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Interval Approximation of Higher Order to the Ranges of Functions

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Abstract—The Bernstein and B-spline forms are generalized to multivariate polynomials. These forms are combined with a type of Taylor form for multivariate functions to generate realizable forms for multivariate functions.

Keywords—Range computations, Bernstein form, B-splines, Interval analysis.

1. INTRODUCTION

Interval approximation theory is strongly focussed on the problem of computing good inclusions to the range of a function over a finite interval. A great deal of work has been done in the area, mainly inspired by the development of centered forms as defined by Moore [1]. Centered forms for multivariate polynomials were defined in [2], and later in [3], it was shown that the number of possible multivariate centered forms was very large. A survey of the results in the area up to the time of publication is given in [4].

These outer approximations to the range of a function have application in the solution of equations, in optimization and in a variety of other areas.

In this paper, some of the previously obtained results given in [5] are generalized and extended to higher order approximations for multivariate polynomials and functions. In Sections 2 and 3, the Bernstein and the B-spline forms of multivariate polynomials are discussed. In Section 4, we define a multivariate Taylor form constructed using the ideas of Cornelius-Lohner [6]. Finally, in Section 5 the results of the earlier sections are combined to obtain realizable approximations of higher order for multivariate functions.

2. THE MULTIVARIATE BERNSTEIN FORM

The fundamental idea of using Bernstein polynomials for computing the range of a polynomial over an interval was presented in [7]. Later, the idea was expanded upon in [8–10]. In this section, the idea is further extended to the multivariate case.

Let $p(x_1, \dots, x_s)$ be a polynomial in s real variables with the maximum degree $n_1 + \cdots + n_s$, that is,

$$p(x_1, \dots, x_s) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} a_{i_1 \dots i_s} x_1^{i_1} \cdots x_s^{i_s}, \tag{1}$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where $(x_1, \dots, x_s) \in [a_1, b_1] \times \dots \times [a_s, b_s]$. We also assume that $[a_1, b_1] = \dots [a_s, b_s] = [0, 1]$ in this section without loss of generality since any finite interval can be mapped to $[0, 1]$ by a linear transformation.

We introduce the Bernstein basis functions

$$B_j^k(x) = \binom{k}{j} x^j (1-x)^{k-j}, \qquad x \in [0, 1]$$

and it is easily shown that [7]

$$(1) \quad B_j^k(x) \geq 0, \quad \sum_{j=0}^{k} B_j^k(x) \equiv 1, \qquad x \in [0, 1],$$

$$(2) \qquad x^i = \sum_{j=i}^{k} \frac{\binom{j}{i}}{\binom{k}{i}} B_j^k(x), \qquad x \in [0, 1], \quad i = 0, 1, \dots, n \leq k.$$

From equation (1), it now follows that

$$\begin{aligned} p(x_1, \dots, x_s) &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} a_{i_1 \dots i_s} \sum_{j_1=i_1}^{k_1} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} B_{j_1}^{k_1}(x_1) \cdots \sum_{j_s=i_s}^{k_s} \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}} B_{j_s}^{k_s}(x_s) \\ &= \sum_{i_1=0}^{n_1} \sum_{j_1=i_1}^{k_1} \cdots \sum_{i_s=0}^{n_s} \sum_{j_s=i_s}^{k_s} a_{i_1 \dots i_s} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}} B_{j_1}^{k_1}(x_1) \cdots B_{j_s}^{k_s}(x_s) \\ &= \sum_{j_1=0}^{k_1} \sum_{i_1=0}^{\min(j_1, n_1)} \cdots \sum_{j_s=0}^{k_s} \sum_{i_s=0}^{\min(j_s, n_s)} a_{i_1 \dots i_s} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}} B_{j_1}^{k_1}(x_1) \cdots B_{j_s}^{k_s}(x_s) \qquad (2) \\ &= \sum_{j_1=0}^{k_1} \cdots \sum_{j_s=0}^{k_s} \left(\sum_{i_1=0}^{\min(j_1, n_1)} \cdots \sum_{i_s=0}^{\min(j_s, n_s)} a_{i_1 \dots i_s} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}} \right) B_{j_1}^{k_1}(x_1) \cdots B_{j_s}^{k_s}(x_s) \\ &= \sum_{j_1=0}^{k_1} \cdots \sum_{j_s=0}^{k_s} b_{j_1 \dots j_s} B_{j_1}^{k_1}(x_1) \cdots B_{j_s}^{k_s}(x_s), \end{aligned}$$

