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An Existence Theorem for a Class of BVP without Restrictions of the Bernstein–Nagumo Type*

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0. INTRODUCTION

Since the beginning of this century, until the present time, many theorems about the existence of solutions for the boundary problems of the following form have been proved

$$x'' = f(t, x, x), \qquad x \in E,$$

where $f\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and E is a subspace of $C^2([0, 1], \mathbb{R})$ of codimension two. Most of these results assume a Bernstein–Nagumo type of growth restriction on f. For details see [1–4, 6]. However, Granas *et al.* [5] have remarked that this restriction is not too natural, since the Dirichlet problems

$$x'' = x'^2 + \pi^2 \qquad x'' = x'^2 - \pi^2$$

$$x(0) = x(1) = 0 \qquad x(0) = x(1) = 0$$

have virtually identical growth as $|x'| \to \infty$; but the first problem has no solutions, while the second one has one solution. In fact, these authors has proven the following result:

0.1. THEOREM. *The Dirichlet problem* $[x'' = q(x'), x(0) = x(1) = 0]$ *has one solution if* $q\colon \mathbb{R} \to \mathbb{R}$ *is a continuous function which has two zeros of opposite sign.*

From the results in [7], we get that this theorem is true for the equation $x'' = p(x)\, q(x')$, for any continuous function $p\colon \mathbb{R} \to \mathbb{R}$. This assertion is also valid for the Sturm–Liouville boundary conditions. See [9].

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Finally, in [10] we have proved that the problem

$$x'' = q(x')\,p(t, x, x'), \qquad x \in E \tag{0.1}$$

has a solution if $p\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ are continuous functions such that q has two zeros $r_1 < 0 < r_0$ and p is bounded on $[0, 1] \times \mathbb{R} \times [r_1, r_0]$. Here E belongs to a large class of closed linear subspaces of $C^2([0, 1], \mathbb{R})$ of codimension two, which contains Sturm–Liouville and antiperiodic boundary conditions. See Corollary 4.3 of [10].

In this paper we prove that the problem

$$x^{(k)} = q(x^{(k-1)})\,p(t, x, x', ..., x^{(k-1)}), \qquad x \in E \tag{0.2}$$

has one solution if $k \geqslant 2$; $p\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ are continuous; q has two zeros of opposite sign and E is a closed linear subspace of $C^k([0, 1], \mathbb{R})$ of codimension k, such that:

(i) E does not contain polynomials of degree less than k.

(ii) The closure of E in $C^{k-1}([0, 1], \mathbb{R})$ does not contain polynomials of degree less than $k-1$. Here, $C^m([0, 1], \mathbb{R})$ is endowed with the usual C^m-topology.

We prove that this result is sharp, with respect to condition (ii) above, and improves Corollary 4.3 of [10].

1. The Main Result

In the following, C^0 denotes the space of all continuous functions $u\colon [0, 1] \to \mathbb{R}$, equipped with the norm $\|u\|_0 = \sup\{|u(t)| : 0 \leqslant t \leqslant 1\}$. For all integers $m \geqslant 1$, C^m denotes the space of all m-times continuously differentiable functions $u \in C^0$, with the norm $\|u\|_m = \max\{\|u^{(i)}\|_0 : 1 \leqslant i \leqslant n\}$. For all integers $n \geqslant 1$, P_n denotes the subspace of C^m of all polynomials whose degree is strictly less than n.

In the following, E denotes a closed and linear subspace of C^k of codimension k, for some $k \geqslant 2$. The closure of E in C^{k-1} will be denoted by E^1. Notice that E^1 is a closed subspace of C^{k-1}, which contains E. Finally, $p\colon [0, 1] \times \mathbb{R}^k \to \mathbb{R}$, $q\colon \mathbb{R} \to \mathbb{R}$ are continuous and q has two zeros of opposite sign.

1.1. Theorem. *Let $f\colon [0, 1] \times \mathbb{R}^k \to \mathbb{R}$ be continuous and assume that there exists an open and bounded subset U of E^1 containing the origin, such that the problem*

$$x^{(k)} = \lambda f(t, x, x', ..., x^{(k-1)}), \qquad x \in E \tag{1.1}$$

has no solutions in the boundary ∂U *of* U, *for* $0<\lambda<1$. *If* $E \cap P_k = \{0\}$, *then* $(1.1)_1$ *has one solution in the closure of* U.

Proof. Let us define $L_0: C^k \to C^0$ by $L_0(u) = u^{(k)}$, then L_0 is surjective and $\ker L_0 = P_k$. Hence, by the open mapping theorem, the restriction $L: E \to C^0$ of L_0, is an isomorphism which is onto. Remember that E is a closed subspace of C^k of codimension k.

