Asymptotic Behavior of Nonoscillatory Solutions of Second-Order Differential Equations

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Abstract—In this paper, we establish some sufficient conditions for the asymptotic behavior of all nonoscillatory solutions of the differential equation,
\[ \{p(t) \phi(y'(t))\}' + q(t) f(y(t)) = 0, \]
under suitable condition on \( p \in C([t_0, \infty); (0, \infty)) \), \( q \in C([t_0, \infty); \mathbb{R}) \), and \( \phi, f \in C(\mathbb{R}; \mathbb{R}) \). © 2005 Elsevier Ltd All rights reserved.

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1. INTRODUCTION

In this paper, we consider the second-order nonlinear differential equation,
\[ \{p(t) \phi(y'(t))\}' + q(t) f(y(t)) = 0, \quad t \geq t_0, \quad (1.1) \]
where \( p \in C([t_0, \infty); (0, \infty)) \), \( q \in C([t_0, \infty); \mathbb{R}) \), and \( \phi, f \in C(\mathbb{R}; \mathbb{R}) \). We list some conditions on \( p, q, \phi, \) and \( f \) as follows.

(C1) \( f'(y) \geq 0 \) and \( yf(y) > 0 \), for \( y \neq 0 \);
(C2) \( \text{sgn} \phi(u) = \text{sgn} |u| \);
(C3) \( \phi(u) \text{sgn} u \) has the inverse function \( \psi(u) \);
(C4) \( q(t) \geq 0 \) and \( q(t) \neq 0 \), on any nonempty open interval.
If \( \phi(u) = u^\sigma \), where \( \sigma \) is a positive quotient of even over odd integers, then equation (1.1) reduces to the equation,

\[
\left\{ p(t) \left[ y'(t) \right]^\sigma \right\}' + q(t) f(y(t)) = 0, \quad t \geq t_0. \tag{1.2}
\]

In [1], Bai established some sufficient conditions of asymptotic behavior of nonoscillatory solutions of equation (1.2) if

\[
\int_0^\infty \frac{1}{p^{1/\sigma}(t)} \, dt = \infty.
\]

The purpose of this paper is to extend Bai's results from equation (1.2) to equation (1.1), and establish some new results if

\[
\int_0^\infty \psi \left( \frac{1}{p(t)} \right) \, dt < \infty
\]

holds.

2. MAIN RESULTS

In this section, we discuss the asymptotic behavior of nonoscillatory solutions of equation (1.1)

**Lemma 2.1.** Let \( y(t) \) be a nonoscillatory solution and \((C_4)\) hold. Then, \( y'(t) \) is nonoscillatory.

**Proof.** Suppose to the contrary that \( y'(t) \) is oscillatory. Without loss of generality, we assume that \( y(t) > 0 \) for \( t \geq T \) for some \( T \geq t_0 \). It follows from \((C_4)\) and (1.1) that \( p(t) \phi(y'(t)) \) is nonincreasing on \([T, \infty)\). This implies that there is a \( T_1 \geq T \) such that \( p(t) \phi(y'(t)) = 0 \) for \( t \geq T_1 \). Hence, \( q(t) = 0 \) on \([T_1, \infty)\), which contradicts \((C_4)\). Then, \( y'(t) \) is nonoscillatory.

**Theorem 2.1.** Let \((C_1), (C_2), (C_4),\) and

\[
\int_0^\infty q(t) \, dt = \infty. \tag{2.1}
\]

hold. If \( y(t) \) is an eventually positive solution of equation (1.1), then \( \lim_{t \to \infty} y(t) = 0 \).

**Proof.** Let \( y(t) > 0 \) on \([T, \infty)\) for some \( T \geq t_0 \). It follows from Lemma 2.1 and (1.1) that \( y'(t) \) is nonoscillatory and

\[
\left( \frac{p(t) \phi(y'(t))}{f(y(t))} \right)' = -q(t) - \frac{p(t) \phi(y'(t)) f'(y(t)) y'(t)}{f^2(y(t))},
\]

for \( t \geq T \). We separate our proof into two cases.