where $b_{j_1 \dots j_s}$ is defined to be

$$b_{j_1 \dots j_s} = \sum_{i_1=0}^{\min(j_1, n_1)} \cdots \sum_{i_s=0}^{\min(j_s, n_s)} a_{i_1 \dots i_s} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}}$$

with the assumption that $k_1 \geq n_1, \dots, k_s \geq n_s$.

We can now prove the following theorem.

THEOREM 1. *For $b_{j_1 \dots j_s}$, $j_1 = 0, \dots, k_1, \dots, j_s = 0, \dots, k_s$, we have that*

$$\left| b_{j_1 \dots j_s} - p\left(\frac{j_1}{k_1}, \dots, \frac{j_s}{k_s} \right) \right| = O\left(\frac{1}{k_1} + \dots + \frac{1}{k_s} \right).$$

PROOF.

$$\left|b_{j_1\dots j_s} - p\left(\frac{j_1}{k_1},\dots,\frac{j_s}{k_s}\right)\right|$$

$$= \left|\sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} a_{i_1\dots i_s} \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}} - \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} a_{i_1\dots i_s} \left(\frac{j_1}{k_1}\right)^{i_1} \cdots \left(\frac{j_s}{k_s}\right)^{i_s}\right|$$

$$\leq \sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} |a_{i_1\dots i_s}| \cdot \left|\left(\frac{j_1}{k_1}\right)^{i_1} \cdots \left(\frac{j_s}{k_s}\right)^{i_s} - \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}}\right|$$

$$+ \left|\sum_{i_1=\min(j_1,n_1)+1}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_s=0}^{n_s} a_{i_1\dots i_s} \left(\frac{j_1}{k_1}\right)^{i_1} \cdots \left(\frac{j_s}{k_s}\right)^{i_s}\right|$$

$$+ \cdots + \left|\sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_{s-1}=0}^{\min(j_{s-1},n_{s-1})} \sum_{i_s=\min(j_s,n_s)+1}^{n_s} a_{i_1\dots i_s} \left(\frac{j_1}{k_1}\right)^{i_1} \cdots \left(\frac{j_s}{k_s}\right)^{i_s}\right|$$

$$\leq \sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} |a_{i_1\dots i_s}| \cdot \left|\left(\frac{j_1}{k_1}\right)^{i_1} \cdots \left(\frac{j_s}{k_s}\right)^{i_s} - \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \left(\frac{j_2}{k_2}\right)^{i_2} \cdots \left(\frac{j_s}{k_s}\right)^{i_s}\right|$$

$$+ \cdots + \sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} |a_{i_1\dots i_s}|$$

$$\times \left|\frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_{s-1}}{i_{s-1}}}{\binom{k_{s-1}}{i_{s-1}}} \left(\frac{j_s}{k_s}\right)^{i_s} - \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}} \cdots \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}}\right| + O\left(\frac{1}{k_1^2} + \cdots + \frac{1}{k_s^2}\right)$$

$$= \sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} |a_{i_1\dots i_s}| \cdot \left|\left(\frac{j_1}{i_1}\right)^{i_1} - \frac{\binom{j_1}{i_1}}{\binom{k_1}{i_1}}\right|$$

$$+ \cdots + \sum_{i_1=0}^{\min(j_1,n_1)} \cdots \sum_{i_s=0}^{\min(j_s,n_s)} |a_{i_1\dots i_s}| \cdot \left|\left(\frac{j_s}{i_s}\right)^{i_s} - \frac{\binom{j_s}{i_s}}{\binom{k_s}{i_s}}\right| + O\left(\frac{1}{k_1^2} + \cdots + \frac{1}{k_s^2}\right)$$

$$= O\left(\frac{1}{k_1} + \cdots + \frac{1}{k_s}\right).$$

This means that the quantities $b_{j_1\dots j_s}$ $j_1 = 0,\dots,k_1,\dots,j_s = 0,\dots,k_s$ can be used to construct a Bernstein form for multivariate polynomials for approximating the range $\overline{p}([0,1],\dots,[0,1])$ of the polynomial (1). For this we define

$$B_p^{k_1\cdots k_s}([0,1],\dots,[0,1]) = \left[\min_{j_1\cdots j_s} b_{j_1\cdots j_s}, \max_{j_1\cdots j_s} b_{j_1\cdots j_s}\right]. \tag{3}$$