Now let us define $N: E^1 \to C^0$ and $j: E \to E^1$ by $j(u) = u$ and $N(u)(t) = f(t, u(t), ..., u^{(k-1)}(t))$. Then j, N are continuous; j is compact and N sends bounded sets into bounded sets. From this, $K: j \circ L^{-1} \circ N$ is compact and $(1.1)_\lambda$ is equivalent to $[x = \lambda K(x)]$. The proof follows now from the homotopy theorem of the Leray–Schauder degree theory.

1.2. Theorem. *If* $E \cap P_k = E^1 \cap P_{k-1} = \{0\}$ *and* $q(a) = q(b) = 0$ *for some* $a<0<b$, *then* (0.2) *has one solution* u *such that* $a \leqslant u^{(k-1)}(t) \leqslant b$, *for* $0 \leqslant t \leqslant 1$.

Proof. Let us fix $a<0<b$ as above and define

$$U = \{u \in E^1 : a < u^{(k-1)}(t) < b; 0 \leqslant t \leqslant 1\}.$$

It is clear that U is an open subset of E^1 which contains $u = 0$.

Claim 1. *U is a bounded subset of E^1.*

Proof. Assume that there is a sequence (x_n) in U such that $\|x_n\|_{k-1} \to \infty$ as $n \to \infty$, and define $y_n = \|x_n\|_{k-1}^{-1} x_n$. Then $\|y_n\|_{k-1} = 1$ and

$$y_n^{(k-1)} \to 0 \qquad (\text{as } n \to \infty) \text{ in } C^0. \tag{1.2}$$

By Ascoli's theorem we can assume, without loss of generality, that there exists $y \in C^{k-2}$ such that $y_n \to y$ in C^{k-2}, and by (1.2) we conclude that $y \in C^{k-1}$, $y_n \to y$ in C^{k-1}, and $y^{(k-1)} \equiv 0$. Thus, $\|y\|_{k-1} = 1$ and $y \in E^1 \cap P_{k-1} = \{0\}$. This contradiction proves Claim 1.

Now, we assume that q is continuously differentiable in $[a, b]$, and we prove that the problem

$$x'' = \lambda q(x^{(k-1)})\, p(t, x, ..., x^{(k-1)}), \qquad x \in E \tag{1.3$_\lambda$}$$

has no solutions in ∂U for $0<\lambda<1$. To show this, let u be a solution to $(1.3)_\lambda$ in the closure of U, for some $\lambda > 0$.

Claim 2. $u^{(k-1)}(t) < b$, $0 \leqslant t \leqslant 1$.

Proof. Assume $u^{(k-1)}(\tau)=b$, for some $\tau \in [0,1]$ and define $v(t)=u^{(k-1)}(t)$, $w(t)\equiv b$, $h(t)=\lambda p(t, u(t), ..., u^{(k-1)}(t))$. Then v, w are solutions to the initial value problem

$$z' = h(t)\, q(z), \qquad z(\tau)=b$$

and so $v\equiv w$, since q belongs to class $C^1[a,b]$. From this, u is a polynomial function of degree $k-1$ and so, $u \in E \cap P_{k-1} = \{0\}$. Hence, $b=0$ and this contradicion proves Claim 2.

Analogously, $a < u^{(k-1)}(t)$, for $0 \leqslant t \leqslant 1$, and so $(1.3)_\lambda$ has no solutions in ∂U for any $\lambda > 0$. Thus, the proof follows from Theorem 1.1, if q belongs to $C^1[a,b]$.

To prove the general case, let $\{q_n : \mathbb{R} \to \mathbb{R}\}$ be a sequence of continuous functions which are continuously differentiable on $[a,b]$, and let it converge to q uniformly on $[a,b]$. Without loss of generality, we can assume that each q_n has two zeros of oppositive sign on $[a,b]$; see [10, Corollary 4.3]; and by the arguments above, the problem

$$x'' = q_n(x^{(k-1)})\, p(t, x, ..., x^{(k-1)}), \qquad x \in E \tag{1.4}$$

has a solution u_n such that $a \leqslant u_n^{(k-1)}(t) \leqslant b$, for $0 \leqslant t \leqslant 1$. By the arguments in Claim 1, we have that (u_n) is a bounded sequence in C^{k-1}, and by (1.4), (u_n) is bounded in C^k. Thus, we can suppose that there is u in C^{k-1}, such that $u_n \to u$ in C^{k-1}. From (1.4), $u_n^{(k)} \to q(u^{(k-1)})\, p(., u, ..., u^{(k-1)})$ in C^0 and hence, u is a solution to our problem.

EXAMPLES. (a) Let us fix $t_1 < \cdots < t_s$ in $[0,1]$, and positive integers $n_1, ..., n_s$ such that $n_1 + \cdots + n_s = k$. Then, the subspace E of C^k, defined by

$$u^{(i)}(t_j)=0, \qquad j=1, ..., s, \qquad 0 \leqslant i \leqslant n_s$$

satisfies the assumption in Theorem 1.2.