**Case 1.** Suppose that \( y'(t) \) is eventually positive. We assume without loss of generality that \( y'(t) > 0 \) for \( t \geq T \). By \((C_1), (C_2),\) and (2.2), we have

\[
\left( \frac{p(t) \phi(y'(t))}{f(y(t))} \right)' \leq -q(t), \tag{2.3}
\]

for \( t \geq T \). Integrating it from \( T \) to \( t \), we get

\[
\frac{p(t) \phi(y'(t))}{f(y(t))} \leq \frac{p(T) \phi(y'(T))}{f(y(T))} - \int_T^t q(s) \, ds.
\]

Letting \( t \to \infty \) and using condition (2.1), we obtain a contradiction. Hence, \( y'(t) \) is eventually negative.

**Case 2** Suppose that \( y'(t) < 0 \) is eventually negative. We assume without loss of generality that \( y'(t) < 0 \) for \( t \geq T \), and there exists a number \( \alpha \geq 0 \), such that

\[
\lim_{t \to \infty} y(t) = \alpha. \tag{2.4}
\]
We claim that $a = 0$. Otherwise, $y(t) \geq a > 0$ for $t \geq T$. By (1.1) and (C1), we yield

$$\left[ p(t) \phi(y'(t)) \right]' = -q(t) f(y(t)) \leq -q(t) f(a),$$

for $t \geq T$. Integrating it from $T$ and $t$, we have

$$p(t) \phi(y'(t)) \leq p(T) \phi(y'(T)) - f(a) \int_T^t q(s) \, ds,$$

for $t \geq T$. By (2.1), the right side of (2.6) tends to $-\infty$ as $t \to \infty$ whereas the left side of (2.6) is positive, which is a contradiction. This contradiction completes the proof of the theorem.

**Theorem 2.2.** Let (C1)-(C4) and

$$\int_0^{\infty} \psi \left( \frac{k}{p(t)} \right) \sgn k \, dt = \infty, \quad \text{for every } k \neq 0,$$

hold. If $y(t)$ is an eventually negative solution of equation (1.1), then $\lim_{t \to \infty} y(t) = -\infty$.

**Proof.** Let $y(t) < 0$ on $[T, \infty)$ for some $T > t_0$. By Lemma 2.1, $y'(t)$ is nonoscillatory. We separate our proof into two cases.

**Case 1.** Suppose that $y'(t)$ is eventually positive. We assume without loss of generality that $y'(t) > 0$ on $[T, \infty)$. It follows from (1.1) that $p(t)\phi(y'(t))$ is nondecreasing on $[T, \infty)$. Then,

$$p(t) \phi(y'(t)) \geq p(T) \phi(y'(T)), \quad t \geq T.$$

Hence,

$$y'(t) \geq \psi \left( \frac{p(T) \phi(y'(T))}{p(t)} \right),$$

this implies

$$y(t) \geq y(T) + \int_T^t \psi \left( \frac{p(T) \phi(y'(T))}{p(t)} \right) \, dt.$$

It follows from (2.7) that $\lim_{t \to \infty} y(t) = \infty$, which contradicts that $y(t) < 0$ on $[T, \infty)$. Hence, $y'(t)$ is eventually negative.

**Case 2.** Suppose that $y'(t)$ is eventually negative. We assume without loss of generality that $y'(t) < 0$ on $[T, \infty)$. It follows from (1.1) that $p(t)\phi(y'(t))$ is nondecreasing on $[T, \infty)$. Then,

$$p(t) \phi(y'(t)) \geq p(T) \phi(y'(T)), \quad t \geq T.$$

Hence,

$$y'(t) = \psi(-\phi(y'(t))) \leq \psi \left( -\frac{p(T) \phi(y'(T))}{p(t)} \right),$$

this implies

$$y(t) \leq y(T) + \int_T^t \psi \left( -\frac{p(T) \phi(y'(T))}{p(t)} \right) \, dt.$$

It follows from (2.7) that $\lim_{t \to \infty} y(t) = -\infty$. This completes our proof of the theorem.

**Theorem 2.3.** Let (C1), (C2), and (C4) hold. Suppose that $\phi(u)$ has the inverse function $\psi(u)$ on $[0, \infty)$ and

$$\int_0^{\infty} \psi \left( \frac{k}{p(t)} \right) \, dt < \infty, \quad \text{for every } k > 0,$$

holds. If $y(t)$ is an eventually positive solution of equation (1.1), then $y(t)$ is bounded.

