003, \tag{66}$$

given in $m_\pi$ units for the mass-dimension quantities. Although there are large variations in the $\mathcal{O}(p^6)$ LECs (especially $r_2$), we verified that the effect on the global uncertainties in the current scattering-length determinations [20] is negligible. Nevertheless, the lack of control on the $r_k$ avoids any improvement of the errors in an analysis based on Ref. [14] beyond these values even if the accuracy in the remaining inputs is considerably increased.

This exercise shows that the extraction of the these couplings requires of a very subtle analysis and a closer examination of the resonance Lagrangian. The Lagrangian in Eqs. (41) and (42) provides only a rough approximation and there can be more resonance contributions to the $\mathcal{O}(p^6)$ LECs beside the scalar tadpole [21]. These variations due to unheeded contributions just point out the level of theoretical uncertainty that comes into play from our ignorance of the resonance Lagrangian.

We presented in this note a new method to calculate $\chi$PT low-energy constants in terms of resonance parameters in a model independent way, without relying on any particular form of the resonance Lagrangian. This technique provides a convenient procedure of implementing the high and low-energy constraints and can be useful for future studies.

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## References

[1] Z.H. Guo, J.J. Sanz-Cillero, H.Q. Zheng, JHEP 0706 (2007) 030.
[2] G. Ecker, et al., Nucl. Phys. B 321 (1989) 311.
[3] G. Ecker, et al., Phys. Lett. B 223 (1989) 425.
[4] K. Kawarabayashi, M. Suzuki, Phys. Rev. Lett. 16 (1966) 255; Riazuddin, Fayazuddin, Phys. Rev. 147 (1966) 1071; J.L. Basdevant, J. Zinn-Justin, Phys. Rev. D 3 (1971) 1865; See also, S. Rudaz, Phys. Rev. D 10 (1974) 3857; G. Kramer, W.F. Palmer, Phys. Rev. D 36 (1987) 154.
[5] Analogous predictions for the LECs can be found in A.A. Bolokhov, et al., Phys. Rev. D 48 (1993) 3090.
[6] S. Weinberg, Physica A 96 (1979) 327.
[7] J. Gasser, H. Leutwyler, Ann. Phys. 158 (1984) 142.
[8] J. Gasser, H. Leutwyler, Nucl. Phys. B 250 (1985) 465.
[9] See also Z.X. Sun, et al., Mod. Phys. Lett. A 22 (2007) 711; Z.G. Xiao, H.Q. Zheng, Mod. Phys. Lett. A 22 (2007) 55.
[10] Z.G. Xiao, H.Q. Zheng, Nucl. Phys. A 695 (2001) 273.
[11] B.R. Martin, D. Morgan, G. Shaw, Pion Pion Interactions in Particle Physics, Academic Press, London, 1976.
[12] H.Q. Zheng, et al., Nucl. Phys. A 733 (2004) 235; Z.Y. Zhou, H.Q. Zheng, Nucl. Phys. A 755 (2006) 212; J.Y. He, Z.G. Xiao, H.Q. Zheng, Phys. Lett. B 536 (2002) 59; J.Y. He, Z.G. Xiao, H.Q. Zheng, Phys. Lett. B 549 (2002) 362, Erratum.
[13] V. Bernard, N. Kaiser, U.G. Meissner, Nucl. Phys. B 357 (1991) 129.
[14] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Nucl. Phys. B 508 (1997) 263; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Nucl. Phys. B 517 (1998) 639, Erratum.
[15] J. Bijnens, P. Dhonte, P. Talavera, JHEP 0401 (2004) 050.
[16] J. Bijnens, G. Colangelo, P. Talavera, JHEP 9805 (1998) 014; J. Bijnens, G. Colangelo, G. Ecker, Ann. Phys. 280 (2000) 100.
[17] J. Gasser, Ch. Haefeli, M.A. Ivanov, M. Schmid, Phys. Lett. B 652 (2007) 21.
[18] J.J. Sanz-Cillero, Phys. Rev. D 70 (2004) 094033.
[19] M. Jamin, J.A. Oller, A. Pich, Nucl. Phys. B 622 (2002) 279.
[20] G. Colangelo, J. Gasser, H. Leutwyler, Nucl. Phys. B 603 (2001) 125.
[21] V. Cirigliano, et al., Nucl. Phys. B 753 (2006) 139.