(b) Let us fix $t_0, ..., t_{k-1}$ in $[0,1]$. Then the subspace of C^k defined by

$$u^{(i)}(t_i)=0, \qquad 0 \leqslant i < k \tag{1.5}$$

satisfies the hypotheses in Theorem 1.2. In particular, the initial value problem

$$x^{(k)} = q(x^{(k-1)})\, p(t, x, ..., x^{(k-1)}), \qquad x^{(i)}(\tau)=0, \qquad i=1, ..., k$$

has a solution u defined in $[0,1]$, for each fixed τ in $[0,1]$.

Remark. If $u \in P_k$, and $u^{(k-1)}(\tau)=0$ for some τ, then $u \in P_{k-1}$. From this, if $u \in P_k$ and it satifies (1.5), then $u \equiv 0$.

(c) Let $(a_1, ..., a_4)$, $(b_1, ..., b_4)$ be linearly independent vectors in $\mathbb{R}^4$, such that

$$(a_1+a_3)(b_2+b_3+b_4) \neq (b_1+b_3)(a_2+a_3+a_4).$$

Then, the subspace E of C^2, defined by the equations

$$a_1 x(0)+a_2 x'(0)+a_3 x(1)+a_4 x'(1)=0$$
$$b_1 x(0)+b_2 x'(0)+b_3 x(1)+b_4 x'(1)=0$$

satisfies the assumptions in Theorem 1.2. In particular, this assertion holds for the Sturm–Liouville boundary conditions $a_3=a_4=b_1=b_3=0$, $a_1 \geqslant 0 \geqslant a_2$, $b_3 \geqslant 0$, $b_4 \geqslant 0$, $a_1 > a_2$, $a_1+b_3>0$, and $b_3+b_4>0$.

Remark. Let us fix $a<0<b$ such that $q(a)=q(b)=0$ and u_0 in P_k such that $a<u_0^{(k-1)}<b$. Then, Theorem 1.2 remains true if we replace E by u_0+E. The proof of this assertion follows from the change of variables $y=x-u_0(t)$.

1.3. COUNTER-EXAMPLE. We prove that Theorem 1.2 fails if we omit the assumption on E^1. To this end, let $q: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $q(z)=(1-z^2)^{1/2}$ if $|z| \leqslant 1$ and let $E=\{u \in C^2([0,\pi], \mathbb{R}): u(i)=u''(i), i=0,\pi\}$. Then, $q(1)=q(-1)=0$ and $E \cap P_2=\{0\}$. We show that the problem

$$x''=q(x'), \qquad x \in E \tag{1.6}$$

has no solutions u such that $|u'(t)| \leqslant 1$ for $0 \leqslant t \leqslant \pi$.

Proof. It is easy to check that

$$u(t)=\begin{cases} \alpha-t+a, & \text{if } t<\alpha \\ \sin(\alpha-t)+a, & \text{if } \alpha \leqslant t<\alpha+\pi \\ t-\alpha-\pi+a, & \text{if } t>\alpha+\pi, \end{cases}$$

where $\alpha, a \in \mathbb{R}$ represents all solutions of the equation $x''=q(x')$, such that $|u'(t)| \leqslant 1$. Considering the cases $\alpha \leqslant -\pi$, $-\pi<\alpha \leqslant 0$, $0<\alpha<\pi$, and $\alpha \geqslant \pi$, it is straightforward to see that (1.6) has no solutions u such that $|u'(t)| \leqslant 1$.

2. THE CASE $N=2$

In this section, we improve Corollary 4.3 of [10]. We say that $u \in C^1$ is admissible [8, 10] if there exists $t_0 \in [0, 1]$ such that

$$|u(t_0)| = \|u\|_0 \quad \text{and} \quad u'(t_0) = 0. \tag{2.1}$$

In this case, $u(t_0)\, u''(t_0) \leqslant 0$, if $u \in C^2$. We say that a linear and closed subspace E of C^k, $k=1, 2$, is admissible if all u in E are admissible and E has codimension two in C^k. See [8, 10].

EXAMPLES. (a) Let $p_i\colon [0, 1] \to \mathbb{R}$, $i=0, 1$, be continuous such that $\int_0^1 |p_i(s)|\, ds < 1$, $i=0, 1$. Then the subspace of C^2, defined by

$$x(i) = \int_0^1 p_i(s)\, x(s)\, ds, \qquad i=0, 1,$$

is admissible. In fact, if $x \in E$ and $x \not\equiv 0$, then $|x(i)| < \|x\|_0$ for $i=0, 1$.