**Proof.** Let $y(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. By Lemma 2.1, $y'(t)$ is nonoscillatory. Hence, $y'(t)$ is either eventually negative or eventually positive. If $y'(t)$ is eventually negative, $y(t)$ is
bounded. If \( y'(t) \) is eventually positive, we assume without loss of generality that \( y'(t) > 0 \) for \( t \geq T \). It follows from (1.1) that \( p(t) \phi(y'(t)) \) is nonincreasing on \( [T, \infty) \). Then,
\[
p(t) \phi(y'(t)) \leq p(T) \phi(y'(T)), \quad t \geq T.
\]
Hence,
\[
y'(t) \leq \psi\left(\frac{p(T) \phi(y'(T))}{p(t)}\right),
\]
this implies
\[
y(t) \leq y(T) + \int_{T}^{t} \psi\left(\frac{p(T) \phi(y'(T))}{p(s)}\right) ds.
\]
It follows from (2.12) that \( y(t) \) is bounded. This completes our proof of the theorem. 

**THEOREM 2.4.** Let \((C_1), (C_2), \text{ and } (C_4)\) hold. Suppose that \( \phi(u) \) has the inverse function \( \psi(u) \) on \([0, \infty)\). Then, equation (1.1) has an eventually increasing positive bounded solution \( y(t) \) if and only if
\[
\int_{T}^{\infty} \psi\left(\frac{1}{p(s)} \int_{s}^{\infty} q(s)f(\lambda) ds \right) dt < \infty, \quad \text{for some } \lambda > 0.
\]

**PROOF.** If (2.13) holds, there exists a number \( T \geq t_0 \), such that
\[
\int_{T}^{\infty} \psi\left(\frac{1}{p(s)} \int_{s}^{\infty} q(s)f(\lambda) ds \right) dt < \frac{\lambda}{2}.
\]
Define a set \( \Omega \) by
\[
\Omega := \left\{ y \in C([T, \infty) ; \mathbb{R}) \left| \frac{\lambda}{2} \leq y(t) \leq \lambda \text{ on } [T, \infty) \right. \right\}
\]
and an operator \( \varphi \) on \( \Omega \) by
\[
(\varphi y)(t) = \frac{\lambda}{2} + \int_{T}^{t} \psi\left(\frac{1}{p(s)} \int_{s}^{\infty} q(\mu)f(\lambda) d\mu \right) ds, \quad t \geq T.
\]
Clearly, \( \varphi(\Omega) \subset \Omega \).

Define a sequence of functions \( \{y_n(t)\}_{n=0}^{\infty} \) as follows,
\[
y_0(t) = \lambda, \quad t \geq T,
\]
\[
y_n(t) = (\varphi y_{n-1})(t), \quad t \geq T, \quad n = 1, 2, 3, \ldots
\]
By induction, we can prove that
\[
\lambda \geq y_0(t) \geq y_1(t) \geq y_2(t) \geq \cdots \geq \frac{\lambda}{2}, \quad t \geq T.
\]
Then there exists a function \( y \in \Omega \), such that \( \lim_{n \to \infty} y_n(t) = y(t) \) on \([T, \infty)\). It is obvious that \( y(t) > \lambda/2 \) on \([T, \infty)\). By Lebesgue's dominated convergence theorem, we have \( y = \varphi y \), that is, \( y(t) \) is a increasing positive solution of (1.1).

Conversely, let \( y(t) \) be an eventually increasing positive bounded solution of (1.1). Without loss of generality, we assume that \( y(t) > 0 \) and \( y'(t) > 0 \) for \( t \geq T \) for some \( T \geq t_0 \). It follows from (1.1) that \( p(t) \phi(y'(t)) \) is positive and nonincreasing for \( t \geq T \) and
\[
p(\xi) \phi(y'(\xi)) - p(t) \phi(y'(t)) = -\int_{t}^{\xi} q(s)f(y'(s)) ds \leq -\int_{t}^{\xi} q(s)f(y(T)) ds, \quad \xi \geq t \geq T.
\]
This implies that
\[
p(t) \phi(y'(t)) \geq \int_{t}^{\infty} q(s)f(y(T)) ds, \quad t \geq T.
\]
Then,
\[
y(t) - y(T) \geq \int_{T}^{t} \psi\left(\frac{1}{p(s)} \int_{s}^{\infty} q(\mu)f(y(T)) d\mu \right) ds, \quad \text{on } [T, \infty).\]
Hence,
\[
\int_{T}^{\infty} \psi\left(\frac{1}{p(s)} \int_{s}^{\infty} q(\mu)f(y(T)) d\mu \right) ds < \infty.
\]
This completes our proof. 