THEOREM 2. *For (3), we have the following results:*

(1) $\overline{p}([0,1],\dots,[0,1]) \subseteq B_p^{k_1\cdots k_s}([0,1],\dots,[0,1])$,

(2) $w\left(B_p^{k_1\cdots k_s}([0,1],\dots,[0,1])\right) - w\left(\overline{p}([0,1],\dots,[0,1])\right) = O\left(\frac{1}{k_1} + \cdots + \frac{1}{k_s}\right).$

PROOF.

(1) From (2), it follows that

$$\min_{j_1\cdots j_s} b_{j_1\cdots j_s} \le \sum_{j_1=0}^{k_1}\cdots\sum_{j_s=0}^{k_s} b_{j_1\cdots j_s} B_{j_1}^{k_1}(x_1)\cdots B_{j_s}^{k_s}(x_s) = p(x_1,\dots,x_s) \le \max_{j_1\cdots j_s} b_{j_1\cdots j_s}.$$

(2) Let $\overline{p}([0,1],\dots,[0,1]) = [\underline{p}^*,\overline{p}^*]$. Then from Theorem 1, it follows that there exists $(j_1^0/k_1,\dots,j_s^0/k_s) \in [0,1]\times\cdots\times[0,1]$ such that

$$\max_{j_1\cdots j_s} b_{j_1\cdots j_s} - p\left(\frac{j_1^0}{k_1},\dots,\frac{j_s^0}{k_s}\right) = O\left(\frac{1}{k_1}+\cdots+\frac{1}{k_s}\right).$$

From this, it follows that

$$\max_{j_1\cdots j_s} b_{j_1\cdots j_s} - \overline{p}^* = O\left(\frac{1}{k_1}+\cdots+\frac{1}{k_s}\right).$$

In a similar manner, we obtain

$$\underline{p}^* - \min_{j_1\cdots j_s} b_{j_1\cdots j_s} = O\left(\frac{1}{k_1}+\cdots+\frac{1}{k_s}\right).$$

Thus

$$\begin{aligned} &w\left(B_p^{k_1\cdots k_s}([0,1],\dots,[0,1])\right) - w\left(\overline{p}([0,1],\dots,[0,1])\right) \\ &\qquad = \left(\max_{j_1\cdots j_s} b_{j_1\cdots j_s} - \overline{p}^*\right) - \left(\min_{j_1\cdots j_s} b_{j_1\cdots j_s} - \underline{p}^*\right) = O\left(\frac{1}{k_1}+\cdots+\frac{1}{k_s}\right). \end{aligned}$$

3. THE MULTIVARIATE B-SPLINE FORM

The basis functions for the B-splines are [11]

$$N_j^m(x) = \Omega_m\left(k\frac{x-a}{b-a} - \frac{m+1}{2} - j\right), \qquad x\in[a,b],$$

where Ω_m is the m^{th} δ-spline function

$$\Omega_m(x) = \sum_{r=0}^{m+1}(-1)^r\frac{1}{m!}\binom{m+1}{r}\left(x+\frac{m+1}{2}-r\right)_+^m.$$

It is easy to verify that [11]

$$(1)\quad N_j^m(x)\ge 0, \quad \sum_{j=-m}^{k-1} N_j^m(x) \equiv 1, \qquad x\in[a,b],$$

$$(2)\qquad x^i = \sum_{j=-m}^{k-1}\pi_j^{(i)}N_j^m(x), \qquad x\in[a,b],\quad i=0,1,\dots,n,$$

where

$$\pi_j^{(i)} = \frac{\mathrm{Sym}_i(j+1,\dots,j+m)}{k^i\binom{m}{i}}$$

with $\mathrm{Sym}_0(j+1,\dots,j+m) = 1$, $\pi_j^{(0)} = 1$. For $i \geq 1$, we have that $\mathrm{Sym}_i(j+1,\dots,j+m)$ represents the i^{th} elementary symmetric polynomial of $j+1,\dots,j+m$, i.e.,

$$\mathrm{Sym}_i(j+1,\dots,j+m) = \sum_{\nu_1,\dots,\nu_i} \nu_1,\nu_2\cdots,\nu_i, \tag{4}$$

where $\nu_1,\dots,\nu_i$ are i distinct integers arbitrarily chosen from the array $\{j+1,\dots,j+m\}$ and where the number of terms in the sum (4) is $\binom{m}{i}$.