(b) Let α_i, β_i be non-negative real numbers such that

$$\alpha_0 + \alpha_1 > 0 \quad \text{and} \quad \alpha_i + \beta_i > 0 \quad \text{for} \quad i=0, 1$$

if $x \in C^2$ satifies the Sturm–Liouville boundary conditions

$$\alpha_0 x(0) - \beta_0 x'(0) = 0, \qquad \alpha_1 x(1) + \beta_1 x'(1) = 0$$

then, x is admissible. (Assume $x \not\equiv 0$ and $|x(0)| = \|x\|_0$. We see that $x'(0)=0$. If $\alpha_0 = 0$ there is nothing to show. Suppose $\alpha_0 > 0$. Since $x(t)^2$ attains its maximum at $t=0$, we have $x(0)\, x'(0) \leqslant 0$. On the other hand, $\alpha_0 x(0)\, x'(0) = \beta_0 x'(0)^2 \geqslant 0$ and the proof follows easily. Analogously, $x'(1)=0$, if $\|x\|_0 = |x(1)|$. Thus, x is admissible.)

The above examples are special cases of the following one. Let α_i, β_i be non-negative real numbers and let $I_i\colon C^0 \to \mathbb{R}$ be linear continuous functions, $i=0, 1$, such that

$$\alpha_i \geqslant \|I_i\|, \qquad \alpha_i + \beta_i > 0, \qquad \alpha_0 + \alpha_1 > \|I_0\| + \|I_1\|, \qquad I_i \neq e_i,$$

where $e_i\colon C^0 \to \mathbb{R}$ is given by $e_i(x) = x(i)$ and $\|I_i\|$ is the usual norm of I_i. Then, the subspace of C^2 given by

$$\alpha_0 x(0) - \beta_0 x'(0) - I_0(x) = 0, \qquad \alpha_1 x(1) + \beta_1 x'(1) - I_1(x) = 0$$

is admissible. Our proof is rahter long and we omit it.

2.1. PROPOSITION. *If F is an admissible subspace of C^1, then $E := F \cap C^2$ is an admissible subspace of C^2.*

Proof. Since C^2 is a dense subspace of C^1, there exists a plane (subspace of dimension two) P of C^2 such that $F \cap P = \{0\}$. In particular, $C^1 = F + P$.

Given x in C^2, we can write $x = x_F + x_P$, where $x_F \in F$ and $x_P \in P$. On the other hand, $x_F = x - x_P$, and so $x_F \in C^2 \cap F = E$. Consequently, $C^2 = E + P$. But, $E \cap P \subseteq F \cap P = \{0\}$, and thus, E has codimension two in C^2. So, the proof is complete.

2.2. PROPOSITION. *Let E be an admissible subspace of C^2. Then, $E = E^1 \cap C^2$. As above, E^1 denotes the closure of E in C^1.*

Proof. It is easy to prove that all u in E^1 is admissible. Now, let us write $F = E^1 \cap C^2$, and define $L: C^2 \to C^0$ by $L(x) = x'' - x$. If $u \in F$ and $L(u) = 0$, then $u(t_0)^2 = u(t_0)\, u''(t_0) \leqslant 0$, where t_0 satisfies (2.1). From this, $F \cap \ker L = \{0\}$ and so, $\operatorname{codim}(F) \geqslant 2$ in C^2. On the other hand, $E \subseteq F$ and the proof follows easily.

2.3. THEOREM. *Let E be an admissible subspace of C^2 such that $E \cap P_1 = \{0\}$. If q has two zeros of opposite sign, and $p: [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, then* (0, 1) *has one solution.*

Proof. We have $E^1 \cap P_1 = E^1 \cap (C^2 \cap P_1) = (E^1 \cap C^2) \cap P_1 = E \cap P_1 = \{0\}$. On the other hand, if $u \in P_2$, we can write $u(t) = \alpha t + \beta$, for some constants α, β. By (2.1) we get $\alpha = 0$ and so $u \in E \cap P_1 = \{0\}$. The proof follows now from Theorem 1.2.

Remark. If E is an admissible subspace of C^2, then E^1 is an admissible subspace of C^1.

Proof. Let us fix a plane P of C^2 such that $E \cap P = \{0\}$, and notice that $E^1 \cap P = E^1 \cap (C^2 \cap P) = E \cap P = \{0\}$. Given x in C^1 choose a sequence (x_n) in C^2 which converges to x in C^1. Now, let us write $x_n = y_n + z_n$, where $y_n \in E$ and $z_n \in P$. Since (x_n) is bounded in C^1, and C^2 is a topological direct sum of E, P, then (z_n) is bounded in P and so we can suppose that $z_n \to z$ in C^1, because $\dim(P) < \infty$. So, $y_n \to y := x - z$ in C^1 and then $y \in E^1$. Hence, $C^1 = E^1 + P$.

On the other hand, all elements of E^1 are admissible, and the proof is complete.

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