THEOREM 2.5. Let \((C_1), (C_2), \text{ and } (C_4)\) hold. Suppose that \(-\phi(u)\) has the inverse function \(\psi(u)\) on \((\infty, 0]\). Then, equation (1.1) has an eventually decreasing positive solution \(y(t)\) with \(\lim_{t \to \infty} y(t) = \alpha\) if and only if

\[
\int_{T}^{\infty} \psi \left( \frac{-1}{p(t)} \int_{t}^{\infty} q(s) f(\lambda) \, ds \right) \, dt > -\infty, \quad \text{for some } \lambda > 0, \tag{2.14}
\]

holds.

PROOF. If (2.14) holds, there exists a number \(T \geq t_0\), such that

\[
\int_{T}^{\infty} \psi \left( \frac{-1}{p(t)} \int_{t}^{\infty} q(s) f(\lambda) \, ds \right) \, dt > \frac{\lambda}{2}.
\]

Define a set \(\Omega\) by

\[
\Omega := \left\{ y \in C([T, \infty); \mathbb{R}) \mid \frac{\lambda}{2} \leq y(t) \leq \lambda, \text{ on } [T, \infty) \right\},
\]

and an operator \(\phi\) on \(\Omega\) by

\[
(\phi y)(t) = \frac{\lambda}{2} - \int_{T}^{\infty} \psi \left( \frac{-1}{p(s)} \int_{s}^{\infty} q(\mu) f(y(\mu)) \, d\mu \right) \, ds, \quad t \geq T.
\]

Clearly, \(\phi(\Omega) \subseteq \Omega\).

Define a sequence of functions \(\{y_n(t)\}_{n=0}^{\infty}\) as follows,

\[
\begin{align*}
y_0(t) &= \lambda, & t \geq T, \\
y_n(t) &= (\phi y_{n-1})(t), & t \geq T, \quad n = 1, 2, 3, \ldots.
\end{align*}
\]

By induction, we can prove that

\[
\lambda \geq y_0(t) \geq y_1(t) \geq y_2(t) \geq \cdots \geq \frac{\lambda}{2}, \quad t \geq T.
\]

Then, there exists a function \(y \in \Omega\), such that \(\lim_{n \to \infty} y_n(t) = y(t)\) on \([T, \infty)\). It is obvious that \(y(t) \geq \lambda/2\) on \([T, \infty)\). By Lebesgue's dominated convergence theorem, we have \(y = \phi y\), that is, \(y(t)\) is a decreasing positive solution of (1.1).

Conversely, \(y(t)\) is eventually decreasing positive solution of (1.1) with \(\lim_{t \to \infty} y(t) = \alpha > 0\). Without loss of generality, we assume that \(y(t) > 0\) and \(y'(t) < 0\) for \(t \geq T\) for some \(T \geq t_0\). Then, \(y(t) \geq \alpha\) for \(t \geq T\). It follows from (1.1) that

\[
p(\xi) \phi(y'(\xi)) - p(t) \phi(y'(t)) = -\int_{t}^{\xi} q(s) f(y(s)) \, ds \leq -\int_{t}^{\xi} q(s) f(\alpha) \, ds, \quad \xi \geq t \geq T;
\]

which implies

\[
p(t) \phi(y'(t)) \geq \int_{t}^{\infty} q(s) f(\alpha) \, ds, \quad t \geq T.
\]

Then,

\[
y'(t) = \psi(-\phi(y'(t))) \leq \psi \left( \frac{-1}{p(t)} \int_{t}^{\infty} q(s) f(\alpha) \, ds \right), \quad t \geq T.
\]

Integrating it over \([T, \infty)\), we get

\[
\alpha - y(T) \leq \int_{T}^{\infty} \psi \left( \frac{-1}{p(t)} \int_{t}^{\infty} q(s) f(\alpha) \, ds \right) \, dt.
\]

Then

\[
\int_{T}^{\infty} \psi \left( \frac{-1}{p(t)} \int_{t}^{\infty} q(s) f(\alpha) \, ds \right) \, dt \geq -y(T) > -\infty
\]