Hence equation (1) can be written as

$$\begin{aligned} p(x_1,\dots,x_s) &= \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} a_{i_1\cdots i_s}\sum_{j_1=-m_1}^{k_1-1}\pi_{j_1}^{(i_1)}N_{j_1}^{m_1}(x_1)\cdots\sum_{j_s=-m_s}^{k_s-1}\pi_{j_s}^{(i_s)}N_{j_s}^{m_s}(x_s) \\ &= \sum_{j_1=-m_1}^{k_1-1}\cdots\sum_{j_s=-m_s}^{k_s-1}\left(\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} a_{i_1\cdots i_s}\pi_{j_1}^{(i_1)}\cdots\pi_{j_s}^{(i_s)}\right)N_{j_1}^{m_1}(x_1)\cdots N_{j_s}^{m_s}(x_s) \\ &= \sum_{j_1=-m_1}^{k_1-1}\cdots\sum_{j_s=-m_s}^{k_s-1} d_{j_1\cdots j_s}N_{j_1}^{m_1}(x_1)\cdots N_{j_s}^{m_s}(x_s), \end{aligned}$$

where $d_{j_1\cdots j_s}$ to be is defined as

$$d_{j_1\cdots j_s} = \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} a_{i_1\cdots i_s}\pi_{j_1}^{(i_1)}\cdots\pi_{j_s}^{(i_s)}.$$

THEOREM 3. *For $d_{j_1\cdots j_s}$, $j_1 = -m_1,\dots,k_1-1,\dots,j_s = -m_s,\dots,k_s-1$, we have*

$$\left|d_{j_1\cdots j_s} - p(\pi_{j_1},\dots,\pi_{j_s})\right| = O\left(\frac{1}{k_1^2}+\cdots+\frac{1}{k_s^2}\right),$$

where

$$\pi_{j_1} = \frac{1}{k_1}\left(j_1+\frac{m_1+1}{2}\right),\dots,\pi_{j_s} = \frac{1}{k_s}\left(j_s+\frac{m_s+1}{2}\right).$$

PROOF.

$$\begin{aligned} &\left|d_{j_1\cdots j_s} - p(\pi_{j_1},\dots,\pi_{j_s})\right| \\ &\quad = \left|\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} a_{i_1\cdots i_s}\pi_{j_1}^{(i_1)}\cdots\pi_{j_s}^{(i_s)} - \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} a_{i_1\cdots i_s}(\pi_{j_1})^{i_1}\cdots(\pi_{j_s})^{i_s}\right| \\ &\quad \leq \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}\left|a_{i_1\cdots i_s}\right|\cdot\left|\pi_{j_1}^{(i_1)}\cdots\pi_{j_s}^{(i_s)} - (\pi_{j_1})^{i_1}\cdots(\pi_{j_s})^{i_s}\right| \\ &\quad \leq \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}\left|a_{i_1\cdots i_s}\right|\cdot\left|\pi_{j_1}^{(i_1)} - (\pi_{j_1})^{i_1}\right|\cdot\left|\pi_{j_2}^{(i_2)}\cdots(\pi_{j_s})^{i_s}\right| \\ &\quad\quad +\cdots+\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}\left|a_{i_1\cdots i_s}\right|\cdot\left|\pi_{j_1}^{(i_1)} - \left(\pi_{j_{s-1}}\right)^{i_{s-1}}\right|\cdot\left|\pi_{j_s}^{(i_s)} - (\pi_{j_s})^{i_s}\right| \\ &\quad = O\left(\frac{1}{k_1^2}+\cdots+\frac{1}{k_s^2}\right). \end{aligned}$$