This completes our proof of the theorem. \(\blacksquare\)
THEOREM 2.6. Let \((C_1), (C_2), \) and \((C_4)\) hold. Suppose that \(\phi(u)\) has the inverse function \(\psi(u)\) on \([0, \infty)\) and there exist a number \(\alpha > 0\), such that

\[
\int_t^\infty \psi \left( \frac{\alpha}{p(s)} \right) ds \to \infty, \quad \text{as } t \to \infty.
\]

Then, equation \((1.1)\) has an eventually increasing positive solution \(y(t)\) satisfying

\[
y(t) \geq G_\lambda(t) \to \infty, \quad \text{for some } \lambda > \alpha,
\]

if and only if

\[
\int_0^\infty q(t) f(G_\beta(t)) dt < \infty, \quad \text{for some } \beta > \alpha,
\]

holds, where

\[
G_m(t) = \int_t^\infty \psi \left( -\frac{m}{p(s)} \right) ds.
\]

PROOF. By \((2.16)\), there exists two numbers \(\alpha < \lambda < \beta\) and \(T \geq t_0\), such that

\[
\int_T^\infty q(s) f(G_\beta(s)) ds < \beta - \lambda, \quad \text{for } t \geq T.
\]

Define a set \(\Omega\) by

\[
\Omega := \{ y \in C([T, \infty), \mathbb{R}) | G_\lambda(t) \leq y(t) \leq G_\beta(t) \text{ on } [T, \infty) \},
\]

and an operator \(\varphi\) on \(\Omega\) by

\[
(\varphi y)(t) = \int_T^t \psi \left( \frac{1}{p(s)} \left( \lambda + \int_s^\infty q(\mu) f(y(\mu)) d\mu \right) \right) ds, \quad t \geq T.
\]

Clearly, \(\varphi(\Omega) \subset \Omega\).

Define a sequence of functions \(\{y_n(t)\}_{n=0}^\infty\) as follows,

\[
y_0(t) = G_\beta(t), \quad t \geq T,
\]

\[
y_n(t) = (\varphi y_{n-1})(t), \quad t \geq T, \quad n = 1, 2, 3, \ldots.
\]

By induction, we can prove that

\[
G_\beta(t) \geq y_0(t) \geq y_1(t) \geq y_2(t) \geq \ldots \geq G_\lambda(t), \quad t \geq T.
\]

Then, there exists a function \(y \in \Omega\), such that \(\lim_{n \to \infty} y_n(t) = y(t)\) on \([T, \infty)\). It is obvious that \(y(t) \geq \lambda\) on \([T, \infty)\). By Lebesgue’s dominated convergence theorem, we have \(y = \varphi y\), that is, \(y(t)\) is a increasing positive solution of \((1.1)\) and \(y(t) \geq G_\lambda(t)\).

Conversely, let \(y(t)\) be an eventually increasing positive solution \(y(t)\) satisfying \((2.15)\). Let \(\alpha < \beta < \lambda\). Then, \(y(t) \geq G_\beta(t)\). From \((1.1)\), we have

\[
p(\xi) \phi(y'(\xi)) - p(T) \phi(y'(T)) \leq - \int_T^\xi q(s) f(G_\beta(s)) ds, \quad \xi \geq t \geq T.
\]

Then,

\[
p(T) \phi(y'(T)) \geq \int_T^\infty q(s) f(G_\beta(s)) ds, \quad t \geq T
\]

Hence, \((2.16)\) holds.
THEOREM 2.7. Let (C1), (C2), and (C4) hold. Suppose that $\phi(u)$ has the inverse function $\psi(u)$ on $[0, \infty)$. Then, equation (1.1) has an eventually increasing negative solution $y(t)$ satisfying $\lim_{t \to \infty} y(t) = \alpha$ for some $\alpha < 0$ if and only if

$$\int_{T}^{\infty} \psi \left( -\frac{1}{p(t)} \int_{T}^{t} q(s) f(\lambda) \, ds \right) \, dt < \infty, \quad \text{for some } \lambda < 0. \quad (2.17)$$

PROOF. If (2.17) holds, there exists a number $T \geq t_0$, such that

$$\int_{T}^{\infty} \psi \left( -\frac{1}{p(t)} \int_{T}^{t} q(s) f(\lambda) \, ds \right) \, dt < -\frac{\lambda}{2}.$$  

Define a set $\Omega$ by

$$\Omega := \left\{ y \in C([T, \infty); \mathbb{R}) \mid \lambda \leq y(t) \leq \frac{\lambda}{2} \text{ on } [T, \infty) \right\}$$

and an operator $\varphi$ on $\Omega$ by

$$(\varphi y)(t) = \frac{\lambda}{2} - \int_{T}^{\infty} \psi \left( -\frac{1}{p(s)} \int_{t}^{s} q(\mu) f(y(\mu)) \, d\mu \right) \, ds, \quad t \geq T.$$  

Clearly, $\varphi(\Omega) \subset \Omega$.

Define a sequence of functions $\{y_n(t)\}_{n=0}^{\infty}$ as follows,

$$y_0 (t) = \frac{\lambda}{2}, \quad t \geq T,$$

$$y_n (t) = (\varphi y_{n-1})(t), \quad t \geq T, \quad n = 1, 2, 3, \ldots.$$  

By induction, we can prove that

$$\frac{\lambda}{2} = y_0 (t) \geq y_1 (t) \geq y_2 (t) \geq \cdots \geq \lambda, \quad t \geq T.$$  

Then, there exists a function $y \in \Omega$, such that $\lim_{n \to \infty} y_n(t) = y(t)$ on $[T, \infty)$. It is obvious that $y(t) \leq \lambda/2$ on $[T, \infty)$. By Lebesgue’s dominated convergence theorem, we have $y = \varphi y$, that is, $y(t)$ is an increasing negative solution of (1.1).

Conversely, let $y(t)$ be an eventually increasing negative solution of (1.1) satisfying

$$\lim_{t \to \infty} y(t) = \alpha < 0.$$  

Without loss of generality, we assume that $y(t) < 0$ and $y'(t) > 0$ for $t \geq T$ for some $T \geq t_0$. It follows from (1.1) that

$$p(t) \phi(y'(t)) - p(T) \phi(y'(T)) = -\int_{T}^{t} q(s) f(y(s)) \, ds \geq -\int_{T}^{t} q(s) f(\alpha) \, ds, \quad t \geq T.$$  

This implies that

$$p(t) \phi(y'(t)) \geq -\int_{T}^{t} q(s) f(\alpha) \, ds, \quad t \geq T.$$  

Then,

$$\alpha - y(T) \geq \int_{T}^{\infty} \psi \left( -\frac{1}{p(s)} \int_{T}^{s} q(\mu) f(\alpha) \, d\mu \right) \, ds, \quad \text{on } [T, \infty).$$  

Hence,

$$\int_{T}^{\infty} \psi \left( -\frac{1}{p(s)} \int_{T}^{s} q(\mu) f(\alpha) \, d\mu \right) \, ds < \infty.$$  

This completes our proof.
Lemma 2.2. Let (C2) and (C3) hold. If
\[ \phi(uv) \leq \phi(u) \phi(v), \] (2.18)
for all \( u, v \in R, \) then
\[ \psi(xy) \operatorname{sgn} xy \geq \psi(x) \psi(y) \operatorname{sgn} xy, \] (2.19)
for all \( x, y \in R. \)

Proof. By (C2), \( \phi(t) \) is decreasing on \( (-\infty, 0] \) and increasing on \([0, \infty). \) Thus, \( \psi(t) \) is increasing.

Case 1. If \( xy = 0, \) (2.19) is obvious.

Case 2. If \( x > 0 \) and \( y > 0, \) there exist two positive numbers \( \alpha \) and \( \beta, \) such that \( \phi(\alpha) = x \) and \( \phi(\beta) = y. \) Then, \( \psi(x) = \alpha, \psi(y) = \beta, \) and
\[ \psi(xy) = \psi(\phi(\alpha) \phi(\beta)) \geq \psi(\phi(\alpha\beta)) = \alpha\beta = \psi(x) \psi(y). \]

Case 3. If \( x < 0 \) and \( y < 0, \) there exist two negative numbers \( \alpha \) and \( \beta, \) such that \( -\phi(\alpha) = x \) and \( -\phi(\beta) = y. \) Then, \( \psi(x) = \alpha, \psi(y) = \beta, \) and
\[ \psi(xy) = \psi(-\phi(\alpha)\phi(\beta)) \leq \psi(-\phi(\alpha\beta)) = \alpha\beta = \psi(x)\psi(y). \]