We now assume that in this section

$$\left[-\frac{m_1-1}{2k_1}, 1+\frac{m_1-1}{2k_1}\right] \subseteq [a_1,b_1],\dots,\left[-\frac{m_s-1}{2k_s}, 1+\frac{m_s-1}{2k_s}\right] \subseteq [a_s,b_s] \tag{5}$$

without loss of generality which means that we can construct the B-spline form as an including approximation to the range $\overline{p}([a_1,b_1],\ldots,[a_s,b_s])$ of the polynomial given in equation (1) as follows:

$$S_p^{k_1\cdots k_s}\left([a_1,b_1],\ldots,[a_s,b_s]\right)=\left[\min_{j_1\cdots j_s} d_{j_1\cdots j_s},\ \max_{j_1\cdots j_s} d_{j_1\cdots j_s}\right]. \tag{6}$$

In a similar manner as in the proof of Theorem 2, we can prove the following theorem.

THEOREM 4. *For the estimate given by equation (6), we have*

$$\text{(1)}\quad \overline{p}\left([a_1,b_1],\ldots,[a_s,b_s]\right)\subseteq S_p^{k_1\cdots k_s}\left([a_1,b_1],\ldots,[a_s,b_s]\right),$$

$$\text{(2)}\quad w\left(S_p^{k_1\cdots k_s}\left([a_1,b_1],\ldots,[a_s,b_s]\right)\right)-w\left(\overline{p}\left([a_1,b_1],\ldots,[a_s,b_s]\right)\right)=O\left(\frac{1}{k_1^2}+\cdots+\frac{1}{k_s^2}\right).$$

4. A MULTIVARIATE TAYLOR FORM

In this section, we consider a multivariate Taylor form along the lines of the form developed in [6]. We assume that the real function $f(x_1,\ldots,x_s)$, $f:[a_1,b_1]\times\cdots\times[a_s,b_s]\longrightarrow R^s$ is $n+1$ times differentiable on the s-dimensional interval $[a_1,b_1]\times\cdots\times[a_s,b_s]$ and that $(x_1,\ldots,x_s)\in[a_1,b_1]\times\cdots\times[a_s,b_s]$. The Taylor expansion of f is then

$$f(x_1,\ldots,x_s)=p(x_1,\ldots,x_s)+r(\xi_1,\ldots,\xi_s),$$

where

$$p(x_1,\ldots,x_s)=\sum_{i_1+\cdots+i_s=0}^{n} a_{i_1\cdots i_s}(c_1,\ldots,c_s)(x_1-c_1)^{i_1}\cdots(x_s-c_s)^{i_s}, \tag{7}$$

$$(c_1,\ldots,c_s)\in[a_1,b_1]\times\cdots\times[a_s,b_s],$$

$$a_{i_1\cdots i_s}(z_1,\ldots,z_s)=\frac{1}{i_1!\cdots i_s!}\frac{\partial f^{(i_1+\cdots+i_s)}(z_1,\ldots,z_s)}{\partial x_1^{i_1}\cdots\partial x_s^{i_s}},$$

$$r(\xi_1,\ldots,\xi_s)=\sum_{i_1+\cdots+i_s=n+1} a_{i_1\cdots i_s}(\xi_1,\ldots,\xi_s)(x_1-c_1)^{i_1}\cdots(x_s-c_s)^{i_s},$$

$$(\xi_1,\ldots,\xi_s)\in[a_1,b_1]\times\cdots\times[a_s,b_s].$$

For $X_1\times\cdots\times X_s\subseteq[a_1,b_1]\times\cdots\times[a_s,b_s]$, the Taylor form can be expressed as

$$F(X_1,\ldots,X_s)=\overline{p}(X_1,\ldots,X_s)+r(X_1,\ldots,X_s), \tag{8}$$

where

(i) $\overline{p}(X_1,\ldots,X_s)$ is the range of p over $X_1\times\cdots\times X_s$ and

$$\text{(ii)}\quad r(X_1,\ldots,X_s)=\sum_{i_1+\cdots+i_s=n+1} a_{i_1\cdots i_s}(X_1,\ldots,X_s)(X_1-c_1)^{i_1}\cdots(X_s-c_s)^{i_s}. \tag{9}$$

We have the following theorem for the form defined by (8).