Case 4. If \( xy < 0, \) we assume without loss of generality that \( x > 0 \) and \( y < 0. \) There exist two numbers \( \alpha > 0 \) and \( \beta < 0, \) such that \( \phi(\alpha) = x \) and \( -\phi(\beta) = y. \) Then, \( \psi(x) = \alpha, \psi(y) = \beta, \) and
\[ \psi(xy) = \psi(-\phi(\alpha)\phi(\beta)) \leq \psi(-\phi(\alpha\beta)) = \alpha\beta = \psi(x)\psi(y). \]

Theorem 2.8. Let (C1)-(C4), and (2.18) hold. Suppose that \( q(t) \) satisfies
\[ \int_{T}^{\infty} q(t) \, dt < \infty \] (2.20)
and
\[ \int_{T}^{\infty} \psi \left( \frac{1}{p(t)} \int_{t}^{\infty} q(s) \, ds \right) \, dt = \infty. \] (2.21)

If
\[ \int_{T}^{\infty} \frac{dy}{\psi(f(y))} < \infty, \quad \text{for each } \epsilon > 0, \] (2.22)
and \( y(t) \) is an eventually positive solution of equation (1.1), then \( \lim_{t \to \infty} y(t) = 0 \)

Proof. Let \( y(t) > 0 \) on \([T, \infty) \) for some \( T \geq t_0. \) By Lemma 2.1, \( y'(t) \) is nonoscillatory. We separate our proof into two cases.

Case 1. Suppose that \( y'(t) > 0 \) for \( t \geq T. \) It follows from (2.3) that \( (p(t)\phi(y'(t)))/(f(y(t))) \) is positive and decreasing on \([T, \infty) \) and
\[ 0 \leq \frac{p(\xi) \phi(y'(\xi))}{f(y(\xi))} + \int_{\xi}^{T} q(s) \, ds \leq \frac{p(t) \phi(y'(t))}{f(y(t))}, \]
for \( \xi \geq t \geq T. \) Letting \( \xi \to \infty, \) we obtain
\[ \int_{T}^{\infty} q(s) \, ds \leq \frac{p(t) \phi(y'(t))}{f(y(t))}, \]
for \( t \geq T. \) It follows from Lemma 2.2 that
\[ \frac{y'(t)}{\psi(f(y(t)))} = \frac{\psi(\phi(y'(t)))}{\psi(f(y(t)))} \geq \psi \left( \frac{\phi(y'(t))}{f(y(t))} \right) \geq \psi \left( \frac{1}{p(t)} \int_{t}^{\infty} q(s) \, ds \right). \]
Thus,
\[
\int_{T}^{t} \psi \left( \frac{1}{p(s)} \int_{s}^{\infty} q(\tau) \, d\tau \right) \, ds \leq \int_{T}^{t} \frac{y'(s)}{\psi(f(y(s)))} \, ds = \int_{y(T)}^{y(t)} \frac{du}{\psi(f(u))},
\]
(2.23)
Letting \( t \to \infty \), the right side of (2.23) is finite whereas the left side of (2.23) tends to \( \infty \). Hence, we obtain a contradiction.

**CASE 2.** Suppose that \( y'(t) < 0 \) for \( t \geq T \). Then, (2.4) holds. We claim that \( \alpha = 0 \). Otherwise, there exists a number \( \beta \geq \alpha > 0 \), such that \( y(t) \geq \beta \) for \( t \geq T \). It follows from (1.1) and (C1) that
\[
0 \leq p(\xi) \phi(y'(\xi)) \leq p(t) \phi(y'(t)) - f(\beta) \int_{t}^{T} q(s) \, ds, \quad \xi \geq t \geq T.
\]
Letting \( \xi \to \infty \), we have
\[
p(t) \phi(y'(t)) \geq f(\beta) \int_{t}^{\infty} q(s) \, ds, \quad t \geq T.
\]
It follows from Lemma 2.2 that
\[
y'(t) = \psi(-\phi(y'(t))) \leq \psi \left( -f(\beta) \int_{t}^{\infty} q(s) \, ds \right) \leq \psi(-f(\beta)) \psi \left( \frac{1}{p(t)} \int_{t}^{\infty} q(s) \, ds \right).
\]
Thus,
\[
y(t) \leq y(T) + \psi(-f(\beta)) \int_{t}^{\infty} \psi \left( \frac{1}{p(s)} \int_{s}^{\infty} q(\tau) \, d\tau \right) \, ds.
\]
By (2.21), \( \lim_{t \to \infty} y(t) = -\infty \), which is a contradiction. Hence, \( \lim_{t \to \infty} y(t) = 0 \). \[ \square \]