THEOREM 5. *Assume that the Taylor form is defined by (8). Then*

$$\text{(i)}\quad \overline{f}(X_1,\ldots,X_s)\subseteq F(X_1,\ldots,X_s),$$

$$\text{(ii)}\quad w(F(X_1,\ldots,X_s))-w(\overline{f}(X_1,\ldots,X_s))=O\left(w^*\left((X_1,\ldots,X_s)^{n+1}\right)\right),$$

where $w^*(X_1,\ldots,X_s)=\max\{w(X_1),\ldots,w(X_s)\}$.

PROOF. The proof of (i) follows from the definition.

For (ii), let

$$\overline{f}(X_1,\dots,X_s) = \left[f\left(\underset{*}{x}_1,\dots,\underset{*}{x}_s\right), f\left(\overset{*}{x}_1,\dots,\overset{*}{x}_s\right)\right],$$

$$\overline{p}(X_1,\dots,X_s) = \left[p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right), p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right)\right],$$

where $(\underset{*}{x}_1,\dots,\underset{*}{x}_s)$ and $(\underset{*}{y}_1,\dots,\underset{*}{y}_s)$ are the minimum points of f and p, respectively, on $X_1\times\cdots\times X_s$ and where $(\overset{*}{x}_1,\dots,\overset{*}{x}_s)$ and $(\overset{*}{y}_1,\dots,\overset{*}{y}_s)$ are the maximum points of f and p, respectively, on $X_1\times\cdots\times X_s$. We also define $r(X_1,\dots,X_s)=[\underline{r},\overline{r}]$.

Thus

$$\begin{aligned}
&w\left(F\left(X_1,\dots,X_s\right)\right) - w\left(\overline{f}\left(X_1,\dots,X_s\right)\right)\\
&\quad = p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right) + \overline{r} - p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) - \underline{r} - f\left(\overset{*}{x}_1,\dots,\overset{*}{x}_s\right) + f\left(\underset{*}{x}_1,\dots,\underset{*}{x}_s\right)\\
&\quad = \left[p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right) + \overline{r} - f\left(\overset{*}{x}_1,\dots,\overset{*}{x}_s\right)\right] + \left[f\left(\underset{*}{x}_1,\dots,\underset{*}{x}_s\right) - p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) - \underline{r}\right]\\
&\quad \le \left[p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right) + \overline{r} - f\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right)\right] + \left[f\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) - p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) - \underline{r}\right]\\
&\quad \le \left[p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right) + \overline{r} - \left(p\left(\overset{*}{y}_1,\dots,\overset{*}{y}_s\right) + \underline{r}\right)\right] + \left[\left(p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) + \overline{r}\right) - p\left(\underset{*}{y}_1,\dots,\underset{*}{y}_s\right) - \underline{r}\right]\\
&\quad = 2\left(\overline{r} - \underline{r}\right) = 2w\left(r\left(X_1,\dots,X_s\right)\right)\\
&\quad = 2\cdot w\left[\sum_{i_1+\cdots i_s=n+1} a_{i_1\cdots i_s}\left(X_1,\dots,X_s\right)\left(X_1-c_1\right)^{i_1}\cdots\left(X_s-c_s\right)^{i_s}\right]\\
&\quad = O\left(w^*\left(X_1,\dots,X_s\right)^{n+1}\right),
\end{aligned}$$

which proves the theorem.

5. A TAYLOR APPROXIMATION OF HIGHER ORDER

Although the approximation formula in the previous section has order $n+1$, it is difficult to realize in practice. The reason for this is that it requires the computation of the range of an n^{th} degree polynomial, a difficult problem in its own right. For this reason, we combine the result in the previous section with the results of Sections 2 and 3 such that the combined method is a very effective approximation method.

The problem is to find an estimate for the s-dimensional function $f(x_1,\dots,x_s)$ over the interval $X_1\times\cdots\times X_s$.