**THEOREM 2.9.** Let (C1)-(C4), and (2.18) hold. Suppose that
\[
\int_{T}^{\infty} \psi \left( \frac{1}{p(s)} \right) \, ds < \infty
\]
(2.24)
and
\[
\Psi(t) = \int_{T}^{\infty} \psi \left( \frac{1}{p(s)} \right) \, ds.
\]
(2.25)
If
\[
\int_{T}^{\infty} \psi \left( \frac{1}{p(s)} \int_{s}^{\infty} q(\mu) f(\lambda \Psi(\mu)) \, d\mu \right) \, ds = -\infty, \quad \text{for every } \lambda < 0,
\]
and \( y(t) \) is an eventually negative solution of equation (1.1), then \( \lim_{t \to \infty} y(t) = -\infty \).

**PROOF.** Let \( y(t) < 0 \) on \([T, \infty)\) for some \( T \geq t_{0} \). By Lemma 2.1, \( y'(t) \) is nonoscillatory. We separate our proof into two cases.

**CASE 1.** Suppose that \( y'(t) \) is eventually positive. We assume without loss of generality that \( y'(t) > 0 \) on \([T, \infty)\). As in the proof of Theorem 2.1, (2.9) holds for \( t \geq T \). By Lemma 2.2 and (2.25), we obtain
\[
-y(t) \geq \alpha - y(t) \geq -\lambda \Psi(t),
\]
for \( t \geq T \), where \( \lambda = -p(T)\phi(y'(T)) < 0 \) and \( \lim_{t \to \infty} y(t) = \alpha \leq 0 \). Then, \( y(t) \leq \lambda \Psi(t) < 0 \) for \( t \geq T \). From (1.1),
\[
[p(t) \phi(y'(t))]' = -q(t)f(y(t)) \geq -q(t)f(\lambda \Psi(t)), \quad \text{on } [T, \infty).
\]
Integrating it over \([T, \infty)\), we get
\[
p(t) \phi(y'(t)) \geq p(t) \phi(y'(t)) - p(T) \phi(y'(T)) \geq -\int_{T}^{t} q(\mu) f(\lambda \Psi(\mu)) \, d\mu.
\]
Then, by Lemma 2.2,
\[ y'(t) = \psi(\phi(y'(t))) \geq \psi\left( \frac{-1}{p(t)} \int_T^t q(\mu) f(\lambda \Psi(\mu)) \, d\mu \right) \geq \psi(-1) \psi\left( \frac{1}{p(t)} \int_T^t q(\mu) f(\lambda \Psi(\mu)) \, d\mu \right). \]
This implies that
\[ y(t) \geq y(T) + \psi(-1) \int_T^t \psi\left( \frac{1}{p(s)} \int_T^s q(\mu) f(\lambda \Psi(\mu)) \, d\mu \right) \, ds. \]

By (2.25), \( \lim_{t \to \infty} y(t) = \infty \), which contradicts the fact that \( y(t) < 0 \) for \( t \geq T \).

**CASE 2.** Suppose that \( y'(t) \) is eventually negative. We assume without loss of generality that \( y'(t) < 0 \) on \( [T, \infty) \). It follows from (1.1) that \( p(t)\phi(y'(t)) \) is nondecreasing on \( [T, \infty) \). By (2.18), we assume without loss of generality that \( y(t) < -\Psi(t) < 0 \) for \( t \geq T \). Similar to the proof of Case 2 in the theorem, we have
\[ y'(t) = \psi(-\phi(y'(t))) \leq \psi\left( \frac{1}{p(t)} \int_T^t q(\mu) f(-\Psi(\mu)) \, d\mu \right). \]
Integrating it over \( [T, \infty) \), we get
\[ y(t) \leq y(T) + \int_T^t \psi\left( \frac{1}{p(s)} \int_T^s q(\mu) f(-\Psi(\mu)) \, d\mu \right) \, ds. \]
By (2.19), \( \lim_{t \to \infty} y(t) = -\infty \). This completes our proof of the theorem. \( \blacksquare \)

**REFERENCES**