The concrete steps in this process are as follows:

1. First find the Taylor polynomial of (7) of f. Then select a linear transformation T such that $X_1\times\cdots\times X_s\to[0,1]\times\cdots\times[0,1]$ or $X_1\times\cdots\times X_s\to[a_1,b_1]\times\cdots\times[a_s,b_s]$ satisfying (5). The transformation T takes $p(x_1,\dots,x_s)$ into $p^*(x_1,\dots,x_s)$.
2. There is now a choice of either finding the Bernstein form $B_{p^*}^{k_1\cdots k_s}$ of $p^*(X_1,\dots,X_s)$ where

$$k_1,\dots,k_s \ge \left[\frac{1}{w^*\left(X_1,\dots,X_s\right)}\right]^{n+1}$$

following Section 2, or the B-spline form $S_{p^*}^{k_1\cdots k_s}$ of $p^*(X_1,\dots,X_s)$, where

$$k_1,\dots,k_s \ge \left[\frac{1}{w^*\left(X_1,\dots,X_s\right)}\right]^{(n+1)/2}$$

following Section 3. Here we will note that the choice of $k_1, \ldots, k_s$ is independent of $w^*(X_1, \ldots, X_s)$.

3. Now find

$$\begin{aligned} F_B(X_1, \ldots, X_s) &= B_{p^*}^{k_1 \cdots k_s} + r(X_1, \ldots, X_s) \quad \text{or} \\ F_S(X_1, \ldots, X_s) &= S_{p^*}^{k_1 \cdots k_s} + r(X_1, \ldots, X_s), \end{aligned} \tag{10}$$

where r is defined as in (9). Equation (10) is then our approximation formula of higher order.

The following theorem relates to the above procedure.

THEOREM 6. *Let the approximation form (10) hold. Then*

$$\begin{aligned} &\text{(i)} \quad \overline{f}(X_1, \ldots, X_s) \subseteq F_B(X_1, \ldots, X_s), \\ &\qquad \overline{f}(X_1, \ldots, X_s) \subseteq F_S(X_1, \ldots, X_s), \\ &\text{(ii)} \quad w(F_B(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) = O\left(w^*(X_1, \ldots, X_s)^{n+1}\right), \\ &\qquad w(F_S(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) = O\left(w^*(X_1, \ldots, X_s)^{n+1}\right). \end{aligned}$$

PROOF. The proof of (i) is obvious. For (ii), let us consider $F_B(X_1, \ldots, X_s)$. From Theorem 2, we have

$$w\left(B_{p^*}^{k_1 \cdots k_s}\right) - w(\overline{p}(X_1, \ldots, X_s)) = O\left(\frac{1}{k_1} + \cdots + \frac{1}{k_s}\right) = O\left(w^*(X_1, \ldots, X_s)^{n+1}\right)$$

and by Theorem 5, we have

$$w(F(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) = O\left(w^*(X_1, \ldots, X_s)^{n+1}\right)$$

and hence

$$\begin{aligned} &w(F_B(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) \\ &\quad \leq w\left(B_{p^*}^{k_1 \cdots k_s} + r(X_1, \ldots, X_s)\right) - w(\overline{f}(X_1, \ldots, X_s)) \\ &\quad = w\left(B_{p^*}^{k_1 \cdots k_s}\right) + w(r(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) \\ &\quad = w\left(B_{p^*}^{k_1 \cdots k_s}\right) - w(\overline{p}(X_1, \ldots, X_s)) \\ &\qquad + w(\overline{p}(X_1, \ldots, X_s)) + w(r(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s)) \\ &\quad = \left[w\left(B_{p^*}^{k_1 \cdots k_s}\right) - w(\overline{p}(X_1, \ldots, X_s))\right] + \left[w(F(X_1, \ldots, X_s)) - w(\overline{f}(X_1, \ldots, X_s))\right] \\ &\quad = O\left(w^*(X_1, \ldots, X_s)^{n+1}\right). \end{aligned}$$

The proof is identical for $F_B(X_1, \ldots, X_s)$.

We should note that if $(w^*(X_1, \ldots, X_s)^{n+1})$ is very small, then $k_1, \ldots, k_s$ will become very large since $B_{p^*}^{k_1 \cdots k_s}$ or $S_{p^*}^{k_1 \cdots k_s}$ is only linearly or quadratically convergent. This means that the computation of $B_{p^*}^{k_1 \cdots k_s}$ or $S_{p^*}^{k_1 \cdots k_s}$ should in general be implemented on a computer due to the extensive computations required.

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